# ON THE LOCAL CONVERGENCE OF A QUASI-NEWTON METHOD FOR THE NONLINEAR PROGRAMMING PROBLEM

Thomas F. Coleman*
Andrew R. Conn**

TR 82-509
August 1982

Department of Computer Science
Upson Hall
Cornell University
Ithaca, New York 14853

*Department of Computer Science, Cornell University, Ithaca, New York

**Computer Science Department, University of Waterloo, Waterloo, Ontario, Canada

# On the Local Convergence of a Quasi-Newton Method

# For the Nonlinear Programming Problem

Thomas F. Coleman

Computer Science Department

Cornell University

Ithaca, NY 14853

Andrew R. Conn

Computer Science Department

University of Waterloo

Waterloo, Ontario, CANADA

N2L 3G1

ABSTRACT

In this paper we propose a new local quasi-Newton method to solve the equality constrained nonlinear programming problem. The pivotal feature of the algorithm is that a **projection** of the Hessian of the Lagrangian is approximated by a sequence of symmetric positive definite matrices. The matrix approximation is updated at every iteration by a projected version of the DFP or BFGS formula. We establish that the method is locally convergent and the sequence of x-values converges to the solution at a 2-step Q-superlinear rate.

## 1. Introduction

Quasi-Newton methods have had a large measure of success in the minimization of smooth nonlinear functions

$$f(x): R^n \rightarrow R^1.$$

In particular, the Davidon-Fletcher-Powell (DFP) and Broyden-Fletcher-Goldfarb-Shanno (BFGS) updating formulae have given solid numerical performances over the past decade and are generally accepted as the best rank-2 updating formulae (for dense problems). In addition to their numerical record, these methods have two significant theoretical properties: they are locally Q-superlinearly convergent and their Hessian approximations remain positive definite.

It is difficult to fully explain the superior numerical performance of the DFP/BFGS method relative to other updates however the maintenance of positive definite Hessian approximations seems crucial - it is also a 'natural' property since the true Hessian at the solution will likely be positive definite (and will certainly be at least positive semi-definite). In addition, positive definiteness allows for a stable implementation (Gill & Murray [1974]) and ensures that search directions are also descent directions.

The situation for minimization in the presence of nonlinear constraints is less satisfactory. Successive quadratic programming (SQP) and projection approaches have recently been in vogue: however, a true Q-superlinear quasi-Newton method for the non-convex case is unknown to the authors. Powell [1978] has adapted the BFGS formula to the nonlinearly constrained case - Powell gives sufficient conditions under which a successive quadratic programming approach will yield a 2-step Q-superlinear convergence rate (assuming convergence) but does not show that his modified BFGS method satisfies these conditions. Instead R-superlinear convergence is proven. Interestingly, the sufficiency conditions given by Powell necessitate that only a **projection** of the Lagrangian Hessian approximations be suitably accurate. We also note that Han [1976] has proven that this SQP/BFGS method exhibits Q-superlinear convergence for the convex case.

Other authors, Boggs,Tolle and Wang [1982], have given sufficiency (and necessary) conditions for Q-superlinear convergence, for the constrained problem, however we are unaware of an updating method which satisfies these conditions. Tanabe [1981] has proposed various projected updating schemes but, to our knowledge, has not established convergence properties.

In our opinion, a major difficulty with most of these approaches is that a full ($n$ by $n$) positive definite Hessian approximation is required by each of the methods but only a **projection** of the Hessian of the Lagrangian need be positive definite at the solution. Therefore we feel that a more natural approach is to recur a positive definite approximation to the projection of the Hessian of the Lagrangian. Gill & Murray [1974] have followed such a strategy in the case where all constraints are linear however there has been little work along these lines for the nonlinearly constrained problem.

Coleman and Conn [1982a,b] have suggested Newton and discrete-Newton methods for nonlinearly constrained problems, which require only a projected Hessian approximation. The method we describe here is a direct extension of the local Newton method given by Coleman and Conn [1982a,b].

The motivating remarks given in these previous papers are applicable here also, however here we present an alternate view. Consider the problem

minimize $f(x)$, subject to

$c_i(x) = 0,\ i=1,\ldots t,$

where all function are twice continuously differentiable. Suppose that our current estimate to the solution is $x$ and let $C$ be the $n$ by $t$ matrix of constraint gradients, evaluated at $x$. Let $Z$ be a $n$ by $(n-t)$ matrix, whose columns form an orthonormal basis for the null space of $C^T$ (assume that $C$ has rank $t$). Finally, let the correction to $x$, $\delta$, be defined as the solution to the following quadratic program:

(1.1) minimize $\nabla f(x)^T\delta + \frac{1}{2}\delta^T ZBZ^T\delta,$

subject to $C^T\delta + c(x) = 0$

where $B$ is an $(n-t)$ by $(n-t)$ positive definite matrix. The solution to (1.1) is given by

(1.2) $\delta = h + v$

where

(1.3) $h = -ZB^{-1}Z^T\nabla f(x)$, and

(1.4) $v = -C(C^TC)^{-1}c(x)$.

Note that B can be considered to be a positive definite approximation to

(1.5) $Z_*^T[\nabla^2 f(x^*) - \Sigma\lambda_i^*\nabla^2 c_i(x^*)]Z_*$

where $C_*^TZ_* = 0$, $Z_*^TZ_* = I$, $C_* = (\nabla c_1(x^*),\ldots,\nabla c_t(x^*))$, $\nabla f(x^*) = C_*\lambda^*$, and $c(x^*) = 0$.

Under second-order sufficiency conditions, the $(n-t)$ by $(n-t)$ matrix (1.5) is positive definite. The method we propose in section 3 uses a projected form of the DFP(BFGS) update to recur a positive definite approximation to (1.5). The correction to $x$ that we analyze differs slightly from (1.2) in that (1.4) is replaced with

(1.6) $v = -C(C^TC)^{-1}c(x+h)$.

**We emphasize that all results given in section 3 are valid if (1.4) replaces (1.6).** We have carried out the analysis using (1.6) because of a result given in Coleman and Conn [1982b] which states that (1.3) + (1.6) guarantee that a certain exact penalty function will decrease, provided $x$ is sufficiently close to $x^*$. The result is valid in the discrete-Newton case and is not true if (1.4) is used instead of (1.6). We have not yet proven that a similar result is true for the case when $B$ is a quasi-Newton approximation but it is this possibility which prompted the use of (1.6).

