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ON THE LIPSCHITZ BEHAVIOR OF OPTIMAL SOLUTIONS IN PARAMETRIC PROBLEMS OF QUADRATIC OPTIMIZATION AND LINEAR COMPLEMENTARITY

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PREFACE

The Adaptation and Optimization Project, part of the System and Decision Sciences Program, is concerned with the development of methods and algorithms for treating stochastic optimization problems. To construct such methods and algorithms, however, often requires preliminary results in optimization theory.

In this paper, Diethard Klatte, one of the participants in the 1983 Young Scientists' Summer Program, studies the Lipschitz behavior of (generally non-polyhedral) optimal set mappings in certain parametric optimization problems. He shows that, under mild assumptions, the corresponding value functions are Lipschitzian on bounded convex sets.

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ABSTRACT

In this paper S.M. Robinson's result concerning the upper Lipschitz continuity of polyhedral multifunctions is used to study the Lipschitz behavior of (generally non-polyhedral) optimal set mappings in certain parametric optimization problems. Under mild assumptions, the corresponding value functions are shown to be Lipschitzian on bounded convex sets.

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1. INTRODUCTION

This paper is concerned with certain classes of parametric optimization problems, and in particular with the possible Lipschitz dependence of both the set-valued functions of global optimal solutions and the minimum value functions upon the parameters in these problems.

We shall begin by introducing a standard *parametric quadratic optimization problem*. Let A be an $m\times n$ matrix, and let C be some symmetric $n\times n$ matrix. We consider the problem

$$\min_x \{f(x,p) \mid x\in M(b)\}, \quad (p,b)\in R^n \times R^m, \tag{1}$$

where p and b are regarded as parameters, M is a set-valued function from R^m to R^n defined by

$$M(b) = \{x\in R^n \mid Ax \leqslant b\}, \quad b\in R^m,$$

and $f: R^n \times R^n \to R$ is given by

$$f(x,p) = \frac{1}{2} x^T Cx + p^T x; \quad (x,p)\in R^n \times R^n .$$

For fixed $(p,b)\in R^n \times R^m$, the necessary conditions for optimality associated with (1) are

$$\begin{aligned} Cx + A^T u + p &= 0 \\ Ax \quad - b &\leqslant 0 \\ u &\geqslant 0 \\ u^T(Ax - b) &= 0 . \end{aligned} \tag{2}$$

We are interested in the properties of some set-valued functions (also called *multifunctions*) related to the parametric problem given above. The multifunction $\psi: R^n \times R^m \to R^n$,

$$\psi(p,b) = \{x\in M(b) \mid f(x,p) \leqslant f(z,p) \quad \text{for all} \quad z\in M(b)\}, \tag{3}$$

assigns to each parameter vector the set of all global optimal solutions of a (generally non-convex) quadratic optimization problem; this is called the *optimal set mapping* of (1). The function $\phi: R^n \times R^m \to R \cup \{-\infty, +\infty\}$,

$$\phi(p,b) = \inf \{f(x,p) \mid x\in M(b)\} \tag{4}$$

is the *infimum function* of (1), while the multifunction $KT: R^n \times R^m \to R^n \times R^m$

$$KT(p,b) = \{(x,u)\in R^n \times R^m \mid (x,u) \quad \text{satisfies} \quad (2)\} \tag{5}$$

is the *Kuhn-Tucker set mapping* associated with (1).

We recall that problem (1) is closely related to the problem of parametric optimization with linear complementary constraints:

$$\min_{(x,u)} \{p^T x - b^T u \mid (x,u)\in KT(p,b)\}, \quad (p,b)\in R^n \times R^m . \tag{6}$$

Let the infimum function ϕ_{KT} of (6) and the optimal set mapping ψ_{KT} of (6) be defined analogously to (4) and (3), respectively. Then a well-known result from quadratic optimization theory tells us that if $\psi(p,b) \neq \phi$, then $\psi_{KT}(p,b) \neq \phi$ and

$$\psi(p,b) = \pi_n(\psi_{KT}(p,b)) , \tag{7}$$

where π_n is the canonical projection from $R^n \times R^m$ to R^n . Of course, in this case the optimal values $\phi(p,b)$ and $\phi_{KT}(p,b)$ coincide.

