On the logarithmic factor in error term estimates in certain additive congruence problems

by

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1. Introduction. In additive number theory an important topic is the problem of finding an asymptotic formula for the number of solutions of a given congruence. In many additive congruences the error term estimates of asymptotic formulas contain logarithmic factors. The aim of the present paper is to illustrate application of double exponential sums and a multi-dimensional smoothing argument in removing these factors for a class of additive problems.

Let g be a primitive root modulo an odd prime number p and let K, N and M be any integers with $1 \leq K, N < p$. We start by recalling the well known formula of Montgomery [6]:

$$(1) \qquad J = \frac{KN}{p} + O(p^{1/2} \log^2 p),$$

where J denotes the number of integers $x \in [H + 1, H + K]$ such that $g^x \in [M + 1, M + N]$. In this paper we establish the following statement.

THEOREM 1. *The following estimate holds*:

$$(2) \qquad J - \frac{KN}{p} \ll p^{1/2} \log^2(KNp^{-3/2} + 2).$$

We recall that the notations $A \ll B$ and $A = O(B)$ are both equivalent to $|A| \leq cB$ for some absolute positive constant c.

Estimate (2) gives the asymptotic formula $J \sim KN/p$ in the range

$$KNp^{-3/2} \to \infty \quad \text{as } p \to \infty,$$

while formula (1) gives the same asymptotic formula when

$$KNp^{-3/2} \log^{-2} p \to \infty \quad \text{as } p \to \infty.$$

Moreover, if $KN \ll p^{3/2}$, then our estimate guarantees the bound $J \ll p^{1/2}$, while formula (1) provides the bound $J \ll p^{1/2} \log^2 p$. Also note that

estimate (2) improves (1) in the range $KN \leq p^{3/2+o(1)}$ and coincides with (1) for larger values of KN.

The method that we use to prove Theorem 1 is applicable to a class of other well known additive problems. For a given integer $h \not\equiv 0 \pmod{p}$, denote by J_1 the number of solutions of the congruence

$$g^x - g^y \equiv h \pmod{p}, \quad 1 \leq x, y \leq N.$$

In [7] (see also [10]) the asymptotic formula

$$(3) \qquad J_1 = \frac{N^2}{p} + O(p^{1/2} \log^2 p)$$

has been established. In the present paper we prove

THEOREM 2. *The following estimate holds*:

$$J_1 - \frac{N^2}{p} \ll N^{2/3} \log^{2/3}(Np^{-3/4} + 2) + p^{1/2}.$$

We see that Theorem 2 provides the asymptotic formula $J_1 \sim N^2/p$ in the range

$$Np^{-3/4} \to \infty \quad \text{as } p \to \infty,$$

while (3) gives the same asymptotic formula when

$$Np^{-3/4} \log^{-2} p \to \infty \quad \text{as } p \to \infty.$$

We mention that in a series of recent works it has been proved that any residue class $h \pmod{p}$ is representable in the form

$$h \equiv g^x - g^y \pmod{p}, \quad 1 \leq x, y \leq cp^{3/4},$$

for a suitably chosen constant c (see [2, 5, 9]). In [3] it is shown that one can take $c = 2^{5/4}$.

The following result has been obtained in [8]:

Let $\mathcal{U}, \mathcal{V} \subset \{0, 1, \ldots, p-1\}$ with u and v elements respectively, and let S and T be any integers with $1 \leq T \leq p$. If J_2 denotes the number of solutions of the congruence

$$xy \equiv z \pmod{p}, \quad x \in \mathcal{U}, \ y \in \mathcal{V}, \ S+1 \leq z \leq S+T,$$

then

$$(4) \qquad \left| J_2 - \frac{uvT}{p} \right| < 2(puv)^{1/2} \log p.$$

Our approach leads to

THEOREM 3. *The following estimate holds*:

$$J_2 - \frac{uvT}{p} \ll (puv)^{1/2} \log(uvT^2p^{-3} + 2).$$

From Theorem 3 we derive the asymptotic formula $J_2 \sim uvT/p$ under the condition