In section 2 we present conditions which are sufficient to give a 2-step Q-superlinear convergence rate (assuming convergence). These conditions are slightly more general than those given by Powell [1978] in that they do not presuppose a particular algorithm class. These conditions are in the spirit of the superlinearity characterization, for unconstrained optimization, given by Dembo, Eisenstat and Steihaug [1982].

In section 3 we describe the algorithm and then establish that the method is locally 2-step Q-superlinearly convergent. The method of proof is similar to that used by Broyden, Dennis, and More [1973] and Dennis and More [1974] for the unconstrained case.

In section 4 we give our concluding observations and discuss future work.

## 2. Sufficient Conditions For 2-Step Q-Superlinear Convergence

Consider the following equality constrained nonlinear programming problem:

minimize $f(x)$

subject to

$c_i(x) = 0, \quad i=1,\ldots,t$

where **all functions are twice continuously differentiable on an open convex set D of** $R^n$ **and map** $D \to R^1$. Let $x^* \in D$ be a local solution to the equality constrained nonlinear programming problem. The question we address in this section is this: given that a sequence of points $\{x^k\}$ converges to $x^*$, when can we be assured that a 2-step Q-superlinear convergence rate is achieved? That is, what reasonable conditions ensure that

$||x^{k+1} - x^*|| = o\,||x^{k-1} - x^*||$?

### Definitions and Assumptions

Unless stated otherwise, the results given in this paper will all be subject to the following assumptions.

Let $C(x)$ denote the $n$ by $t$ matrix $(\nabla c_1(x),\ldots,\nabla c_t(x))$ and let $c(x)$ denote the vector $(c_1(x),\ldots,c_t(x))^T$. Define $C_*$ and $c^*$ to be $C(x^*)$ and $c(x^*)$ respectively. For any $x$ in $D$, define $\lambda = \lambda(x)$ to be the vector $[C(x)^T C(x)]^{-1} C(x)^T \nabla f(x)$. **We will assume that there is an open convex set** $D$ containing $x^*$ such that for all $x$ in $D$, the singular values of $C(x)$ are uniformly bounded on $D$, above and below, by positive scalars.

An $n$ by $(n-t)$ matrix $Z(x)$ is defined to be a **Lipschitz continuous function of x** satisfying

(2.1) $\quad Z(x)^T Z(x) = I$

and

(2.2) $\quad C(x)^T Z(x) = 0,$

where I represents the identity matrix. (See Coleman and Sorensen [1982] for remarks concerning the computation of Z(x).) Uniquely define vectors $u(x)$, and $w(x)$ by

(2.3) $\quad x - x^* = C(x)w(x) + Z(x)u(x).$

Since $x^*$ is a solution, it follows that the gradient of $f$ can be expressed as a linear combination of the gradients of the constraint functions. That is, there exists a vector $\lambda^* \epsilon R^t$, such that

(2.4) $\quad \nabla f(x^*) = C_* \lambda^*.$

Define $L(x)$ to be the Lagrangian function $f(x) - c(x)^T \lambda^*$. **It will be assumed that the second-order sufficiency conditions hold at $x^*$. Thus the matrix**

$$H(x^*,x^*) = Z(x^*)^T [\nabla^2 f(x^*) - \Sigma \lambda_i^* \nabla^2 c_i(x^*)] Z(x^*),$$

**is positive definite.** We note that this implies that the the eigenvalues of the (n-t) by (n-t) matrix H(x,y), defined by

$$H(x,y) = Z(x)^T [\nabla^2 f(y) - \Sigma \lambda_i^* \nabla^2 c_i(y)] Z(x),$$

are uniformly bounded below by a positive scalar on a open convex region ($(D,D)$, say) containing $(x^*,x^*)$. The above implication is a consequence of the following: The eigenvalues of the matrix $H(x,y)$ are continuous functions of the elements of the matrix (see Ortega[1972] page 45, for example). The elements of $H(x,y)$ vary continuously with $(x,y)$ due to $Z$ and $\nabla^2 f$, $\nabla^2 c_i$ being continuous functions of $x$ and $y$ respectively. Finally, the result follows from observing that $H(x^*,x^*)$ is positive definite. **We assume that the radius of $D$ is sufficiently small so that the eigenvalues of the Hessian matrices**

$$\nabla^2 f(x),\ \nabla^2 c_i(x),\ i=1,\ldots,t$$

**are uniformly bounded above on $D$ by a positive scalar and that the Hessian matrices satisfy a Lipschitz condition on $D$.**

When the above quantities are evaluated at a particular point $x^k$, then the argument $x^k$ will be abbreviated to a simple subscript or superscript. For example, $C(x^k)$ will be written $C_k$ and $w(x^k)$ becomes $w^k$. We will denote $H(x^k,x^k)$ *by* $H_k$. The symbol '*' will be used to denote a function evaluated at $x^*$: for example, $\nabla f^*$ represents $\nabla f(x^*)$.

Let $\delta^k$ represent $x^{k+1} - x^k$, and define

$$r_L^k = Z_k^T \nabla f(x^k) + H_k Z_k^T \delta^k, \text{ and}$$

$$r_c^k = c^k + C_k^T \delta^k.$$

Unless noted otherwise, the symbol $||\cdot||$ will denote the vector or matrix 2-norm. **One final assumption: we assume that finite convergence does not occur - $x^k \neq x^*$, for all $k$.**

**Theorem 2.1**

If $x^k$ converges to $x^*$, $||x^{k+1} - x^*|| = O||x^k - x^*||$, and

$$(2.5) \qquad ||r_L^k|| + ||r_c^k|| = o(||Z_k^T \nabla f^k|| + ||c^k||)$$

then

$$||x^{k+1} - x^*|| = o||x^{k-1} - x^*||.$$

**Proof:**

The proof is divided into three parts: In the first part it is established that $||w^{k+1}|| = o||x^k - x^*||$; in part two, it is proven that $||u^{k+1}|| = o||x^{k-1} - x^*||$; finally, in part three, the desired result is obtained.

**Part 1**

Clearly we can write

$$(2.6) \qquad C_k^T(x^{k+1} - x^*) = C_k^T(x^k - x^*) - c^k + c^k + C_k^T \delta^k.$$

If we add $C_{k+1}^T(x^{k+1} - x^*)$ to both sides of (2.6) , re-arrange, and then take norms, we obtain

$$(2.7) \qquad ||w^{k+1}|| \leq ||(C_{k+1}^T C_{k+1})^{-1}|| [|| C_*^T(x^k - x^*) + (C_k - C_*)^T(x^k - x^*) - c^k || + ||c^k + C_k^T \delta^k|| + ||(C_{k+1} - C_k)^T (x^{k+1} - x^*)||].$$

By Taylor's Theorem, and $c^* = 0$,

$$c^k = C_*^T(x^k - x^*) + o||x^k - x^*||,$$

and therefore the first term in (2.7) is $o||x^k-x^*||$. By (2.5),

(2.8) $||r_c^k|| = o(||Z_k^T\nabla f^k|| + ||c^k||)$.