We shall now give some definitions and notation used when dealing with multifunctions. For the main part we follow Robinson (1981) and the monograph by Bank et al. (1982).

Let $F: Y \subset R^r \to R^s$ be a multifunction. The set

$$\text{graph } F = \{(\mu,x) \in Y \times R^s \mid x \in F(\mu)\}$$

is called the *graph* of F . The *effective domain* of F is

$$\text{dom } F = \{\mu \in Y \mid F(\mu) \neq \phi\} .$$

The set $X \subset R^r \times R^s$ is said to be polyhedral if it is a union of finitely many polyhedral convex sets, called components. F is *polyhedral* (*closed, convex*) if its graph is polyhedral (closed, convex). F is *locally upper Lipschitzian at* $\mu^o \in Y$ with modulus γ (shortened to U.L.(γ) in what follows), if for some neighborhood U of μ^o and all $\mu \in U \cap Y$,

$$F(\mu) \subset F(\mu^o) + \gamma \|\mu-\mu^o\| B ,$$

where B denotes the closed Euclidean unit ball, $\|.\|$ is the Euclidean norm, and $+$ represents the Minkowski sum of two sets. We say that F is *locally upper Lipschitzian on* Y if there exists a number $\gamma > 0$ such that F is locally U.L.(γ) at all points $\mu^o \in Y$. For further (semi-) continuity terminology we refer to Bank et al. (1982), Chapter 2.

Building on work by Hoffman (1952) and Walkup and Wets (1969a), Robinson (1976, 1981) has shown that a polyhedral multifunction $F: R^r \to R^s$ is always locally upper Lipschitzian on R^r. It is easy to see that the solution set mapping KT of the parametric linear complementary problem (2) is polyhedral.

The close connection between problems (1) and (6) suggests that it would be interesting to explore the Lipschitzian properties of the functions ψ and ϕ (or ψ_{KT} and ϕ_{KT}) by studying a more general parametric optimization problem of the type

$$\min_z \{\lambda^T z \mid z \in \Gamma(\mu)\} , \quad (\lambda,\mu) \in R^s \times R^r \tag{8}$$

where Γ is a polyhedral multifunction from R^r to R^s. Obviously, the parametric optimization problem (6) is a special case of problem (8). Let $\tilde{\psi}$ and $\tilde{\phi}$ denote the optimal set mapping and infimum function of (8), respectively, i.e.,

$$\tilde{\phi}(\lambda,\mu) = \inf \{\lambda^T z \mid z \in \Gamma(\mu)\} ,$$

$$\tilde{\psi}(\lambda,\mu) = \{z \in \Gamma(\mu) \mid \lambda^T z = \tilde{\phi}(\lambda,\mu)\} . \tag{9}$$

The functions $\tilde{\phi}$ and $\tilde{\psi}$ are in general far from being continuous, let alone Lipschitzian. Consider the simple example $\min_{z=(x,u)} \{-u \mid x = \mu , u \leqslant 1 , x.u = 0\}$ where $\mu \in R$ is a parameter. Here we have $\tilde{\phi}(0) = -1$ but $\tilde{\phi}(\mu) = 0$ if $\mu \neq 0$. Further, $\tilde{\psi}(0) = \{(0,1)\}$ but $\tilde{\psi}(\mu) = \{(\mu,0)\}$ if $\mu \neq 0$. Thus neither the infimum function nor the optimal solution function is continuous at $\mu = 0$. This example also shows that $\tilde{\psi}$ is not in general polyhedral. However, the functions ψ, $\tilde{\psi}$, ϕ and $\tilde{\phi}$ can display Lipschitz behavior in various special cases.

