On the Long-Time Limit of Semiclassical (Zero Dispersion Limit) Solutions of the Focusing Nonlinear Schrödinger Equation: Pure Radiation Case

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Abstract

In a previous paper [13] we calculated the leading-order term $q_0(x, t, \varepsilon)$ of the solution of $q(x, t, \varepsilon)$, the focusing nonlinear (cubic) Schrödinger (NLS) equation in the semiclassical limit ($\varepsilon \to 0$) for a certain one-parameter family of initial conditions. This family contains both solitons and pure radiation. In the pure radiation case, our result is valid for all times $t \geq 0$. The aim of the present paper is to calculate the long-term behavior of the semiclassical solution $q(x, t, \varepsilon)$ in the pure radiation case. As before, our main tool is the Riemann-Hilbert problem (RHP) formulation of the inverse scattering problem and the corresponding system of "moment and integral conditions," known also as a system of "modulation equations." © 2006 Wiley Periodicals, Inc.

Contents

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1 Introduction

The general initial-value problem for the cubic nonlinear Schrödinger (NLS) equation was solved by the method of inverse scattering in [15]. The characterization of the map between the initial data and the scattering data was obtained in [16]. In [13] we studied the semiclassical limit of the focusing NLS

$$i\epsilon q_t + \left(\frac{\epsilon^2}{2}\right) q_{xx} + |q|^2 q = 0, \tag{1.1}$$

subject to a one-parameter family of initial conditions

$$q(x, 0, \varepsilon) = A(x) e^{iS(x)/\epsilon} \tag{1.2}$$

with $A(x) = -\operatorname{sech} x$, $S' = -\mu \tanh x$, and $S(0) = 0$, where $\mu \geq 0$. Earlier numerical studies [1, 2, 11] showed that, in spite of the modulational instability [8], orderly oscillatory structures appear as the system evolves. A rigorous approach to (1.1) with real analytic, pure soliton initial data [10], which relied however on the numerically confirmed assumption of the breaking of the genus 0 solution, produced the modulated algebrogeometric oscillatory evolution similar to that of small-dispersion KdV [5] (see also [14]). The breaking of the genus 0 solution was proved for initial data (1.2) in [13], where it was shown that breaking occurs at each point of a curve $t = t_0(x)$ in the case $\mu > 0$. In the latter work, basic properties of the breaking curve were established and formulae (together with error estimates) that give the leading asymptotic behavior of the solution to (1.1)–(1.2) as $\varepsilon \to 0$ were derived in the following regimes: (a) for any values $-\infty < x < \infty$, $t \geq 0$, except on the breaking curve when the initial data are solitonless ($\mu/2 \geq 2$), and (b) up to the second break when the initial data contain solitons. The scattering data of the one-parameter family of initial data (1.2) was derived explicitly in [12].

The algebrogeometric solutions referred to above are controlled by $2N + 1$ constants $\alpha_0, \alpha_2, \dots, \alpha_{2N}$, in the upper complex half-plane and by their complex conjugates. They are given explicitly by formulae that involve theta functions that arise from the radical

$$R(z) = \sqrt{\prod_{j=0}^{N} (z - \alpha_{2j})(z - \bar{\alpha}_{2j})}.$$

Modulations of these are solutions that agree to leading order with an algebrogeometric one in which a large-scale dependence $\alpha_j = \alpha_j(x, t)$, $N = N(x, t)$ (the oscillations occur at space-time order ε), is allowed. We refer to the curves on the (x, t)–plane on which the integer N has a jump as *breaking curves*. They may be viewed as *fully nonlinear caustics*.