Since $\nabla L(x)$ and $c(x)$ are Lipschitz continuous, $c^*=0$, $Z_k^T\nabla f^k = Z_k^T\nabla L^k$, and $\nabla L^*=0$, it follows that

(2.8.1) $||Z_k^T\nabla f^k|| + ||c^k|| = O||x^k-x^*||$,

and therefore, combining (2.8.1) with (2.8) it follows that the second term of (2.7) is $o||x^k-x^*||$. But clearly by assumption, $||x^{k+1}-x^*|| = O||x^k-x^*||$, and by Lipschitz continuity of $C(x)$ we have $||C_{k+1}-C_k|| = O||x^{k+1}-x^k||$. However,

$$||x^{k+1}-x^k|| \le ||x^{k+1}-x^*|| + ||x^k-x^*||$$

and hence $||C_{k+1}-C_k|| = O||x^k-x^*||$. It follows that the third term of (2.7) is $o||x^k-x^*||$ and Part 1 is established: $||w^{k+1}|| = o||x^k-x^*||$.

**Part 2**

Clearly we can write

(2.9) $Z_k^T(x^{k+1}-x^*) = H_k^{-1}[H_kZ_k^T(x^k-x^*) - Z_k^T(\nabla f^k-\nabla f^k) + H_kZ_k^T\delta^k]$.

If we add $Z_{k+1}^T(x^{k+1}-x^*)$ to both sides, re-arrange and take norms, we obtain

(2.10) $||u^{k+1}|| \le T_1 + T_2 + T_3 + T_4$

where

$T_1 = ||H_k^{-1}[-Z_k^T\nabla f^k + H(x^k,x^*)Z_k^T(x^k-x^*)]||$,
$T_2 = ||-H_k^{-1}(H(x^k,x^*) - H_k)Z_k^T(x^k-x^*)||$,
$T_3 = ||H_k^{-1}[Z_k^T\nabla f^k + H_kZ_k^T\delta^k]||$, and
$T_4 = ||(Z_{k+1}-Z_k)^T(x^{k+1}-x^*)||$.

By Taylor's Theorem, and $\nabla L^*=0$,

$$\nabla L^k = \nabla^2L^*(x^k-x^*) + o||x^k-x^*||,$$

and thus, using (2.3),

$$Z_k^T\nabla f^k = H(x^k,x^*)Z_k^T(x^k-x^*) + Z_k^T\nabla^2L^*C_kw^k + o||x^k-x^*||.$$

But $||\nabla^2L^*C_k||$ and $||H_k^{-1}||$ are bounded above and $||w^k||$ is $o||x^{k-1}-x^*||$, by Part 1;

therefore, $T_1$ is $o\,||x^{k-1}-x^*||$. By convergence, $||H(x^k,x^*)-H_k|| \rightarrow 0$, and this along with the fact that $||H_k^{-1}||$ is bounded above implies that $T_2$ is $o\,||x^k-x^*||$, and thus $T_2$ is $o\,||x^{k-1}-x^*||$.

Assumption (2.5) implies that

$$(2.11)\quad ||Z_k^T\nabla f^k + H_k Z_k^T\delta^k|| = o(||Z_k^T\nabla f^k|| + ||c^k||).$$

By (2.8.1) and the boundedness of $H_k^{-1}$, it follows that $T_3$ is $o\,||x^k-x^*||$ which implies that $T_3$ is $o\,||x^{k-1}-x^*||$. Finally, since $||x^{k+1}-x^*||$ is $O\,||x^{k-1}-x^*||$, and $||Z_{k+1}-Z_k|| \rightarrow 0$, $T_4$ is $o\,||x^{k-1}-x^*||$. We have established that $T_i = o\,||x^{k-1}-x^*||$ for i=1,2,3,4, which, in light of (2.10), implies that $||u^{k+1}|| = o\,||x^{k-1}-x^*||$.

**Part 3**

By definition,

$$x^{k+1}-x^* = C_{k+1}w^{k+1} + Z_{k+1}u^{k+1},$$

which implies

$$||x^{k+1}-x^*|| \leq ||C_{k+1}||\cdot||w^{k+1}|| + ||Z_{k+1}||\cdot||u^{k+1}||.$$

But $||x^k-x^*||$ is $O\,||x^{k-1}-x^*||$, and therefore, by Part 1, $||w^{k+1}|| = o\,||x^{k-1}-x^*||$. Part 2 establishes that $||u^{k+1}|| = o\,||x^{k-1}-x^*||$ and since $||C_k||$ and $||Z_k||$ are bounded above, it follows that $||x^{k+1}-x^*|| = o\,||x^{k-1}-x^*||$. □

The conditions given here are closely related to those given by Powell [1978]: however, the above conditions do not presuppose a paricular algorithm class. In the next section we employ Theorem 2.1 to establish the local convergence rate property.

### 3. The Algorithm and Its' Properties

In this section we develop and analyze, in detail, a projected DFP updating procedure. We have chosen to focus on the DFP updating scheme, instead of the BFGS update, in order to follow more closely the results of Broyden, Dennis and More [1973] and Dennis and More [1974]. It is not difficult to see, as Dennis and More [1974, section 4] indicate for the unconstrained case, that the results are equally true for the projected BFGS update specified below by 3.5'.

The method we are concerned with is defined by

(3.1) $h^k \leftarrow -Z_k B_k^{-1} Z_k^T \nabla f^k$

(3.2) $x^{k+} \leftarrow x^k + h^k$

*If* $(h^k = 0)$ go to (3.6)

(3.3) $s^k \leftarrow Z_k^T (x^{k+} - x^k)$

(3.4) $y^k \leftarrow Z_k^T [(\nabla f^{k+} - C_{k+} \lambda^k) - \nabla f^k]$

(3.5) $$B_{k+1} \leftarrow B_k + \frac{[y^k - B_k s^k](y^k)^T + y^k [y^k - B_k s^k]^T}{(s^k)^T y^k} - \frac{(s^k)^T (y^k - B_k s^k) y^k (y^k)^T}{((s^k)^T y^k)^2}$$

(3.6) $v^k \leftarrow -C_k (C_k^T C_k)^{-1} c^{k+}$

(3.7) $x^{k+1} \leftarrow x^{k+} + v^k$

(3.8) $\lambda^{k+1} \leftarrow (C_{k+1}^T C_{k+1})^{-1} C_{k+1}^T \nabla f^{k+1}$

Update (3.5) is just a projected version of the DFP formula. The corresponding projected BFGS formula is

(3.5') $$B_{k+1} \leftarrow B_k + \frac{y^k (y^k)^T}{(y^k)^T s^k} - \frac{B_k s^k (s^k)^T B_k}{(s^k)^T B_k s^k}.$$

Note that if $h^k = v^k = 0$, then $x^k = x^*$. Once again we remark that (3.6) could be replaced with (1.4) and all results in this paper remain valid. In the next three lemmas we establish some useful bounds.