As mentioned above, $\tilde{\psi}$ is locally U.L. on the whole parameter space, assuming that $\tilde{\psi}$ is polyhedral. Moreover, in this case $\tilde{\phi}$ is Lipschitzian on each bounded convex subset of dom $\tilde{\psi}$ (cf. Robinson, 1981, Proposition 4). These results can be immediately applied to the functions ψ and ϕ of a parametric quadratic program of type (1) if the matrix C is positive semidefinite (cf. Robinson, 1981). Under this convexity assumption the set dom ψ of (1) is a polyhedral convex cone (cf. Eaves, 1971 , or Bank et al., 1982). When $C = 0$, (1) reduces to a parametric linear optimization problem; for analogous properties of ϕ, ψ and dom ψ in this special case we refer, e.g., to Walkup and Wets (1969a,b), Nožička et al. (1974), Kleinmann (1978) and Mangasarian (1982).

The purpose of this paper is to show which of these results concerning Lipschitz behavior and polyhedrality are "conserved" in a general parametric quadratic program of type (1) and in parametric problems (8) with a polyhedral constraint set function. It will be shown that $\tilde{\psi}$ and $\tilde{\phi}$ still keep the Lipschitz properties mentioned above when we require $\tilde{\phi}$ to be upper semicontinuous rather than requiring $\tilde{\psi}$ to be polyhedral. This result will be applied to the optimal set mapping ψ and the infimum function ϕ of the quadratic program (1). We also provide a theorem which shows that the set dom ψ is polyhedral. It is worth noting here that the mapping ψ of (1) is not in general polyhedral. A counterexample will be given.

2. MAIN CONTINUITY RESULTS

Throughout this section we are concerned with the parametric optimization problem (8):

$$\min_z \{\lambda^T z \mid z \in \Gamma(\mu)\} , \quad (\lambda,\mu) \in R^s \times R^r$$

where Γ is a polyhedral multifunction.

Let $G_i \subset R^r \times R^s$ $(i=1,2,\ldots,N)$ be nonempty polyhedral convex sets satisfying

$$\text{graph } \Gamma = \bigcup_{i=1}^{N} G_i . \tag{10}$$

Then there exist vectors $b^{i1}, b^{i2},\ldots,b^{iN_i} \in R^s$ and $d^{i1},d^{i2},\ldots,d^{iN_i} \in R^r$ and real numbers $a_{i1},a_{i2},\ldots,a_{iN_i}$ such that

$$G_i = \{(\mu,z)\in R^r \times R^s \mid b^{ijT} z \leqslant d^{ijT}\mu + a_{ij},\ j=1,2,\ldots,N_i\}. \qquad (11)$$

The decomposition (10) of graph Γ into its components suggests splitting the multifunction Γ into the multifunctions Γ_i, where

$$\Gamma_i(\mu) = \{z\in R^s \mid (\mu,z)\in G_i\},\quad \mu\in R^r,\ i\in\{1,2,\ldots,N\}.$$

This leads to the following parametric linear optimization problems:

$$\min_z \{\lambda^T z \mid z\in\Gamma_i(\mu)\},\quad (\lambda,\mu)\in R^s \times R^r, \qquad (12)_i$$

or, equivalently,

$$\min_z \{\lambda^T z \mid b^{ijT} z \leqslant d^{ijT}\mu + a_{ij},\ j\in\{1,2,\ldots,N_i\}\},$$

$$(\lambda,\mu)\in R^s \times R^r.$$

We shall use $\tilde{\phi}_i$ and $\tilde{\psi}_i$ to denote the infimum function and the optimal set mapping of the problem $(12)_i$ $(i=1,2,\ldots,N)$; the definition is analogous to (9). Obviously, for any $(\lambda,\mu)\in$ dom $\tilde{\psi}$ there is an index set $I(\lambda,\mu) \subset \{1,2,\ldots,N\}$, $I(\lambda,\mu) \neq \emptyset$, such that

$$\tilde{\psi}(\lambda,\mu) = \bigcup_{i\in I(\lambda,\mu)} \tilde{\psi}_i(\lambda,\mu) \quad \text{and} \quad \tilde{\psi}_i(\lambda,\mu) \neq \emptyset \quad \text{if} \quad i\in I(\lambda,\mu).$$

Now we state the main result of this paper.