$$uvT^2p^{-3} \to \infty \quad \text{as } p \to \infty,$$

while estimate (4) gives the same formula only when

$$uvT^2p^{-3}\log^{-2}p \to \infty \quad \text{as } p \to \infty.$$

We remark that estimate (4) (even with constant 2 on the right hand side replaced by 1) is a consequence of the Vinogradov double exponential sum estimate (see Lemma 5 below) and the inequality

$$\sum_{a=1}^{p-1}\Big|\sum_{n=S+1}^{S+T}e^{2\pi ian/p}\Big| < p^{1/2}\log p$$

(see, for example, the proof of Lemma 5 in [4, p. 109]).

THEOREM 4. *Let* $h \not\equiv 0 \pmod{p}$ *and let* J_3 *denote the number of solutions of the congruence*

$$xy \equiv h \pmod{p}, \quad 1 \le x, y \le N.$$

Then

$$J_3 - \frac{N^2}{p} \ll p^{1/2}\log^2(Np^{-3/4} + 2).$$

In particular, the asymptotic formula $J_3 \sim N^2/p$ holds when $Np^{-3/4} \to \infty$ as $p \to \infty$. In passing we remark that the argument of our paper can be used in a series of other related problems.

For more information on very recent results on distribution properties of special sequences related to our paper we refer the reader to [1]–[3], [5], [7], [9], [10] and references therein.

2. Lemmas

LEMMA 5. *Let* m *be a positive integer, and let* a *be an integer coprime to* m. *Then*

$$\Big|\sum_{x=0}^{m-1}\sum_{y=0}^{m-1}\nu(x)\varrho(y)e^{2\pi iaxy/m}\Big| \le \sqrt{mXY}$$

for any complex numbers $\nu(x), \varrho(y)$ *with*

$$\sum_{x=0}^{m-1}|\nu(x)|^2 = X, \quad \sum_{y=0}^{m-1}|\varrho(y)|^2 = Y.$$

The proof of this lemma can be found in [11, p. 142].

LEMMA 6. *Let* L_1, L_2, A, B *and* m *be any integers,* $1 \le A, B \le m$. *Then*

$$W := \sum_{a=0}^{m-1} \Big| \sum_{x=L_1+1}^{L_1+A} e^{2\pi i a x/m} \Big| \Big| \sum_{y=L_2+1}^{L_2+B} e^{2\pi i a y/m} \Big| \ll mA \log(BA^{-1} + 2).$$

Proof. If $A \ge B$, then applying the Cauchy inequality we obtain

$$W^2 \le \sum_{a=0}^{m-1} \Big| \sum_{x=L_1+1}^{L_1+A} e^{2\pi i a x/m} \Big|^2 \sum_{a=0}^{m-1} \Big| \sum_{y=L_2+1}^{L_2+B} e^{2\pi i a y/m} \Big|^2 = m^2 AB \le m^2 A^2,$$

whence the result.

Let $A < B$. Then

$$W \le 2W_1 + 2W_2 + 2W_3,$$

where

$$W_1 = \sum_{0 \le a \le m/B} \Big| \sum_{x=L_1+1}^{L_1+A} e^{2\pi i a x/m} \Big| \Big| \sum_{y=L_2+1}^{L_2+B} e^{2\pi i a y/m} \Big|,$$

$$W_2 = \sum_{m/B < a \le \min\{m/A, m/2\}} \Big| \sum_{x=L_1+1}^{L_1+A} e^{2\pi i a x/m} \Big| \Big| \sum_{y=L_2+1}^{L_2+B} e^{2\pi i a y/m} \Big|,$$

$$W_3 = \sum_{\min\{m/A, m/2\} < a \le m/2} \Big| \sum_{x=L_1+1}^{L_1+A} e^{2\pi i a x/m} \Big| \Big| \sum_{y=L_2+1}^{L_2+B} e^{2\pi i a y/m} \Big|.$$