**Lemma 3.0**

Provided $x^k \in D$ there exists a positive scalar $K_0$ such that

$$||\lambda^k - \lambda^*|| \leq K_0 \{ ||x^k - x^*|| + ||x^k - x^*||^2 + ||x^k - x^*||^3 \}.$$

**Proof:**

$$||\lambda^k-\lambda^*|| = ||(C_k^TC_k)^{-1}C_k^T\nabla f^k - (C_*^TC_*)^{-1}C_*^T\nabla f^*||$$

$$\leq ||(C_*^TC_*)^{-1}C_*^T[\nabla f^k-\nabla f^*]||$$

$$+ ||(C_*^TC_*)^{-1}[C_k^T-C_*^T]\nabla f^k||$$

$$+ ||(C_k^TC_k)^{-1}[C_*^TC_* - C_k^TC_k](C_*^TC_*)^{-1}||\cdot||C_k^T\nabla f^k||.$$

But

$$||C_*^TC_* - C_k^TC_k||$$

$$= ||(C_k-C_*)^T(C_k-C_*) - C_k^T(C_k-C_*) - (C_k-C_*)^TC_k||$$

$$\leq ||C_k-C_*||^2 + 2||C_k||\cdot||C_k-C_*||.$$

Hence, by considering that C(x) and $\nabla f(x)$ are Lipschitz continuous on $D$, $||(C(x)^TC(x))^{-1}||$ and $||C(x)||$ are bounded above on $D$, and

$$||\nabla f^k|| \leq ||\nabla f^*|| + O||x^k-x^*||$$

we obtain the required result. □

**Lemma 3.1**

For some positive constant $K_1$ and any $Z(x)$, $x \in D$, $x^k \in D$,

$$||Z(x)^T(\nabla^2 f^k - \Sigma\lambda_i^k\nabla^2 c_i^k)Z(x) - H(x,x^*)|| \leq K_1\{||x^k-x^*|| + ||x^k-x^*||^2 + ||x^k-x^*||^3\}.$$

**Proof:**

Clearly, by using $||Z|| = 1$, the Lipschitz continuity of the Hessian matrices, and the upper boundedness of $\nabla^2 c_i, \nabla^2 f$,

$$||Z(x)^T(\nabla^2 f^k - \Sigma\lambda_i^k\nabla^2 c_i^k)Z(x) - H(x,x^*)|| \leq ||(\nabla^2 f^k - \Sigma\lambda_i^k\nabla^2 c_i^k) - (\nabla^2 f^* - \Sigma\lambda_i^*\nabla^2 c_i^*)|| \leq \tau_1||x^k-x^*|| + \tau_2||\lambda^k-\lambda^*||,$$

for some positive scalars $\tau_1$ and $\tau_2$. Hence, by considering Lemma 3.0, the result follows. □

**Lemma 3.2**

For some positive constant $K_2$ and all $x$ *in* $D$, $x^k \in D$,

$$\| Z_k^T[\nabla^2 f(x) - \Sigma\lambda_i(x)\nabla^2 c_i(x)]Z_k - Z_*^T[\nabla^2 f(x) - \Sigma\lambda_i(x)\nabla^2 c_i(x)]Z_* \|$$
$$\leq$$
$$K_2\{\|x^k - x^*\| + \|x^k - x^*\|^2 + \|x^k - x^*\|^3 + \|x^k - x^*\|^4 + \|x^k - x^*\|^5\}.$$

**Proof:**

Let $A$ denote the matrix $\nabla^2 f(x) - \Sigma\lambda_i(x)\nabla^2 c_i(x)$. Clearly,

$$\| Z_k^T A Z_k - Z_*^T A Z_* \|$$
$$= \| (Z_k^T - Z_*) A (Z_k - Z_*) - 2 Z_*^T A Z_* + Z_k^T A Z_* + Z_*^T A Z_k \|$$
$$\leq \| Z_k - Z_* \|^2 \| A \| + 2 \| Z_k - Z_* \| \cdot \| A \|.$$

But, by assumption, Z(x) is Lipschitz continuous and since

$$A(x) = \nabla^2 L(x) + O\|\lambda(x) - \lambda^*\|,$$

the result follows from the boundedness of $\|\nabla^2 L(x)\|$ on D, and Lemma 3.0. □

The following result utilizes the bounds established in the previous three lemmas and, in conjunction with Lemma 3.4, will yield the convergence rate result.

**Lemma 3.3**

Assuming that $\|x^{k+1} - x^*\| = O\|x^k - x^*\|$, there exists a positive scalar $\hat{\epsilon}$ such that if $\|x^k - x^*\| \leq \hat{\epsilon}$, then

$$\| M y^k - M^{-1} s^k \| \leq \frac{1}{3} \| M^{-1} s^k \|,$$

where $M = H_*^{-1/2}$.

**Proof:**

Clearly,

$$\text{(3.9)} \quad \| M y^k - M^{-1} s^k \| \leq \| M \| \cdot \| y^k - H_* s^k \|.$$

By Taylor's Theorem, and Lipschitz continuity of $\nabla^2 f$, $\nabla^2 c_i$,

$$\text{(3.9.2)} \quad (\nabla f^{k+} - C_{k+}\lambda^k) = (\nabla f^k - C_k \lambda^k) + (\nabla^2 f^k - \Sigma\lambda_i^k \nabla^2 c_i^k)(x^{k+} - x^k)$$

$$+ \; E_k(x^{k+}-x^k),$$

where $||E_k|| = O||x^{k+}-x^k||$. But,

(3.9.3) $\quad x^{k+}-x^k = Z_k Z_k^T(x^{k+1}-x^k) = Z_k s^k$

and therefore, combining (3.9.2) and (3.9.3) and multiplying by $Z_k^T$ yields

(3.10) $\quad y^k = Z_k^T[\nabla^2 f^k - \Sigma\lambda_i^k \nabla^2 c_i^k]Z_k s^k + Z_k^T E_k Z_k s^k.$