As in the earlier work on KdV [5], the following are key elements of the analysis in [13]:

(1) For each $N = 0, 1, \dots$, we derive a set of $4N + 2$ real pointwise equations for determining the $2N + 1$ complex α_{2j} ($j = 0, 1, \dots, N$) as functions of

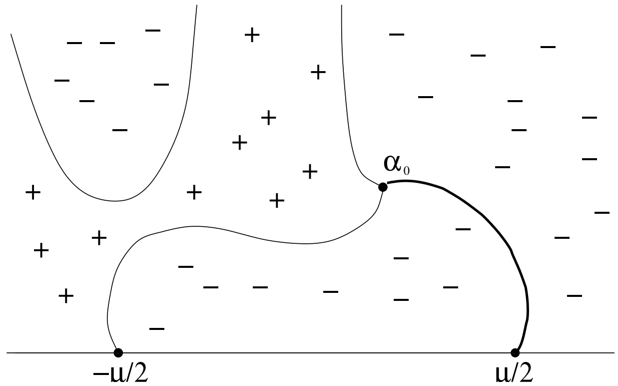

FIGURE 1.1. Zero level curves and signs of $\operatorname{Im} h$, prebreak.

x and t; from the method of their derivation, we refer to these equations as the set of the moment and integral conditions (MI conditions). Our system of equations may be viewed as the solution to the corresponding Whitham-type modulation system for focusing NLS in implicit function form. In the case $N = 1$ the MI conditions for (1.1)–(1.2) are listed below.

(2) We have a mechanism involving a set of inequalities that allows us to select the correct value of $N = N(x, t)$. These inequalities are requirements for the sign of $\operatorname{Im} h(z)$ in different parts of the upper half-plane, where

$$h'(z) = \frac{1}{2} R(z) \int_{|\zeta| \geq T} \frac{\operatorname{sign} \zeta \, d\zeta}{(\zeta - z)|R(\zeta)|} \tag{1.3}$$

and $\operatorname{Im} h(\mu/2) = 0$.

The leading-order term $q_0(x, t, \varepsilon)$ of the solution $q(x, t, \varepsilon)$ to (1.1)–(1.2) at the point (x, t) as $\varepsilon \to 0$ is given by a formula [13] that, up to phase shifts, involves only the local α_j's, i.e., $\alpha_j(x, t)$. Expressions for q_0 below the breaking curve $t_0(x)$, i.e., in the genus 0 region (where genus is equal to $2N$), is given by

$$q_0(x, t, \varepsilon) = \operatorname{Im} \alpha_0(x, t) e^{-(2i/\varepsilon)} \int_0^x \Re \alpha_0(s, t) ds, \tag{1.4}$$

whereas in the genus 2 region above the breaking curve the expression for q_0 is considerably more complicated (see [13]). The breaking curve $t_0(x)$ is asymptotic to $t = x/(2\mu)$ as $x \to \infty$. The change of genus across the breaking curve is caused by changes in the topology of zero level curves of $\operatorname{Im} h$; see Figures 1.1 through 1.3.

The present paper is devoted to the study of the behavior of $q(x, t, \varepsilon)$ when $t \to \infty$ along the rays $x = \xi t$ in the genus 2 region, i.e., when $\xi \in [0, 2\mu)$. Here we assume that $\mu \geq 2$, i.e., that we are in the pure radiation case. In this case [13], the genus 2 region covers the whole area above the breaking curve $t_0(x)$. The

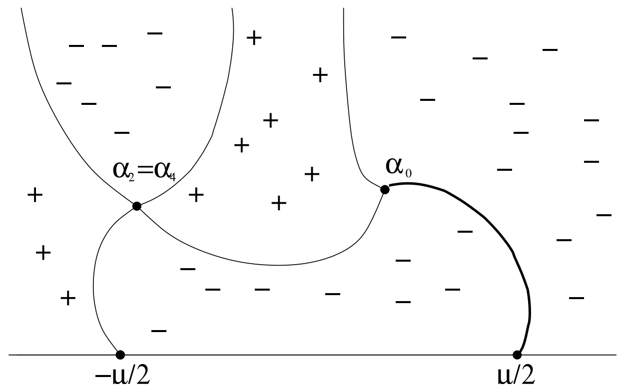

FIGURE 1.2. Zero level curves and signs of $\operatorname{Im} h$, breaking point.