The trivial estimate shows

$$W_1 \ll (m/B)AB \le mA.$$

To estimate W_2 we recall that for $1 \le a \le m/2$,

$$\Big| \sum_{y=L_2+1}^{L_2+B} e^{2\pi i a y/m} \Big| \ll \frac{m}{a}.$$

Then, estimating the sum over x trivially, we obtain

$$W_2 \ll A \sum_{m/B < a \le m/A} \frac{m}{a} \ll mA \log(BA^{-1} + 2).$$

Finally, for W_3 we have

$$W_3 \ll \sum_{a > m/A} \frac{m^2}{a^2} \ll mA.$$

Therefore, $W \ll mA \log(BA^{-1} + 2)$. ∎

LEMMA 7. *Let* L_1, L_2, A, B *be any integers with* $1 \leq A, B \leq p-1$. *Then for any integer* a *with* $(a, p) = 1$,

$$I := \Big| \sum_{x=L_1+1}^{L_1+A} \sum_{y=L_2+1}^{L_2+B} e^{2\pi i a g^{x+y}/p} \Big| \ll p^{1/2} A \log(BA^{-1} + 2).$$

The same estimate holds if in the exponent the function g^{x+y} *is replaced by* g^{x-y}.

Proof. Applying the smoothing argument, we obtain

$$I = \frac{1}{p-1} \Big| \sum_{b=0}^{p-2} \sum_{x=L_1+1}^{L_1+A} \sum_{y=L_2+1}^{L_2+B} \sum_{z=1}^{p-1} e^{2\pi i a g^z/p} e^{2\pi i b(x+y-z)/(p-1)} \Big|$$

$$\leq \frac{1}{p-1} \sum_{b=0}^{p-2} \Big| \sum_{z=1}^{p-1} e^{2\pi i a g^z/p} e^{-2\pi i b z/(p-1)} \Big| \Big| \sum_{x=L_1+1}^{L_1+A} \sum_{y=L_2+1}^{L_2+B} e^{2\pi i b(x+y)/(p-1)} \Big|.$$

The sum over z is a Gauss sum, so its absolute value is equal to $p^{1/2}$ for any integer $b \not\equiv 0 \;(\mathrm{mod}\,(p-1))$ and is equal to 1 for $b \equiv 0 \;(\mathrm{mod}\,(p-1))$. Thus,

$$I \ll p^{-1/2} \sum_{b=0}^{p-2} \Big| \sum_{x=L_1+1}^{L_1+A} e^{2\pi i b x/(p-1)} \Big| \Big| \sum_{y=L_2+1}^{L_2+B} e^{2\pi i b y/(p-1)} \Big|$$

$$\ll p^{1/2} A \log(BA^{-1} + 2),$$

where we have also used Lemma 6 with $m = p-1$.

The estimate of the sum with g^{x-y} in the exponent instead of g^{x+y} is completely analogous. ∎

3. Proof of Theorem 1. If $N > p/2$ then J is equal to K minus the number of integers x for which

$$H + 1 \leq x \leq H + K, \quad g^x \in [M+N+1, M+p] \;(\mathrm{mod}\,p),$$

where now $p - N < p/2$. For this reason it is sufficient to consider the case $N < p/2$. By the same argument we may suppose that $K < p/2$. Also note that if $K \leq 10$ or $N \leq 10$, then the estimate becomes trivial, since in this case we have $J \leq 10$. Therefore, we may assume that $10 \leq K, N < p/2$.

Let K_1, N_1 be some positive integers with $K_1 < K$ and $N_1 < N$. Denote by J' the number of solutions of the congruence

$$g^{x+z} \equiv y + t \;(\mathrm{mod}\,p)$$

subject to the conditions

$$H + 1 \leq x \leq H + (K - K_1), \quad 1 \leq z \leq K_1,$$
$$M + 1 \leq y \leq M + (N - N_1), \quad 1 \leq t \leq N_1.$$

It is obvious that for fixed integers z and t the corresponding number of solutions of the above congruence (in variables x and y) is not greater than J. Therefore,

$$(5) \qquad J \geq \frac{J'}{K_1 N_1}.$$

Similarly, let J'' be the number of solutions to the congruence

$$g^{x-z} \equiv y - t \pmod{p}$$

subject to the conditions

$$H + 1 \leq x \leq H + K + K_1, \qquad 1 \leq z \leq K_1,$$
$$M + 1 \leq y \leq M + N + N_1, \qquad 1 \leq t \leq N_1.$$