But $(h^k)^T v^k = 0$ implies $||x^{k+}-x^k|| \leq ||x^{k+1}-x^k||$. Hence,

$$||E_k|| = O||x^{k+1}-x^k|| = O(max\{||x^{k+1}-x^*||,||x^k-x^*||\}).$$

Therefore using (3.10) there exists a positive constant $K_3$ such that

$$||y^k - H_* s^k||$$

$$\leq (||Z_k^T[\nabla^2 f^k - \Sigma\lambda_i^k\nabla^2 c_i^k]Z_k - H_*|| + K_3\max\{||x^{k+1}-x^*||,||x^k-x^*||\})\cdot||s^k||,$$

$$\leq ||Z_k^T[\nabla^2 f^k - \Sigma\lambda_i^k\nabla^2 c_i^k]Z_k - Z_*^T[\nabla^2 f^k - \Sigma\lambda_i^k\nabla^2 c_i^k]Z_*||\cdot||s^k||$$

$$+ ||Z_*^T[\nabla^2 f^k - \Sigma\lambda_i^k\nabla^2 c_i^k]Z_* - H_*||\cdot||s^k||$$

$$+ K_3\max\{||x^{k+1}-x^*||,||x^k-x^*||\}\cdot||s^k||.$$

Hence, in light of Lemmas 3.1, 3.2 and provided $\hat{\epsilon}$ is sufficiently small,

(3.10.1) $\quad ||y^k-H_* s^k|| \leq (2K_1 + 2K_2 + K_3)\max\{||x^{k+1}-x^*||,||x^k-x^*||\}\cdot||s^k||.$

Since $||x^{k+1}-x^*|| = O||x^k-x^*||$ (by assumption) it follows that for $\hat{\epsilon}$ sufficiently small,

$$||y^k - H_* s^k|| \leq \frac{||s^k||}{3||M||^2}$$

which implies, by (3.9)

$$||My^k - M^{-1}s^k|| \leq \frac{1}{3}||M^{-1}s^k||. \quad \square$$

Dennis and More [1974, Lemma 3.1] established the following 'bounded deterioration' result. For completeness, we reproduce it here.

**Lemma 3.4**

Let $M$ be a nonsingular symmetric matrix of order $n-t$ such that $||My^k-M^{-1}s^k|| \leq \frac{1}{3}||M^{-1}s^k||$ for some vectors $y^k$ and $s^k$ in $R^{n-t}$ with $s^k \neq 0$. Then

$(y^k)^T s^k > 0$ and thus $B_{k+1}$ is well-defined by the update formula (3.5). Moreover, there are positive constants $\alpha_0, \alpha_1$, and $\alpha_2$ (depending only on $M$ and $n-t$) such that for any symmetric matrix $A$ of order n-t,

$$||B_{k+1}-A||_M \leq [(1-\alpha_0\theta^2)^{1/2} + \frac{\alpha_1 ||My^k - M^{-1}s^k||}{||M^{-1}s^k||}]\cdot||B_k - A||_M$$
$$+ \alpha_2 \frac{||y^k - As^k||}{||M^{-1}s^k||}$$

where $||Q||_M = ||MQM||_F$ (F denotes the Frobenius norm), $\alpha_0 \in (0,1]$ and

$$\theta = \frac{||M[B_k - A]s^k||}{||B_k - A||_M ||M^{-1}s^k||} \text{ for } B_k \neq A,$$

$$= 0 \qquad \text{otherwise} . \quad \Box$$

We are now ready to prove, in Theorem 3.5, Lemma 3.6, and Theorem 3.7, that a 2-step Q-superlinear convergence rate is exhibited, provided we assume that the sequence converges. These results follow almost directly from the results of Dennis and More [1974] and Theorem 2.1.

**Theorem 3.5**

Assuming that $\Sigma||x^k - x^*|| < \infty$, $||x^{k+1} - x^*|| = O||x^k - x^*||$, and that $B_0$ is symmetric positive definite, then the algorithm defined by (3.1)-(3.8) produces a sequence of matrices $B_k$ and vectors $x^k$ which satisfy

$$\frac{||[B_k - H_*]Z_k^T(x^{k+1} - x^k)||}{||x^{k+1} - x^k||} \to 0. \tag{3.11}$$

**Proof :**

Initially, asssume that $s^k \neq 0$ for all $k$. Clearly for k sufficiently large, Lemma 3.3 is applicable and therefore the assumptions of Lemma 3.4 are valid. But, for $M^{-2} = H_*$,

$$||My^k - M^{-1}s^k|| \leq ||M||\cdot||y^k - H_* s^k||,$$

and using (3.10.1) and $||x^{k+1} - x^*|| = O||x^k - x^*||$ ( by assumption),

$$||y^k - H_* s^k|| \leq K_4||x^k - x^*||\cdot||s^k||$$

for some positive $K_4$. Therefore, taking $A = M^{-2}$ in Lemma 3.4,

$$||B_{k+1}-H_*||_M \leq [(1-\alpha_0\theta_k^2)^{1/2} + \alpha_1\sigma_k]||B_k-H_*||_M + \alpha_2\sigma_k,$$

where $\sigma_k = O||x^k-x^*||$,

$$\text{and} \quad \theta_k = \frac{||M(B_k-H_*)s^k||}{||B_k-H_*||_M||M^{-1}s^k||} \text{ for } B_k \neq H_*,$$

$$= 0 \qquad \text{otherwise,}$$

and $\alpha_0 \in (0,1]$.

It is clear that Lemma 3.3 and Theorem 3.4 in Dennis and More [1974] can now be directly applied to establish that (3.11) is true if $s^k \neq 0$ for all $k$. However, if $s^k = 0$ and $x^k \neq x^*$, then

$$\frac{||(B_k-H_*)Z_k^T(x^{k+1}-x^k)||}{||x^{k+1}-x^k||} = 0,$$

and the result is established. □

**Lemma 3.6**

Under the assumption that $\Sigma||x^k-x^*|| < \infty$, $||x^{k+1}-x^*|| = O||x^k-x^*||$, and that $B_0$ is positive definite, the algorithm given by (3.1) - (3.8) produces a sequence of iterates with the property

$$||r_L^k|| + ||r_c^k|| = o(||Z_k^T\nabla f^k|| + ||c^k||).$$

**Proof:**

By definition,

$$w^k = (C_k^TC_k)^{-1}C_k^T(x^k-x^*) ,$$

and by Taylor's theorem and $c^* = 0$,

$$c^k = C_k^T(x^k-x^*) + o||x^k-x^*||.$$

Considering that $||(C_k^TC_k)^{-1}||$ is bounded above, it follows that

$$(3.12) \qquad ||w^k|| = O||c^k|| + o||x^k-x^*||.$$

Furthermore, by Taylor's theorem and using $\nabla L^* = 0$,

$$\nabla L^k = -\nabla^2 L^k(x^*-x^k) + o||x^k-x^*||,$$

which implies, using (2.3),

$$Z_k^T\nabla f^k = H_k u^k + Z_k^T\nabla^2 L^k C_k w^k + o||x^k-x^*||.$$

But $||H_k^{-1}||$ and $||Z_k^T\nabla^2 L^k C_k||$ are bounded above and therefore,