Theorem 1. *Consider the parametric optimization problem (8). Suppose that Λ is a nonempty subset of dom $\tilde{\psi}$ and that the infimum function $\tilde{\phi}$ (restricted to the domain Λ) is upper semicontinuous on Λ. Then there exists a constant γ such that the optimal set mapping $\tilde{\psi}$ is locally U.L.(γ) on Λ.*

Proof

Let the sets G_i, the functions $\tilde{\phi}_i$ and $\tilde{\psi}_i$ and the parametric optimization problems $(12)_i$ be defined as above. Writing $y = (\lambda,\mu)$, we let $I(y)$ denote the index sets:

$$I(y) = \{i\in\{1,2,\dots,N\} \mid \emptyset\neq\tilde{\psi}_i(y) \subset \tilde{\psi}(y)\}\ ,\quad y\in\Lambda\ .$$

Choose any $y^o = (\lambda^o,\mu^o)\in\Lambda$. To show the existence of a neighborhood $U = U(y^o)$ such that

$$\tilde{\psi}(y)\subset \bigcup_{i\in I(y^o)} \tilde{\psi}_i(y) \quad \text{for all} \quad y\in U \cap \Lambda\ , \tag{13}$$

we assume, arguing by contradiction, that

$$\emptyset \neq \tilde{\psi}_j(y^k) \cap \tilde{\psi}(y^k) \tag{14}$$

for some $j\notin I(y^o)$ and some sequence $\{y^k\} \subset U \times \Lambda$ converging to y^o . In particular, $\tilde{\phi}(y^k) = \tilde{\phi}_j(y^k)$ for all k .

The classical results of parametric linear optimization theory tell us that the set dom $\tilde{\psi}_j$ is closed and $\tilde{\phi}_j$ is continuous on dom $\tilde{\psi}_j$ (cf., for example, Nožička et al., 1974). It follows that

$$\tilde{\psi}_j(y^o) \neq \emptyset \quad \text{and} \quad \tilde{\phi}_j(y^o) = \lim_{k\to\infty} \tilde{\phi}_j(y^k)\ .$$

Taking (14) and the assumption that $\tilde{\phi}$ is upper semicontinuous into account, we thus have

$$\tilde{\phi}(y^o) < \tilde{\phi}_j(y^o) = \lim_{k\to\infty} \tilde{\phi}_j(y^k) = \lim_{k\to\infty} \tilde{\phi}(y^k) \leqslant \tilde{\phi}(y^o)\ ,$$

which is a contradiction. Hence (13) is true. From $(12)_i$ we know that each of the multifunctions $\tilde{\psi}_i$ $(i=1,2,\dots,N)$ is polyhedral, and so the multifunction

$$\Big(\bigcup_{i\in I(y^o)}\psi_i\Big)(y) = \bigcup_{i\in I(y^o)} \psi_i(y)$$

is also polyhedral. Hence, $\Big(\bigcup_{i\in I(y^o)}\psi_i\Big)$ is locally U.L. with some modulus γ on $R^s \times R^r$ (cf. Robinson, 1981, Proposition 1). Then (13) and the definition of $I(y^o)$ lead to the inclusions

$$\tilde{\psi}(y) \subset \bigcup_{i\in I(y^o)} \tilde{\psi}_i(y) \subset \tilde{\psi}(y^o) + \gamma\|y-y^o\| B$$

for all $y\in\Lambda$ near y^o. Since $y^o\in\Lambda$ was chosen arbitrarily, this completes the proof.

One consequence of Theorem 1 is that a certain Lipschitz property of the infimum function $\tilde{\phi}$ of (8) can be derived by adapting some of Robinson's (1981) results to our case. The following corollary is a modification of Proposition 2 in Robinson (1981). It is important in investigating the Lipschitz behavior of $\tilde{\phi}$ over bounded subsets of parameter space.