Then we have

$$(6) \qquad J \leq \frac{J''}{K_1 N_1}.$$

We claim that

$$\frac{J'}{K_1 N_1} - \frac{KN}{p} \ll p^{1/2} \log^2(KN p^{-3/2} + 2)$$

and

$$\frac{J''}{K_1 N_1} - \frac{KN}{p} \ll p^{1/2} \log^2(KN p^{-3/2} + 2)$$

for some K_1, N_1. To prove it we express J' by means of trigonometric sums:

$$J' = \frac{1}{p} \sum_{a=0}^{p-1} \sum_{x=H+1}^{H+K-K_1} \sum_{z=1}^{K_1} \sum_{y=M+1}^{M+N-N_1} \sum_{t=1}^{N_1} e^{2\pi i a(g^{x+z} - y - t)/p}.$$

Isolating the term corresponding to $a = 0$ we find

$$J' = \frac{K_1 N_1 (K - K_1)(N - N_1)}{p}$$
$$+ \frac{1}{p} \sum_{a=1}^{p-1} \sum_{x=H+1}^{H+K-K_1} \sum_{z=1}^{K_1} \sum_{y=M+1}^{M+N-N_1} \sum_{t=1}^{N_1} e^{2\pi i a(g^{x+z} - y - t)/p}.$$

For $1 \leq a \leq p - 1$ we have, according to Lemma 7,

$$\left| \sum_{x=H+1}^{H+K-K_1} \sum_{z=1}^{K_1} e^{2\pi i a g^{x+z}/p} \right| \ll p^{1/2} K_1 \log(K K_1^{-1} + 2).$$

Therefore,

$$J' - \frac{K_1 N_1 (K - K_1)(N - N_1)}{p}$$

$$\ll p^{-1/2} K_1 \log(K K_1^{-1} + 2) \sum_{a=1}^{p-1} \Big| \sum_{y=M+1}^{M+N-N_1} e^{2\pi i a y/p} \Big| \Big| \sum_{t=1}^{N_1} e^{2\pi i a t/p} \Big|.$$

According to Lemma 6 the sum over a is $\ll p N_1 \log(N N_1^{-1} + 2)$. Hence,

$$J' - \frac{K_1 N_1 (K - K_1)(N - N_1)}{p} \ll p^{1/2} K_1 N_1 \log(K K_1^{-1} + 2) \log(N N_1^{-1} + 2),$$

whence

(7)
$$\frac{J'}{K_1 N_1} = \frac{(K - K_1)(N - N_1)}{p}$$
$$+ O(p^{1/2} \log(K K_1^{-1} + 2) \log(N N_1^{-1} + 2)).$$

If $KN < 100 p^{3/2}$, then we choose $K_1 = [K/2]$, $N_1 = [N/2]$ and obtain

$$\frac{J'}{K_1 N_1} = O(p^{1/2}) = \frac{KN}{p} + O(p^{1/2} \log^2(K N p^{-3/2} + 2)).$$

If $KN > 100 p^{3/2}$, then we put

$$V = K N p^{-3/2} \log^{-2}(K N p^{-3/2}),$$

and observe that $2 < V \le \min\{K, N\}$. Thus, we can choose K_1 and N_1 to be

$$K_1 = [K/V], \quad N_1 = [N/V].$$

Therefore, from (7) we obtain

$$\frac{J'}{K_1 N_1} - \frac{KN}{p} \ll \frac{KN}{pV} + p^{1/2} \log^2 V \ll p^{1/2} \log^2(K N p^{-3/2}).$$