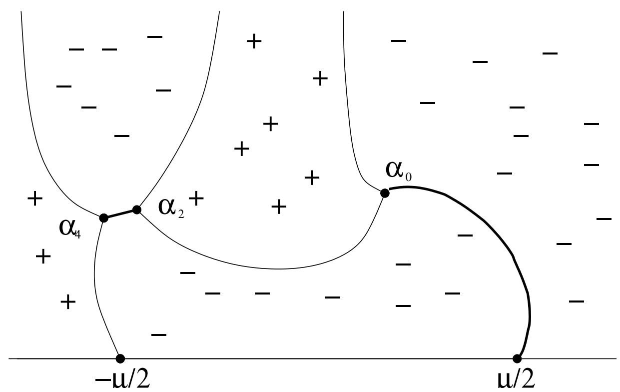

FIGURE 1.3. Zero level curves and signs of $\operatorname{Im} h$, postbreak.

key element in the present study are the MI conditions for the genus 2 region (see Section 2):

$$\int_{|\zeta|\geq T} \frac{\operatorname{sign}\zeta \, d\zeta}{|R(\zeta)|} = 0, \tag{M$_0$}$$

$$\int_{|\zeta|\geq T} \frac{\zeta \operatorname{sign}\zeta \, d\zeta}{|R(\zeta)|} = 8t, \tag{M$_1$}$$

$$\int_{|\zeta|\geq T} \frac{\zeta^2 \operatorname{sign}\zeta \, d\zeta}{|R(\zeta)|} = 2x + 8t\sum_{j=0}^{2} a_{2j}, \tag{M$_2$}$$

$$\int_{|\zeta|\geq T} \frac{[\zeta^3 \operatorname{sign}\zeta - |R(\zeta)|]d\zeta}{|R(\zeta)|} = 2x\sum_{j=0}^{2} a_{2j} + 8tQ(\alpha) - \mu + 2T, \tag{M$_3$}$$

$$(\mathrm{I}_1) \qquad \operatorname{Im} \int_{a_2}^{\alpha_2} \int\limits_{|\zeta| \geq T} \frac{R(z)\,\mathrm{sign}(\zeta) d\zeta\, dz}{(\zeta - z)|R(\zeta)|} = \pi \left(\frac{\mu}{2} - |\hat{a}_2| \right) \mathrm{sign} \left(\frac{\mu}{2} - a_2 \right),$$

$$(\mathrm{I}_2) \qquad \operatorname{Im} \int_{a_4}^{\alpha_4} \int\limits_{|\zeta| \geq T} \frac{R(z)\,\mathrm{sign}(\zeta) d\zeta\, dz}{(\zeta - z)|R(\zeta)|} = \pi \left(\frac{\mu}{2} - |\hat{a}_4| \right) \mathrm{sign} \left(\frac{\mu}{2} - a_4 \right).$$

Here $T = \sqrt{\mu^2/4 - 1}$, $\alpha_j = a_j + ib_j$, the quadratic form $Q(\alpha) = \frac{1}{2}\sum_{j<k}(a_{2j} + a_{2k})^2 - \frac{1}{2}\sum_{j=0}^{2} b_{2j}^2$, and $\hat{a}$ denotes $\max\{a, T\}$ when $a \geq 0$ and $\min\{a, -T\}$ when $a < 0$. The integrals in M_2 and M_3 are principal value integrals. This system determines the values of α_0, α_2, and α_4 for a given pair (x, t).

In Section 4 we study the MI conditions in the limit $t \to \infty$ along the rays $x = \xi t$ in order to derive the asymptotics of α_0, α_2, and α_4. The main result of this section is the following:

THEOREM 1.1 *If $\mu \geq 2$ and $\xi \in [0, 2\mu)$, then the branch points α_0, α_2, and α_4 converge to $\mu/2$, $-\xi/4$, and $-\mu/2$, respectively, as $t \to \infty$, $x = \xi t$. The convergence of a_0 and a_4 to $\mu/2$ and $-\mu/2$, respectively, is exponentially fast. Moreover,*

$$\ln b_4 = -4\left(\frac{\mu}{2} - \frac{\xi}{4}\right)t + O(1), \qquad \ln b_0 = -4\left(\frac{\mu}{2} + \frac{\xi}{4}\right)t + O(1),$$

$$b_2 = \frac{\sqrt{\mu/2 - \max\{\xi/4, T\}}}{\sqrt{2t}}(1 + O(t^{-1/2})),$$

$$a_2 = -\frac{\xi}{4} + \kappa\left(\frac{\xi}{4}\right)\frac{\ln t}{8t} + O\left(\frac{1}{t}\right),$$

where $\kappa(s) = 0, \frac{1}{2}, 1$ if $s < T$, $s = T$, or $s > T$, respectively, in the case $T > 0$; $\kappa(0) = 0$ and $\kappa(s) = 1$ if $s > 0$ in the case $T = 0$.