Corollary 1. Assume that the assumptions of Theorem 1 hold, and that $\Lambda \subset \operatorname{dom} \tilde{\psi}$ is closed. If Q is any bounded subset of Λ then

$$\left.\begin{array}{l}\text{there is a constant } \beta = \beta(Q) \text{ such that} \\ \tilde{\psi}(\lambda,\mu) \cap \beta.B \neq \phi \text{ for all } (\lambda,\mu)\in Q .\end{array}\right\} \qquad (15)$$

Proof

Since Λ is closed the closure of Q, $\operatorname{cl} Q$, is a subset of Λ. As above, we construct the multifunctions Γ_i and $\tilde{\psi}_i$ and the problems $(12)_i$ $(i = 1,2,\ldots,N)$. Choose any $y^o=(\lambda^o,\mu^o)\in \operatorname{cl} Q$. Under the assumptions of Theorem 1, there is a neighborhood $U = U(y^o)$ such that (13) holds, i.e.,

$$\tilde{\psi}(y) \subset \bigcup_{i\in I(y^o)} \tilde{\psi}_i(y) \quad \text{for all } y\in U \cap \operatorname{cl} Q .$$

Without loss of generality U may be assumed to be a compact convex polyhedron. First let $i\in I(y^o)$ be fixed and consider any $y = (\lambda,\mu)\in U \cap \operatorname{dom} \tilde{\psi}_i$. Since $\tilde{\psi}_i(y)$ is the set of optimal solutions of a linear program, it must be a closed face of the convex polyhedron $\Gamma_i(\mu)$ of feasible solutions of this program. This means that

$$\tilde{\psi}_i(\lambda,\mu) = \Gamma_{i,J}(\mu) \underset{Df}{=} \{z\in\Gamma_i(\mu) \mid b^{ijT}z = d^{ijT}\mu + a_{ij} ,\ j\in J\}$$

for some index set $J \subset \{1,2,\dots,N_i\}$ (cf. (11)). Obviously the set

$$W_{i,J} \underset{Df}{=} \pi_r(U \cap \operatorname{dom} \tilde{\psi}_i) \cap \operatorname{dom} \Gamma_{i,J}$$

is a bounded convex polyhedron, where π_r denotes the canonical projection to R^r . The function

$$h_{i,J}(\tilde{\mu}) = \min \{\|z\| \mid z \in \Gamma_{i,J}(\tilde{\mu})\}\ ,\ \tilde{\mu} \in W_{i,J}\ ,$$

is convex on $W_{i,J}$, and hence attains a maximum on $W_{i,J}$. If Z is the set of all those index sets $J \subset \{1,2,\dots,N_i\}$ for which $W_{i,J} \neq \phi$ then

$$\tau_i = \max_{J \in Z} \max_{\tilde{\mu} \in W_{i,J}} h_{i,J}(\tilde{\mu})$$

is well defined.

Now let $y = (\lambda,\mu)$ be any element of $U \cap \operatorname{cl} Q$. From the definition of U, $\tilde{\psi}_i(y) \cap \tilde{\psi}(y) \neq \phi$ for some $i \in I(y^o)$. This implies, in particular, that $\tilde{\psi}_i(y) \subset \tilde{\psi}(y)$. Hence, there is a point $z(y) \in \tilde{\psi}_i(y)$ satisfying $z(y) \in \tilde{\psi}(y)$ and

$$\|z(y)\| = \min \{\|z\| \mid z \in \tilde{\psi}_i(y)\}\ .$$

Because $\tilde{\psi}_i(y) = \Gamma_{i,J}(\mu)$ for some $J \subset \{1,2,\dots,N_i\}$, we have

$$\|z(y)\| = h_{i,J}(\mu) \leqslant \tau_i\ .$$

Therefore

$$\tilde{\psi}(y) \cap (\max_{i \in I(y^o)} \tau_i) \cdot B \neq \phi \quad \text{for all} \quad y \in U \cap \operatorname{cl} Q$$

and the assertion follows from the compactness of $\operatorname{cl} Q$ (we omit this standard argument here).