Thus, in both cases we have

$$\frac{J'}{K_1 N_1} - \frac{KN}{p} \ll p^{1/2} \log^2(K N p^{-3/2} + 2),$$

whence, in view of (5), we deduce the bound

(8)
$$J \ge \frac{KN}{p} + O(p^{1/2} \log^2(K N p^{-3/2} + 2)).$$

The above argument applied to J'' leads to

$$\frac{J''}{K_1 N_1} - \frac{KN}{p} \ll p^{1/2} \log^2(K N p^{-3/2} + 2),$$

which, due to (6), implies

(9)
$$J \le \frac{KN}{p} + O(p^{1/2} \log^2(K N p^{-3/2} + 2)).$$

The result now follows from (8) and (9). ∎

4. Proof of Theorem 2. We may suppose that $N > 10$ and also, due to (3) for example, that $N < p/2$.

Let N_1 be a positive integer to be chosen later, $N_1 \leq N/4$. Denote by J_1' the number of solutions of the congruence

$$g^{x+z} - g^y \equiv hg^{-t} \pmod{p}$$

subject to the conditions

$$1 \leq x \leq N - 2N_1, \quad 1 \leq z \leq N_1, \quad 1 \leq y \leq N - N_1, \quad 1 \leq t \leq N_1.$$

Let J_1'' denote the number of solutions of the congruence

$$g^{x-z} - g^y \equiv hg^t \pmod{p}$$

subject to the conditions

$$1 \leq x \leq N + 2N_1, \quad 1 \leq z \leq N_1, \quad 1 \leq y \leq N + N_1, \quad 1 \leq t \leq N_1.$$

Then

$$\tag{10} \frac{J_1'}{N_1^2} \leq J_1 \leq \frac{J_1''}{N_1^2}.$$

We express J_1' in terms of trigonometric sums and obtain

$$J_1' = \frac{1}{p} \sum_{a=0}^{p-1} \sum_{x=1}^{N-2N_1} \sum_{z=1}^{N_1} \sum_{y=1}^{N-N_1} \sum_{t=1}^{N_1} e^{2\pi i a(g^{x+z} - g^y - hg^{-t})/p}.$$

Isolating the term corresponding to $a = 0$ and applying Lemma 7 to the sum over x and z, we deduce

$$\frac{J_1'}{N_1^2} - \frac{(N - 2N_1)(N - N_1)}{p}$$

$$\ll p^{-1/2} N_1^{-1} \log(NN_1^{-1} + 2) \sum_{a=1}^{p-1} \left| \sum_{y=1}^{N-N_1} e^{2\pi i a g^y/p} \right| \left| \sum_{t=1}^{N_1} e^{2\pi i a h g^{-t}/p} \right|.$$

Application of the Cauchy inequality yields

$$\sum_{a=1}^{p-1} \left| \sum_{y=1}^{N-N_1} e^{2\pi i a g^y/p} \right| \left| \sum_{t=1}^{N_1} e^{2\pi i a h g^{-t}/p} \right|$$

$$\ll \left(\sum_{a=0}^{p-1} \left| \sum_{y=1}^{N-N_1} e^{2\pi i a g^y/p} \right|^2 \right)^{1/2} \left(\sum_{a=0}^{p-1} \left| \sum_{t=1}^{N_1} e^{2\pi i a h g^{-t}/p} \right|^2 \right)^{1/2} \leq p N^{1/2} N_1^{1/2}.$$

Hence,

$$\tag{11} \frac{J_1'}{N_1^2} - \frac{N^2}{p} \ll \frac{NN_1}{p} + p^{1/2} N^{1/2} N_1^{-1/2} \log(NN_1^{-1} + 2).$$

If $N < 100p^{3/4}$ then we let $N_1 = [N/4]$ and obtain

$$\frac{J_1'}{N_1^2} = O(p^{1/2}) = \frac{N^2}{p} + O(N^{2/3} \log^{2/3}(Np^{-3/4} + 2) + p^{1/2}).$$

If $N > 100p^{3/4}$, then we define

$$V = N^{4/3}p^{-1} \log^{-2/3}(Np^{-3/4})$$

and observe that $4 \leq V \leq N$. Now put $N_1 = [N/V]$ and observe that in this case from (11) we again have

$$\frac{J_1'}{N_1^2} = \frac{N^2}{p} + O(N^{2/3} \log^{2/3}(Np^{-3/4} + 2) + p^{1/2}).$$