The analysis of Section 4 is preceded by a detailed study of a number of inequalities that are consequences of the MI conditions (Section 3). These inequalities immediately lead to Theorem 3.2, which states the boundedness of α_0, α_2, and α_4 for any compact subset in the (x, t)–plane. This theorem was formulated in [13, sec. 6.4], but the proof was deferred to the present paper since it follows naturally from the above inequalities.

The MI conditions are used in Section 5 to determine the asymptotics of the double point $\alpha = \alpha_2 = \alpha_4$ as $t \to \infty$ along the breaking curve $t_0(x)$ and to refine the asymptotics of the breaking curve $t_0(x) \sim x/(2\mu)$ obtained in [13]. The main result of this section (Theorem 5.1) is the following:

THEOREM 1.2 *The function $t = t_0(x)$, defining the breaking curve l, has asymptotics*

$$t_0(x) = \frac{x}{2\mu} - \frac{1}{2\mu}\ln\frac{2\mu}{\mu + 2T} - \frac{T/\mu}{\mu + 2T} + O\left(\frac{1}{x}\right), \tag{1.5}$$

as $x \to \infty$. Moreover, along this curve,

$$
\begin{aligned}
b &= \frac{\pi}{8t}\left(1 - \frac{\mu^2/2-1}{4\mu t} + O(t^{-2})\right), \\
a &= -\frac{\mu}{2} - \frac{1}{4t}\left[\ln\frac{2\mu^2}{\mu+2T} + \frac{2T}{\mu+2T}\right] + O(t^{-2}),
\end{aligned}
\tag{1.6}
$$

as $t \to \infty$, where $\alpha = a + ib$ is the double branch point.

Our error estimate analysis (Section 6) show that our results are valid for $t \to \infty$ as long as $\varepsilon = o(te^{-8t[\mu/2+\xi/4]})$. It allows us (Section 7, Theorem 7.5) to calculate the leading-order behavior of the solution $q(x,t,\varepsilon)$ along the rays $x = \xi t$ as $t \to \infty$. Theorem 7.5, which is based on Theorem 1.1, is stated below:

THEOREM 1.3 *If $\mu \geq 2$ and $\xi \in [0, 2\mu)$, then the leading-order behavior of the solution $q(x,t,\varepsilon)$ to (1.1)–(1.2) as $\varepsilon \to 0$, $t \to \infty$, and $\varepsilon = o\left(b_0^2\sqrt{|\ln b_0|}\right)$ along the ray $x = \xi t$ is given by*

$$
\begin{aligned}
&q(x,t,\varepsilon) \\
&\quad = -\sqrt{\frac{\mu/2 - \max\{\xi/4, T\}}{2t}}\, e^{\frac{i}{\varepsilon}(t\xi^2/4 + \ln t[\mu/2 - \max\{\xi/4,T\}])(1+O(t^{-1}))}(1 + O(t^{-1/2})) \\
&\qquad + O\left(\frac{\varepsilon}{b_0^2|\ln b_0|}\right),
\end{aligned}
$$

where $\ln b_0 = -4t\,(\mu/2 + \xi/4) + O(1)$.

The long-term behavior of q along $t_0(x)$, which is based on Theorem 1.2, was also obtained in Section 7. However, no error estimates for that case are included in the paper. Finally, in Section 8, we prove that branches of the zero level curve of $\operatorname{Im} h(z)$ in the upper half-plane approach the real axis and the vertical ray $\operatorname{Re} z = -\xi/4$ as $t \to \infty$ along $x = \xi t$.