A function $g: Q \subset R^r \to R$ is said to be *locally Lipschitzian on* Q if the multifunction $q \mapsto \{g(q)\}$ is locally U.L. on Q ; it is *Lipschitzian on* Q if there is a constant γ such that

$$|g(q') - g(q")| \leq \gamma \|q' - q"\| \quad \text{for all} \quad q', q" \in Q .$$

Corollary 2. Assume that the assumptions of Theorem 1 hold and that $\Lambda \subset \text{dom}\,\tilde{\psi}$ *is closed. If* $Q \subset \Lambda$ *is any bounded set then* $\tilde{\phi}$ *is locally Lipschitzian on* Q *(with a uniform constant* $\gamma = \gamma(Q)$*). Further, if* Q *is bounded and convex then* $\tilde{\phi}$ *is Lipschitzian on* Q *.*

The proof of Corollary 2 is almost identical to that of Proposition 4 in Robinson (1981), and we will not repeat it here. We should just mention that Robinson's Proposition 4 requires $\tilde{\psi}$ to be polyhedral, but this assumption may be weakened without affecting the result. The proof actually makes no use of the polyhedrality condition, but only of the following requirements which are fulfilled both when $\tilde{\psi}$ is polyhedral and when $\tilde{\psi}$ satisfies the assumptions of Theorem 1:

(i) $\tilde{\psi}$ is locally U.L on Λ ,
(ii) for each bounded $Q \subset \Lambda$, $\tilde{\psi}$ has the property (15).

3. APPLICATION TO PARAMETRIC QUADRATIC PROGRAMS

In this section we consider the special case of the parametric quadratic optimization problems defined in (1). First we give an example which shows that, in general, the graph of the optimal set mapping ψ of (1) is not polyhedral.

Take the parameter vector $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ and consider

$$\min \; \{x_1 x_2 \mid x = (x_1, x_2) \in M(\mu)\} ,$$

where

$$M(\mu) = \{x \in R^2 \mid \mu_1 \leq x_1 \leq \mu_2, \; \mu_3 \leq x_2 \leq \mu_4\} .$$

If graph $\psi \subset R^6$ were a union of a finite number of polyhedral convex sets then

$$G = \left\{ (\mu,x) \in \text{graph } \psi \;\middle|\; \begin{array}{lll} x = -1, & \mu_1 = -1, & \mu_3 \leqslant 0 \\ y = 1, & \mu_2 \geqslant 0, & \mu_4 = 1 \end{array} \right\}$$

would also have this property. It can easily be verified that (for $\mu_2 \geqslant 0$, $\mu_3 \leqslant 0$)

$$\psi(-1, \mu_2, \mu_3) = \begin{cases} \left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\} & \text{if } -1 < \mu_2\mu_3 \\ \left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} \mu_2 \\ \mu_3 \end{pmatrix}\right\} & \text{if } -1 = \mu_2\mu_3 \\ \left\{\begin{pmatrix} \mu_2 \\ \mu_3 \end{pmatrix}\right\} & \text{if } -1 > \mu_2\mu_3 \end{cases}$$

and therefore

$$G = \left\{ (\mu,x) \in R^6 \;\middle|\; \begin{array}{llll} x = -1, & \mu_1 = -1, & \mu_3 \leqslant 0, & -1 \leqslant \mu_2\mu_3 \\ y = 1, & \mu_2 \geqslant 0, & \mu_4 = 1 & \end{array} \right\} .$$

G cannot be represented as a union of finitely many polyhedral convex sets, and hence, in this example, ψ is not polyhedral.

However, using Theorem 1 and Corollary 2, we can show that the infimum function ϕ and the optimal set mapping ψ of the parametric quadratic program (1) do have certain Lipschitz properties although ψ is not in general polyhedral.

Theorem 2. The optimal set mapping ψ of the parametric quadratic program (1) is locally U.L. on $R^n \times R^m$. The infimum function ϕ of (1) is locally Lipschitzian on bounded subsets of dom ψ and Lipschitzian on bounded convex subsets of dom ψ.