(3.13) $$||u^k|| = O(||Z_k^T\nabla f^k|| + ||w^k||) + o||x^k-x^*||.$$

Combining (3.12) and (3.13) produces, for k sufficiently large,

(3.14) $$||x^k-x^*|| = O(||c^k|| + ||Z_k^T\nabla f^k||),$$

since $||C_k||$ is bounded above. By definition,

$$r_L^k = Z_k^T\nabla f^k + H_k Z_k^T(x^{k+1}-x^k)$$

$$= Z_k^T\nabla f^k + H_* s^k - (H_*-H_k)s^k$$

$$= -B_k s^k + H_* s^k + (H_k-H_*)s^k.$$

Therefore, taking norms,

$$||r_L^k|| \leq ||[B_k-H_*]s^k|| + ||H_k-H_*||\cdot||s^k||.$$

But $||s^k|| = O||x^k-x^*||$ and $H_k \to H_*$, which along with Theorem 3.5 and (3.14) gives

(3.15) $$||r_L^k|| = o(||Z_k^T\nabla f^k|| + ||c^k||).$$

Finally, by definition,

$$r_c^k = c^k + C_k^T(x^{k+1}-x^k) = c^k + C_k^T(h^k + v^k).$$

But $C_k^T h^k = 0$ and $v^k = -C_k(C_k^T C_k)^{-1}c(x^k + h^k)$. It is now easy to verify, using Taylor's theorem, that

$$||r_c^k|| = o||x^k-x^*||,$$

which implies, by (3.14),

(3.16) $$||r_c^k|| = o(||Z_k^T\nabla f^k|| + ||c^k||).$$

Clearly, by (3.15) and (3.16) the result is established. □

By Theorem 2.1 we have now established that the sequence $x^k$ converges at a 2-step Q-superlinear rate (assuming $\Sigma||x^k-x^*|| < \infty$ and $||x^{k+1}-x^*|| = O||x^k-x^*||$ ). We state this formally in the following theorem.

**Theorem 3.7**

Under the assumptions that $\Sigma||x^k-x^*|| < \infty$, $||x^{k+1}-x^*|| = O||x^k-x^*||$, and $B_0$ is symmetric positive definite, algorithm (3.1)-(3.8) produces a sequence of iterates $\{x^k\}$, with the property

$$\frac{||x^{k+1}-x^*||}{||x^{k-1}-x^*||} \to 0.$$

**Proof:**

The result follows immediately from Lemma 3.6 and Theorem 2.1. □

The remaining results are needed to establish the local convergence properties: $\Sigma||x^k-x^*|| < \infty$, and $||x^{k+1}-x^*|| = O||x^k-x^*||$. Firstly we establish two useful bounds in Lemmas 3.8 and 3.9.

**Lemma 3.8**

Assuming that the smallest eigenvalue of $B_k$ is greater than a positive scalar $K_5$ and that $x^k \epsilon$ D, then there exists a positive scalar $K_6$ such that

$$||h^k|| \leq K_6||x^k-x^*||.$$

**Proof:**

By definition, $h^k = -Z_k B_k^{-1} Z_k^T \nabla L^k$, and since $\nabla L^* = 0$, it follows that

$$||h^k|| \leq ||Z_k B_k^{-1} Z_k^T[\nabla L^k - \nabla L^*]||.$$

But since $\nabla L$ is Lipschitz continuous on D, $||B_k^{-1}||$ is bounded above, and $||Z_k|| = 1$, the result follows. □

**Lemma 3.9**

Under the assumptions of Lemma 3.8, there exists a positive constant $K_7$ such that

$$||v^k|| \leq K_7||x^k-x^*||.$$

**Proof:**

By definition,

$$v^k = -C_k(C_k^TC_k)^{-1}c(x^k+h^k),$$

and since $C_k^Th^k = 0$,

$$c(x^k+h^k) = c^k + o||h^k||.$$

Clearly then,

$$||v^k|| \leq ||C_k||\cdot||(C_k^TC_k)^{-1}||\cdot\{||c^k|| + o||h^k||\}.$$

By the boundedness of $||C_k||,||(C_k^TC_k)^{-1}||$, $c^* = 0$, the Lipschitz continuity of $c(x)$, and Lemma 3.8, the result follows. □

**Corollary 3.10**

Under the assumptions of Lemma 3.8, there exists a positive constant $K_8$ such that

$$||x^{k+1}-x^*|| \leq K_8||x^k-x^*||.$$

**Proof:**

The result is an immediate consequence of Lemmas 3.8 and 3.9. □

We are now ready to show, in Lemmas 3.11, 3.12 and Corollary 3.13, that provided two consecutive points are sufficiently close to $x^*$, then a (2-step) contraction is exhibited.

**Lemma 3.11**

Under the assumptions of Lemma 3.8 and provided $||x^k-x^*||$ is sufficiently small, there exisits a positive constant $K_9$ such that

$$||w^{k+1}|| \leq K_9||x^k-x^*||^2.$$

**Proof:**

By Corollary 3.10 we can assume that $||x^k-x^*||$ is sufficiently small so that $x^{k+1} \epsilon D$. It is easy to verify that, using $c^* = 0$ and $C_k^Th^k = 0$,

$$x^{k+1} = x^k - C_k(C_k^T C_k)^{-1} C_k^T (x^k - x^*) + h^k + p^k,$$

where $p^k$ is a vector satisfying $||p^k|| = O||x^k - x^*||^2$. Therefore, $C_k^T(x^{k+1} - x^*) = C_k^T p^k$.