The present paper can be considered as a natural extension of our previous work [13] for the case of pure radiational initial data. Therefore, although some facts from [13] are repeated here, we frequently refer to [13] for required statements, definitions, and notation. We would also like to mention that additional information, relevant to the present paper, can be found in [13].

2 The MI Conditions

For the solitonless case $\mu \geq 2$, the genus 2 region ($N = 1$) is the whole region above the breaking curve $t = t_0(x)$. Therefore, the above-listed MI conditions with $N = 1$ (i.e., the six real equations for three complex unknown α_{2j}'s—α_0, α_2, and α_4) will occupy us in this study. In general, these six equations are given by (A.6) in the appendix, where $f(z)$ is, roughly speaking, proportional to the logarithm of the reflection coefficient.

The function $f'(z)$ corresponding to our initial data (1.2) is given by (A.7). Since $f'(z)$ is analytic in the upper half-plane, the integrals in (A.7) depend on the endpoints α_0, α_2, and α_4 and their conjugates but not on the contours of integration. For more information about functions $f(z)$, $g(z)$, and $h(z)$ and the MI conditions, see [13, secs. 2–3] and also the Appendix, which contains an overview of part of the work in [13]). The moment conditions (M_0) through (M_3)

$$(\mathrm{M}_0) \qquad \int\limits_{|\zeta|\geq T} \frac{\operatorname{sign}\zeta\, d\zeta}{|R(\zeta)|} = 0,$$

$$(\mathrm{M}_1) \qquad \int\limits_{|\zeta|\geq T} \frac{\zeta\operatorname{sign}\zeta\, d\zeta}{|R(\zeta)|} = 8t,$$

$$(\mathrm{M}_2) \qquad \int\limits_{|\zeta|\geq T} \frac{\zeta^2\operatorname{sign}\zeta\, d\zeta}{|R(\zeta)|} = 2x + 8t\sum_{j=0}^{2} a_{2j},$$

$$(\mathrm{M}_3) \qquad \int\limits_{|\zeta|\geq T} \frac{[\zeta^3\operatorname{sign}\zeta - |R(\zeta)|]d\zeta}{|R(\zeta)|} = 2x\sum_{j=0}^{2} a_{2j} + 8tQ(\alpha) - \mu + 2T,$$

where $T = \sqrt{\mu^2/4 - 1}$, $\alpha_j = a_j + ib_j$, and the quadratic form

$$Q(\alpha) = \frac{1}{2}\sum_{j<k}(a_{2j} + a_{2k})^2 - \frac{1}{2}\sum_{j=0}^{2} b_{2j}^2,$$

are derived from (A.7) in the appendix. The integrals in (M_2) and (M_3) are principal value integrals.

Taking linear combinations of the moment conditions (M_0) to (M_3), we derive the modified set of conditions

$$(\widehat{\mathrm{M}}_0) \qquad \int\limits_{|\zeta|\geq T} s_0(\zeta)s_2(\zeta)s_4(\zeta)\operatorname{sign}\zeta\, d\zeta = 0,$$

$$(\widehat{\mathrm{M}}_1) \qquad \int\limits_{|\zeta|\geq T} p_{2i}(\zeta)s_{2j}(\zeta)s_{2k}(\zeta)\operatorname{sign}\zeta\, d\zeta = 8t,$$

$$(\widehat{\mathrm{M}}_2) \qquad \int\limits_{|\zeta|\geq T} p_{2i}(\zeta)p_{2j}(\zeta)s_{2k}(\zeta)\operatorname{sign}\zeta\, d\zeta = 2x + 8ta_{2k},$$

$$(\widehat{\mathrm{M}}_3) \qquad \int\limits_{|\zeta|\geq T} [1 - p_0(\zeta)p_2(\zeta)p_4(\zeta)\operatorname{sign}\zeta]d\zeta = 4t(b_0^2 + b_2^2 + b_4^2) + \mu - 2T,$$

where the indices i, j, and k in ($\widehat{\mathrm{M}}_1$) and ($\widehat{\mathrm{M}}_2$) are any permutation of 0, 1, and 2, and where the following definitions apply:

$$
\begin{aligned}
p_{2j}(\zeta) &= \frac{\zeta - a_{2j}}{|\zeta - \alpha_{2j}|} = \frac{\zeta - a_{2j}}{\sqrt{(\zeta - a_{2j})^2 + b_{2j}^2}}, \\
s_{2j}(\zeta) &= \frac{1}{|\zeta - \alpha_{2j}|} = \frac{1}{\sqrt{(\zeta - a_{2j})^2 + b_{2j}^2}}.
\end{aligned} \tag{2.1}
$$