Thus, for the N_1 chosen the above asymptotic formula holds, and hence due to (10) we have

$$J_1 \geq \frac{N^2}{p} + O(N^{2/3} \log^{2/3}(Np^{-3/4} + 2) + p^{1/2}).$$

Analogously,

$$\frac{J_1''}{N_1^2} = \frac{N^2}{p} + O(N^{2/3} \log^{2/3}(Np^{-3/4} + 2) + p^{1/2}),$$

whence, in view of (10),

$$J_1 \leq \frac{N^2}{p} + O(N^{2/3} \log^{2/3}(Np^{-3/4} + 2) + p^{1/2}).$$

Therefore,

$$J_1 = \frac{N^2}{p} + O(N^{2/3} \log^{2/3}(Np^{-3/4} + 2) + p^{1/2})$$

and the result follows. ∎

5. Proof of Theorem 3. We remark that if $T \leq 10$, then the statement follows from the trivial estimates $J_2 \leq 10u$, $J_2 \leq 10v$ and the fact that the error term in this case dominates. Furthermore, without loss of generality we may assume that $T < p/2$.

Let $T_1 \leq T/2$ be an integer to be chosen later. Denote by J_2' the number of solutions of the congruence

$$xy \equiv z + t \pmod{p}$$

subject to the conditions

$$x \in \mathcal{U}, \quad y \in \mathcal{V}, \quad S + 1 \leq z \leq S + T - T_1, \quad 1 \leq t \leq T_1.$$

Let J_2'' denote the number of solutions of the congruence

$$xy \equiv z - t \pmod{p}$$

subject to the conditions

$$x \in \mathcal{U}, \quad y \in \mathcal{V}, \quad S+1 \leq z \leq S+T+T_1, \quad 1 \leq t \leq T_1.$$

Then

(12)
$$\frac{J_2'}{T_1} \leq J_2 \leq \frac{J_2''}{T_1}.$$

Expressing J_2' via trigonometric sums, isolating the main term, applying Lemma 5 to the double sum over $x \in \mathcal{U}$, $y \in \mathcal{V}$, and Lemma 6 to the double sum over z and t, and following exactly the same lines of the proofs of Theorems 1 and 2, we obtain

$$\frac{J_2'}{T_1} - \frac{uvT}{p} \ll \frac{uvT_1}{p} + (puv)^{1/2} \log(TT_1^{-1} + 2).$$

If $T^2 uv < 10000 p^3$, then we put $T_1 = [T/2]$, and in this case obtain

$$\frac{J_2'}{T_1} = O((puv)^{1/2}) = \frac{uvT}{p} + O((puv)^{1/2} \log(uvT^2 p^{-3} + 2)).$$

If $T^2 uv > 10000 p^3$, then define

$$V = (uvT^2 p^{-3})^{1/2} \log^{-1}(uvT^2 p^{-3}).$$

Observe that $2 \leq V \leq T$. Let $T_1 = [T/V]$. Then we immediately obtain

(13)
$$\frac{J_2'}{T_1} - \frac{uvT}{p} \ll (puv)^{1/2} \log(uvT^2 p^{-3} + 2).$$

Analogously,

(14)
$$\frac{J_2''}{T_1} - \frac{uvT}{p} \ll (puv)^{1/2} \log(uvT^2 p^{-3} + 2).$$

Putting (12)–(14) together, we deduce Theorem 3. ∎

6. Proof of Theorem 4. For $N \leq 10$ the statement is trivial. Furthermore, we have the well known asymptotic formula

$$J_3 = \frac{N^2}{p} + O(p^{1/2} \log^2 p).$$

Therefore, to prove Theorem 4 we may assume that $10 < N < p/2$.