But, by definition,

$$w^{k+1} = (C_{k+1}^T C_{k+1})^{-1} C_{k+1}^T (x^{k+1} - x^*),$$

which implies that

$$w^{k+1} = (C_{k+1}^T C_{k+1})^{-1} [C_k^T p^k + (C_{k+1}^T - C_k^T)(x^{k+1} - x^*)].$$

But, $||(C_{k+1}^T C_{k+1})^{-1}||, ||C_k^T||$ are bounded above, $||p^k||$ is $O||x^k - x^*||^2$, $||x^{k+1} - x^*||$ is $O||x^k - x^*||$ by Corollary 3.10, and $C(x)$ is Lipschitz continuous on $D$. The result follows immediately. □

**Lemma 3.12**

Provided the smallest eigenvalue of $B_{k-1}$ and $B_k$ is greater than a positive scalar $K_5$ then there exist positive scalars $\bar{\epsilon}$ and $\bar{\Delta}$ such that if

$$||x^{k-1} - x^*|| \leq \bar{\epsilon}, \ ||x^k - x^*|| \leq \bar{\epsilon}, \text{ and}$$

$$||B_k^{-1} - H_*^{-1}||_M \leq \bar{\Delta},$$

then

$$||u^{k+1}|| \leq \frac{1}{4} ||x^{k-1} - x^*||.$$

**Proof:**

Initially choose $\bar{\epsilon}$ so that $||x - x^*|| \leq \bar{\epsilon}$ implies that $x$ is in $D$. By Corollary 3.10 we can reduce $\bar{\epsilon}$, if necessary, so that

$$||x^{k-1} - x^*|| \leq \bar{\epsilon} \Rightarrow x^k \epsilon D \text{ and } x^{k+1} \epsilon D.$$

By (3.1)-(3.8)

$$x^{k+1} = x^k - Z_k H_*^{-1} Z_k^T \nabla L^k + Z_k [H_*^{-1} - B_k^{-1}] Z_k^T \nabla L^k + v^k.$$

However, subtracting $x^*$ from both sides, multiplying by $Z_k^T$ and using Lemma 3.2 yields, for $\bar{\epsilon}$ sufficiently small,

$$u^{k+1} = A_k w^k + Z_k^T p^k + [H_*^{-1} - B_k^{-1}] Z_k^T \nabla L^k + (Z_{k+1} - Z_k)^T (x^{k+1} - x^*),$$

where $A_k = -H_*^{-1}Z_k^T\nabla^2L^*C_k$, and $p^k$ is a vector satisfying $||p^k|| = o||x^k-x^*||$. But $||A_k||$ is bounded above, $||Z_k^T\nabla L^k|| = O||x^k-x^*||$, and

$$||Z_{k+1}-Z_k|| = O(max\{||x^{k+1}-x^*||,||x^k-x^*||\}).$$

Therefore, by Corollary 3.10 and Lemma 3.11, there exists a positive constant $K_{10}$ such that

$$||u^{k+1}|| \leq K_{10}[||x^{k-1}-x^*||+||H_*^{-1}-B_k^{-1}||]||x^{k-1}-x^*||.$$

Therefore, if $\max\{\bar{\epsilon},\bar{\Delta}\} \leq \frac{1}{8K_{10}}$, then

$$||u^{k+1}|| \leq \frac{1}{4}||x^{k-1}-x^*||,$$

which is the required result. □

**Corollary 3.13**

Provided the smallest eigenvalue of $B_{k-1}$ and $B_k$ is greater than a positive scalar $K_5$ then there exist positive scalars $\bar{\epsilon}$ and $\bar{\Delta}$ such that if

$$||x^{k-1}-x^*|| \leq \bar{\epsilon},\ ||x^k-x^*|| \leq \bar{\epsilon}, \text{ and}$$

$$||B_k^{-1}-H_*^{-1}||_M \leq \bar{\Delta},$$

then

$$||x^{k+1}-x^*|| \leq \frac{1}{2}||x^{k-1}-x^*||.$$

**Proof:**

Initially let $\bar{\epsilon}$ and $\bar{\Delta}$ be as defined in Lemma 3.12. Lemma 3.11 and the boundedness of $||C_k||$ allow $\bar{\epsilon}$ to be further restricted, if necessary, until

$$(3.17)\qquad ||C_{k+1}||\cdot||w^{k+1}|| \leq \frac{1}{4K_8}||x^k-x^*||.$$

Combining (3.17) with Lemma 3.12 and Corollary 3.10 produces the desired inequality. □

Borrowing heavily from Broyden, Dennis and More [1973], we now establish the local convergence property.

**Theorem 3.14**

Suppose that the sequence $\{x^k,B_k\}$ is generated by Algorithm (3.1)-(3.8) with starting pair $\{x^0,B_0\}$, and with the matrix $B_0$ being symmetric positive definite. Then, there exist positive scalars $\epsilon_0$ and $\Delta$ such that if $||x^0-x^*|| < \epsilon_0$, and $||B_0-H_*||_M < \Delta$, then

$$\Sigma||x^k-x^*|| < \infty.$$

**Proof:**

Choose positive scalars $\epsilon_0$ and $\Delta$ so that $\epsilon_0 \leq \bar{\epsilon}$ and $\Delta \leq \bar{\Delta}$, where $\bar{\epsilon}$ and $\bar{\Delta}$ are defined in the statement of Corollary 3.13. Further restrict $\Delta$, if necessary, so that

$$(3.18) \quad 2\rho\Delta\gamma \leq \frac{1}{2},$$

where for any matrix A, $||A|| \leq \rho||A||_M$, and $\gamma = ||H_*^{-1}||$. But, by hypothesis, $||B_0-H_*|| < \rho\Delta < 2\rho\Delta$, and by (3.18), the Banach Perturbation Lemma (Ortega & Rheinboldt [1970, p45]) can be applied to give

$$(3.19) \quad ||B_0^{-1}|| \leq \frac{\gamma}{1-(2\rho\Delta\gamma)} \leq 2\gamma.$$

Since $||B_0^{-1}||$ is bounded, Corollary 3.10 can be used, for $k$=0, to give

$$(3.20) \quad ||x^1-x^*|| = O(\epsilon_0).$$

Let $\epsilon_1 = ||x^1-x^*||$ and set $\epsilon = \max\{\epsilon_0,\epsilon_1\}$ . Further restrict $\epsilon_0$ , if necessary, so that $\epsilon \leq \min\{\bar{\epsilon},\hat{\epsilon}\}$. ( $\hat{\epsilon}$ is defined in Lemma 3.3.) If $s^0 = 0$ then $B_1 = B_0$ and (3.21) is trivially true. Otherwise, the assumptions of Lemma 3.4 are valid here and

$$(3.20.1) \quad ||B_1-H_*||_M - ||B_o-H_*||_M \leq \alpha_1 \frac{||My^0-M^{-1}s^0||}{||M^{-1}s^0||}\cdot 2\Delta + \alpha_2 \frac{||y^0-H_*s^0||}{||M^{-1}s^0||},$$

where $M^2 = H_*^{-1}$. But $||My^0-M^{-1}s^0|| \leq ||M||\cdot||y^0-H_*s^0||$, and $||M^{-1}s^0|| \geq \frac{||s^0||}{||M||}$.