The factors $p_{2i}(\zeta)$ are approximated by $\operatorname{sign}(\zeta - a_{2i})$. The inequality

$$
\left| \frac{\zeta}{\sqrt{\zeta^2 + b^2}} - \operatorname{sign}\zeta \right| \leq \frac{b^2}{\zeta^2 + b^2}
$$

provides the error estimate

$$
\left| \int_J f(\zeta) p_{2i}(\zeta) d\zeta - \int_J f(\zeta) \operatorname{sign}(\zeta - a_{2i}) d\zeta \right| \leq \pi b_{2i} \sup_{\zeta \in J} |f(\zeta)| \tag{2.2}
$$

on an interval $J \subset \mathbb{R}$ and its straightforward iterates (recall $|p_{2l}| \leq 1, l = 0, 1, 2$),

$$
\begin{aligned}
&\left| \int_J f(\zeta) p_{2i}(\zeta) p_{2j}(\zeta) d\zeta - \int_J f(\zeta) \operatorname{sign}(\zeta - a_{2i}) \operatorname{sign}(\zeta - a_{2j}) d\zeta \right| \\
&\qquad \leq \pi (b_{2i} + b_{2j}) \sup_{\zeta \in J} |f(\zeta)|
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
&\left| \int_J f(\zeta) p_0(\zeta) p_2(\zeta) p_4(\zeta) d\zeta \right. \\
&\qquad \left. - \int_J f(\zeta) \operatorname{sign}(\zeta - a_0) \operatorname{sign}(\zeta - a_2) \operatorname{sign}(\zeta - a_4) d\zeta \right| \\
&\qquad \leq \pi (b_0 + b_2 + b_4) \sup_{\zeta \in J} |f(\zeta)|.
\end{aligned} \tag{2.4}
$$

In these formulae, f is any integrable function and i and j are equal 0, 1, or 2. Note that estimate (2.2) has the form $|\int fg| \leq |f|_{L^\infty} |g|_{L^1}$. Sharpening this to

$$
\left| \int_J fg \right| \leq |f|_{L^\infty(J \setminus D)} |g|_{L^1(J \setminus D)} + |f|_{L^1(D)} |g|_{L^\infty(D)} \tag{2.5}
$$

where D is a subinterval of J, we obtain stronger versions of (2.2) and (2.3) given by

$$\left|\int_J f(\zeta)p_{2i}(\zeta)d\zeta - \int_J f(\zeta)\operatorname{sign}(\zeta - a_{2i})d\zeta\right| \le \pi b_{2i} \sup_{\zeta\in J\setminus D} |f(\zeta)| + \left(\sup_{\zeta\in D} \frac{b_{2i}^2}{((\zeta - a_{2i})^2 + b_{2i}^2)}\right)\int_D |f(\zeta)|d\zeta, \tag{2.6}$$

$$\begin{aligned}\left|\int_J f(\zeta)p_{2i}(\zeta)p_{2j}(\zeta)d\zeta - \int_J f(\zeta)\operatorname{sign}(\zeta - a_{2i})\operatorname{sign}(\zeta - a_{2j})d\zeta\right| &\le \pi(b_{2i} + b_{2j}) \sup_{\zeta\in J\setminus D} |f(\zeta)| \\ &+ \sup_{\zeta\in D}\left(\frac{b_{2i}^2}{(\zeta - a_{2i})^2 + b_{2i}^2} + \frac{b_{2j}^2}{(\zeta - a_{2j})^2 + b_{2j}^2}\right)\int_D |f(\zeta)|d\zeta.\end{aligned} \tag{2.7}$$