Proof

We shall use Theorem 1 and Corollary 2 to prove Theorem 2. If we set $s = n + m$, $r = n + m$, $\lambda = (p, -b) \in R^n \times R^m$ and $\mu = (p,b) \in R^n \times R^m$, the parametric problem (8) reduces to the special case (6). We thus have $\tilde{\psi} = \psi_{KT}$ and $\tilde{\phi} = \phi_{KT}$. From quadratic optimization theory we know that

$$\text{dom } \psi \subset \text{dom } \psi_{KT}$$

and

$$\phi(p,b) = \phi_{KT}(p,b) \quad \text{for all} \quad (p,b) \in \text{dom } \psi .$$

Since the multifunction M defined by (1) is lower semicontinuous and closed on dom M, the infimum function ϕ is upper semicontinuous on dom ψ and dom ψ is closed (cf., for example, Theorems 4.2.1 and 4.2.2 in Bank et al., 1982). If we define

$$\Lambda = \text{dom } \psi$$

then Theorem 1 and Corollary 2 apply to ϕ and ψ. This implies that ψ is locally upper Lipschitzian on dom ψ and that ϕ has the properties specified in the theorem. If $(\lambda^o, \mu^o) \notin$ dom ψ then there is a neighborhood U of (λ^o, μ^o) such that $\psi(\lambda,\mu) = \emptyset$ for all $(\lambda,\mu) \in U$, because dom ψ is closed. Hence, ψ is, trivially, U.L. on U, and this completes the proof.

As a by-product the preceding theorem provides a new proof of the fact that the optimal set mapping ψ and the infimum function ϕ of (1) are Hausdorff upper semicontinuous and continuous, respectively, on dom ψ (cf. Kummer, 1977). Theorem 2 still holds if the constraint set mapping M of (1) is an arbitrary polyhedral convex multifunction. In fact, the proof of Theorem 2 does not depend upon any special form of the multifunctions M and KT.

We note that Kleinmann also postulated upper Lipschitz continuity of ψ (cf. Kleinmann, 1978, Satz III.2) in a sense even more strongly than here, but the outline proof given in his paper is contradictory and the result, in the form presented there, is incorrect.

We shall conclude this paper with a theorem which gives a deeper insight into the structure of the effective domain of ψ. It is known that dom ψ is a closed cone (Eaves, 1971), and there are simple examples illustrating the fact that dom ψ is not necessarily convex (Bank et al., 1982). However, Theorem 2 suggests that we should look for possible convex subsets of dom ψ.

Theorem 3. The effective domain of the optimal set mapping ψ of (1) is a union of finitely many polyhedral convex cones.

Proof

Eaves (1971) has shown that the parameter vector (p,b) is an element of dom ψ if and only if

$$M(b) \neq \phi \tag{16a}$$

$$\left.\begin{array}{l} A v \leq 0 \\ v \in R^n \end{array}\right\} \Rightarrow v^T C v \geq 0 , \tag{16b}$$

$$\left.\begin{array}{ll} v^T C v = 0 & \\ A v & \leq 0 \\ A x & \leq b \\ v \in R^n, & x \in R^n \end{array}\right\} \Rightarrow v^T (Cx + p) \geq 0 . \tag{16c}$$

If (16b) holds then, obviously, the cone

$$S \underset{Df}{=} \{v \in R^n \mid v^T C v = 0 , Av \leq 0\}$$

is the set of optimal solutions of the optimization problem

$$\min \{v^T C v \mid Av \leq 0\}$$

and hence S is a union of a finite number of polyhedral convex cones. This means that there are finitely many vectors generating S, say $v^1, v^2, \ldots, v^N$. Under assumption (16b), condition (16c) is therefore equivalent to

$$\left.\begin{array}{l} Ax \leqslant b \\ x \in R^n \end{array}\right\} \Rightarrow v^{iT}(Cx + p) \geqslant 0 \quad (i=1,2,\ldots,N). \qquad (16c')$$