Let J_3' be the number of solutions of the congruence

$$(x+u)(y+v) \equiv h \pmod{p}, \quad 1 \leq x, y \leq N-K, \ 1 \leq u, v \leq K,$$

where $K < N$ is a positive integer to be chosen later. By the same argument that we have used in the previous sections, we have the inequality

$$J_3 \geq \frac{J_3'}{K^2}.$$

Next, we express J_3' in terms of trigonometric sums:

$$J_3' = \frac{1}{p} \sum_{a=0}^{p-1} \sum_{x=1}^{N-K} \sum_{y=1}^{N-K} \sum_{u=1}^{K} \sum_{v=1}^{K} e^{2\pi i a(x+u-h(y+v)^{-1})/p}.$$

Using the standard technique, we obtain

(15) $\quad J_3' = \frac{1}{p^2} \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{z=1}^{p-1} \sum_{x=1}^{N-K} \sum_{y=1}^{N-K} \sum_{u=1}^{K} \sum_{v=1}^{K} e^{2\pi i(a(x+u-hz^{-1})+b(z-y-v))/p}.$

From the classical Weil estimate of Kloosterman sums we have

(16) $\qquad \left| \sum_{z=1}^{p-1} e^{2\pi i(bz-ahz^{-1})/p} \right| \le 2p^{1/2}$

for any $a \not\equiv 0 \pmod p$. This also holds if $a \equiv 0 \pmod p$ and $b \not\equiv 0 \pmod p$ (even with the right hand side replaced by 1). Therefore, (16) holds if at least one of the numbers a and b is not divisible by p. Hence, in (15) isolating the term corresponding to $a = b = 0$ and using (16) for other values of a and b, we obtain

$$J_3' = \frac{(N-K)^2 K^2 (p-1)}{p^2} + 2\theta p^{1/2} \left(\frac{1}{p} \sum_{a=0}^{p-1} \left| \sum_{x=1}^{N-K} e^{2\pi i a x/p} \right| \left| \sum_{u=1}^{K} e^{2\pi i a u/p} \right| \right)^2,$$

where $|\theta| \le 1$. We use Lemma 6 to bound the sum over a. This yields

$$J_3' - \frac{(N-K)^2 K^2}{p} \ll p^{1/2} K^2 \log^2(NK^{-1} + 2).$$

Hence,

(17) $\qquad \frac{J_3'}{K^2} - \frac{N^2}{p} \ll \frac{KN}{p} + p^{1/2} \log^2(NK^{-1} + 2).$

If $N < 100p^{3/4}$, then define $K = N - 1$ and deduce that in this case

$$\frac{J_3'}{K^2} - \frac{N^2}{p} \ll p^{1/2} \ll p^{1/2} \log^2(Np^{-3/4} + 2).$$

Let $N > 100p^{3/4}$. Choose

$$V = N^2 p^{-3/2} \log^{-2}(Np^{-3/4})$$

and note that $2 \le V \le N$. Now define $K = [N/V]$ and observe that in this case as well from (17) we have

$$\frac{J_3'}{K^2} - \frac{N^2}{p} \ll p^{1/2} \log^2(Np^{-3/4} + 2).$$

Hence,

(18) $\qquad J_3 \ge \frac{N^2}{p} + O(p^{1/2} \log^2(Np^{-3/4} + 2)).$

To obtain a similar upper bound for J_3, define J_3'' to be the number of solutions of the congruence

$$(x - u)(y - v) \equiv h \;(\mathrm{mod}\,p), \quad 1 \leq x, y \leq N + K, \; 1 \leq u, v \leq K.$$

Then $J_3 \leq K^{-2} J_3''$ and

$$J_3'' = \frac{1}{p^2} \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{z=1}^{p-1} \sum_{x=1}^{N+K} \sum_{y=1}^{N+K} \sum_{u=1}^{K} \sum_{v=1}^{K} e^{2\pi i (a(x-u-hz^{-1})+b(z-y+v))/p}.$$

The argument used to obtain lower bounds for J_3' and J_3 leads to the upper bound

$$J_3 \leq \frac{N^2}{p} + O(p^{1/2} \log^2(Np^{-3/4} + 2)).$$

Combining this with (18), we conclude that

$$J_3 - \frac{N^2}{p} \ll p^{1/2} \log^2(Np^{-3/4} + 2).$$

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