The approximation with the sign functions leads to elementary integrals $F(\zeta) = \int f(\zeta)d\zeta$ in $\hat{M}_2$ (see ($\widehat{\mathrm{M}}_2$)) with $f(\zeta) = s_{2k}$ and in $\hat{M}_3$ (see ($\widehat{\mathrm{M}}_3$)) with $f(\zeta) = p_2$. We pick the f's differently in the special case $a_2 = a_4$, the integrals though remain elementary. The following elementary proposition will be frequently used in the following sections:

PROPOSITION 2.1 *If a function $f(x)$ is continuous (piecewise continuous) on $[a, b]$, points $a \le x_1 \le x_2 \le \cdots \le x_{2n} \le b$ and $F(x) = \int f(\zeta)d\zeta$, then*

$$\int_a^b f(\zeta)\prod_{j=1}^{2n}\operatorname{sign}(\zeta - x_j)d\zeta = F(b) - 2\sum_{k=1}^{n} F(x_{2k}) + 2\sum_{k=1}^{n} F(x_{2k-1}) - F(a). \tag{2.8}$$

Here $n \in \mathbb{N}$ and $-\infty \le a < b \le \infty$. The obvious adjustments to (2.8) can be made for the case of an odd number of points $x_j \in [a, b]$.

To consider the integral conditions we start with the following definition:

DEFINITION 2.2 $\hat{a} = \max\{a, T\}$ when $a \ge 0$ and $\hat{a} = \min\{a, -T\}$ when $a < 0$.

PROPOSITION 2.3 *Let $\alpha = a + ib$ denote any of the α_{2j}. Then,*

$$\operatorname{Im}\int_a^\alpha \int_{|\zeta|\ge T} \frac{R(z)\operatorname{sign}(\zeta)d\zeta\, dz}{(\zeta - z)|R(\zeta)|} = \pi\left(\frac{\mu}{2} - |\hat{a}|\right)\operatorname{sign}\left(\frac{\mu}{2} - a\right). \tag{2.9}$$

PROOF: According to [13, sec. 3.1], the integral conditions can be written in the form $\operatorname{Im} h(\alpha_{2j}) = 0$, where α_{2j} is any branch point of h in the upper half-plane. Equivalently, $\operatorname{Im}\int_{\pm\mu/2}^{\alpha_{2j}} h'(z)dz = 0$, where we choose the contour of integration going horizontally from $\pm\mu/2$ to a_{2j} (the sign of $\mu/2$ coincides with the sign of

a_{2j}) and then vertically from a_{2j} to α_{2j}. Using the definition of $h'(z)$ (see (A.7) or section 3.1 in [13]), we can write these integral conditions as

$$\operatorname{Im}\left\{\int_a^\alpha + \int_{\pm\mu/2}^a\right\}\frac{R(z)dz}{2\pi i}\int_\gamma \frac{f'(\zeta)d\zeta}{(\zeta - z)R(\zeta)} = 0 \tag{2.10}$$

where $\alpha = a + ib$ could be any branch point α_{2j}, $j = 0, 1, 2$.

Since $h = 2g - f$ and $\operatorname{Im} g = 0$ on $\mathbb{R}$, the second integral is equal to

$$\operatorname{Im}\left[h(a) - h\left(\pm\frac{\mu}{2}\right)\right] = -\operatorname{Im} f(a) = -\frac{\pi}{2}\left(\frac{\mu}{2} - |\hat{a}|\right)\operatorname{sign}\left(\frac{\mu}{2} - a\right), \tag{2.11}$$

where the expression for $f(a)$ was obtained in [13]. Combining the previous two equations and using the fact that $\operatorname{Im} f'(\zeta) = -(\pi/2)\operatorname{sign}(\zeta)$, we obtain (2.9). □

Note that one of the above three equations follows from the remaining two and from the moment conditions.

3 A Priori Inequalities for MI Conditions and Control of the α for Finite Time

In the present and following sections we assume that $a_0 = \max\{a_{2j}\}$ and $a_4 = \min\{a_{2j}\}$, where $j = 0, 1, 2$. According to M_0, we always have $a_0 \geq 0$ and $a_4 \leq 0$. This labeling of the branch points α does not necessarily coincide with the "natural" labeling α_0, α_2, and α_4 of the branch points as we move along the contour γ. The use of labeling in the latter sense will always be indicated in these sections.