Eaves' characterization will now be used to derive the result. If $\text{dom}\,\psi = \phi$ there is nothing to be proved. Suppose now that $\text{dom}\,\psi \neq \phi$. Defining

$$\phi_i(b) = \inf \{v^{iT} C\, x \mid Ax \leqslant b\} \quad (i=1,2,\ldots,N)$$

and

$$\text{dom}\,\phi_i = \{b \in \text{dom}\,M \mid \phi_i(b) > -\infty\} \quad (i=1,2,\ldots,N),$$

we deduce from $\text{dom}\,\psi \neq \phi$, taking (16a), (16b) and (16c') into account, that for all $i \in \{1,2,\ldots,N\}$ the set $\text{dom}\,\phi_i$ is non-empty. By linear optimization theory, we thus have (for all i)

$$\text{dom}\,\phi_i = \{b \in \text{dom}\,M \mid \phi_i(b) = \min_{x \in M(b)} v^{iT} C\, x\} = \text{dom}\,M .$$

Since condition (16b) is satisfied independently of (p,b), we may conclude from the foregoing that

$$(p,b) \in \text{dom}\,\psi \quad \Leftrightarrow \quad \left\{\begin{array}{l} b \in \text{dom}\,M \\ \text{and} \\ \phi_i(b) \geqslant -p^T v^i \quad (i=1,\ldots,N). \end{array}\right.$$

The Basic Decomposition Theorem of parametric linear optimization (cf. Nozicka et al., 1974; Walkup and Wets, 1969) states that $\text{dom}\,M$ can be decomposed into finitely many polyhedral convex cones such that $\phi_i (i \in \{1,2,\ldots,N\})$ is linear on each of these components. Therefore, the sets

$$W_i = \{(p,b) \in R^n \times \text{dom } M \mid \phi_i(b) \geqslant -p^T v^i\} \quad (i=1,2,\ldots,N)$$

are polyhedral (but not necessarily convex) cones. Since

$\text{dom } \psi = \bigcap_{i=1}^{N} W_i$, the proof is complete.

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REFERENCES

Bank, B., J. Guddat, D. Klatte , B. Kummer and K. Tammer (1982). Non-linear parametric optimization, Akademie-Verlag, Berlin.

Eaves, B.C. (1971). On quadratic programming. *Management Science*, Vol. 17, pp. 678-711.

Hoffman, A.J. (1952). On approximate solutions of systems of linear inequalities. J. Res. Natl. Bur. Standards, Vol. 49, pp. 263-265.

Kleinmann, P. (1978). Quantitative Sensitivitaetsanalyse bei parametrischen Optimierungsaufgaben, Seminarbericht Nr. 9, Humboldt-Universitaet zu Berlin.

Kummer, B.(1977). Stabilitaet quadratischer Optimierungs-probleme. Wiss. Z. Humboldt-Univ. Berlin, Math.-Naturwiss, Reihe 5, pp. 565-569.

Mangasarian, O.L. (1982). A condition number for linear inequalities and linear programs. Methods of Operations Research, Vol. 43, pp. 3-15, (Verlagsgruppe Athenaeum/ Hain/Scriptor/Hanstein.)

Nožička, F., J. Guddat, H. Hollatz and B. Bank (1974). Theorie der linearen parametrischen Optimierung, Akademie-Verlag, Berlin.

Robinson, S.M. (1976). An implicit-function theorem for generalized variational inequalities. MRC Technical Summary Report No. 1672, Math. Res. Cent. Univ. of Wisconsin-Madison, Madison, Wisconsin.

Robinson, S.M. (1981). Some continuity properties of polyhedral multifunctions. Math. Programming Study 14, North-Holland Publ. Co., Amsterdam, pp. 206-214.

Walkup, D. and R. Wets (1969a). A Lipschitzian characterization of convex polyhedra. Proc. Am. Math. Soc., Vol. 23, pp. 167-173.

Walkup, D. and R. Wets (1969b). Lifting projections of convex polyhedra. Pacific J. Math., Vol. 28, pp. 465-475.