Hence, if we define

$$\alpha_3 = \alpha_1||M||^2\,[2(K_1+K_2)+K_3], \text{ and}$$

$$\alpha_4 = \alpha_2 ||M|| \; [2(K_1 + K_2) + K_3]$$

then (3.10.1) and (3.20.1) imply

(3.21) $||B_1 - H_*||_M - ||B_0 - H_*||_M \leq (2\alpha_3\Delta + \alpha_4)\epsilon.$

Further restrict $\epsilon_0$, if necessary, so that

(3.22) $4(2\alpha_3\Delta + \alpha_4)\epsilon \leq \Delta,$

which implies by (3.21) that

(3.23) $||B_1 - H_*||_M \leq 2\Delta.$

Clearly, by (3.18) and (3.23) the Banach Perturbation Lemma can be applied again, to give

(3.24) $||B_1^{-1}|| \leq 2\gamma.$

Now considering Corollary 3.13 we obtain

(3.25) $||x^2 - x^*|| \leq \frac{1}{2}||x^0 - x^*||.$

We complete the proof with an induction step. Assume that $||B_k - H_*||_M \leq 2\Delta$, $||B_k^{-1}|| \leq 2\gamma$, and $||x^{k+1} - x^*|| \leq \frac{1}{2}||x^{k-1} - x^*||$, for $k=1,\ldots,m-1$.

Clearly, for each $k$ either Lemma 3.4 is applicable or $s^k = 0$. In either case we obtain

(3.26) $||B_{k+1} - H_*||_M - ||B_k - H_*||_M \leq (2\alpha_3\Delta + \alpha_4)\cdot\epsilon\cdot(\frac{1}{2})^{\lfloor \frac{k}{2} \rfloor},$

where $\lfloor x \rfloor$ represents the largest integer less than or equal to x. Therefore, summing both sides of (3.26) from $k=0$ to $k=m-1$ yields

(3.27) $||B_m - H_*||_M \leq ||B_0 - H_*||_M + (2\alpha_3\Delta + \alpha_4)\cdot\epsilon\cdot 4,$

which, by (3.22) gives $||B_m - H_*||_M \leq 2\Delta$. Therefore the Banach Perturbation Lemma will again give $||B_m^{-1}|| \leq 2\gamma$, and Corollary 3.10 will guarantee that $||x^{m+1} - x^*|| \leq \frac{1}{2}||x^{m-1} - x^*||$. It follows that $\Sigma||x^k - x^*|| < \infty$. □

Theorem 3.7, Corollary 3.13, and Theorem 3.14 imply that algorithm (3.1)-(3.8) generates $x$-values which converge to $x^*$ and do so at a 2-step Q-superlinear rate. We state this formally in the following theorem.

**Theorem 3.15**

Suppose that the sequence $\{x^k, B_k\}$ is generated by Algorithm (3.1)-(3.8) with starting pair $\{x^0, B_0\}$, and with the matrix $B_0$ being symmetric positive definite. Then, there exist positive scalars $\epsilon_0$ and $\Delta$ such that if $||x^0 - x^*|| \leq \epsilon_0$, and $||B_0 - H_*||_M \leq \Delta$, then $\{x^k\}$ converges to $x^*$ and does so at a 2-step Q-superlinear rate.

**Proof:**

The result follows immediately from Theorem 3.7, Corollary 3.10 and Theorem 3.14. □

## 4. Conclusions

We have proposed an adaptation of the DFP/BFGS formula to the nonlinearly constrained problem. The central feature of our approach is that a positive definite approximation to a **projected** Hessian is maintained. We have established, without assuming convexity, that the method is locally 2-step Q-superlinearly convergent. The performance of this method in practise is unknown and will be the subject of future work. A detailed discussion of implementation techniques is also postponed: we only remark that the conditions placed on $Z(\mathrm{x})$ can be realized in practise by using a careful implementation of the $QR$ decomposition - details are given in Coleman and Sorensen [1982].

For the inequality constrained problem, it is clear that once the active solution set (of constraints) is identified, either implicitly or explicitly, the results given here are directly applicable. However, the best way to modify projected approximations when the active set is changing is not presently known. Another subject of future work is how to adapt a line search algorithm and generally globalize the local procedure given here.

## REFERENCES

P.T. Boggs, J.W. Tolle and P. Wang [1982], On the local convergence of quasi-Newton methods for constrained optimization, SIAM J. Control and Optimization, 20, 161-171.

C.G. Broyden, J.E. Dennis and J.J. More [1973], On the local and superlinear convergence of quasi-Newton methods, J. Inst. Math. Appl., 12, 223-245.

T.F. Coleman and A.R. Conn [1982a], Nonlinear programming via an exact penalty function: Asymptotic analysis, Mathematical Programming, 24, to appear.

T.F. Coleman and A.R. Conn [1982b], Nonlinear programming via an exact penalty function: Global analysis, Mathematical Programming, 24, to appear.

T.F. Coleman and D.C. Sorensen [1982], A note on the computation of an orthonormal basis for the null space of a matrix, TR 82-510, Computer Science Department, Cornell University.

R.S. Dembo, S.C. Eisenstat and T. Steihaug [1982], Inexact Newton methods, SIAM J. Numer. Anal., 19, 400-408.

J.E. Dennis and J.J. More [1974], A characterization of superlinear convergence and its application to quasi-Newton methods, Math. of Computation, 28, 549-560.

P. Gill and W. Murray [1974], *Numerical Methods for Constrained Optimization*, Academic Press, London.

S-P Han [1976], Superlinearly convergent variable metric algorithms for general nonlinear programming problems, Mathematical Programming, 11, 263-282.

J.M. Ortega and W.C. Rheinboldt [1970], *Iterative Solution of Nonlinear Equations in Several Variables,* Academic Press, New York.

J.M. Ortega [1972],*Numerical Analysis A Second Course,* Academic Press, New York.

M.J.D. Powell [1978], The convergence of variable metric methods for nonlinearly constrained optimization calculations, in: O.L. Mangasarian, R.R. Meyer and S.M. Robinson, eds., *Nonlinear Programming 3,* Academic Press, New York, 27-63.

K. Tanabe [1981], Feasibility-improving gradient-acute-projection methods: a unified approach to nonlinear programming, Lecture Notes in Num. Appl. Anal., 3, 57-76.