THEOREM 3.1 *The following a priori inequalities hold for the MI equations*:

(1) REDUCTION OF INTEGRAL CONDITIONS. *For any* $j = 0, 1, 2$, *we have*

$$\left|\frac{\mu}{2} - |\hat{a}_{2j}|\right| \leq 2tb_{2j}^2 + \frac{4b_{2j}}{\pi}[1 + 2\ln(1 + \sqrt{2})]. \tag{3.1}$$

(2) REDUCTION OF $\hat{M}_1$. *For any pair* $i, j \in \{0, 1, 2\}$, $i \neq j$, *we have*

$$b_{2i}b_{2j} \leq \left(\frac{\pi}{8t}\right)^2;\ \textit{if additionally } b_{2i} \leq b_{2j},\ \textit{then } b_{2i} \leq \frac{\pi}{8t}. \tag{3.2}$$

(3) REDUCTION OF $\hat{M}_3$.

(i) $$\left||\hat{a}_0 - \alpha_2| + |\hat{a}_4 - \alpha_2| - \frac{1}{2}|T - \alpha_2| - \frac{1}{2}|T + \alpha_2| - \frac{\mu}{2} - 2t(b_0^2 + b_2^2 + b_4^2)\right| \leq \frac{\pi}{2}(b_0 + b_4);$$

(ii) $$\left|\hat{a}_0 - \hat{a}_4 - \max\{|a_2|, T\} - \frac{\mu}{2} - 2t(b_0^2 + b_2^2 + b_4^2)\right| \leq \left(\frac{\pi}{2} + 1\right)(b_0 + b_2 + b_4).$$

If $\alpha_2 = \alpha_4$, *that is,* $a_2 = a_4 = a$ *and* $b_2 = b_4 = b$, *and if additionally* $a < -T$, *then*

$$\text{(iii)} \quad \left| 8tb^2 - \pi b + (\mu - 2a_0) + b\left(2\tan^{-1}\frac{a_0 - a}{b} - \tan^{-1}\frac{T - a}{b} + \tan^{-1}\frac{T + a}{b}\right)\right| < \pi b_0 + 4tb_0^2.$$

(4) Reduction of $\hat{M}_2$. *If* $a_0 - \max\{a_2, T\} > 2\delta > 0$, *then*

$$|2x + 8ta_0 + 2\ln b_0| \le \frac{\pi(b_2 + b_4)}{\delta} + \frac{2(b_2^2 + b_4^2)}{\delta^2}\tanh^{-1}\frac{\delta}{\sqrt{\delta^2 + b_0^2}} + 6\max\left\{\left|\ln\left(\delta + \sqrt{\delta^2 + b_0^2}\right)\right|, \left|\ln\left(a_0 - \hat{a}_4 + \sqrt{(a_0 - \hat{a}_4)^2 + b_0^2}\right)\right|\right\}. \tag{3.3}$$

If $\min\{a_2, -T\} - a_4 > 2\delta > 0$, *then*

$$|2x + 8ta_4 - 2\ln b_4| \le \frac{\pi(b_2 + b_0)}{\delta} + \frac{2(b_2^2 + b_0^2)}{\delta^2}\tanh^{-1}\frac{\delta}{\sqrt{\delta^2 + b_4^2}} + 6\max\left\{\left|\ln\left(\delta + \sqrt{\delta^2 + b_4^2}\right)\right|, \left|\ln\left(\hat{a}_0 - a_4 + \sqrt{(\hat{a}_0 - a_4)^2 + b_4^2}\right)\right|\right\}. \tag{3.4}$$

If $2\delta = \min\{a_2 - a_4, a_0 - a_2\}$, *then*

$$|2x + 8ta_2 - (F(T) + F(-T))| \le \frac{\pi(b_0 + b_4)}{\delta} + \frac{2(b_0^2 + b_4^2)}{\delta^2}\tanh^{-1}\frac{\delta}{\sqrt{\delta^2 + b_2^2}} + 4\max\left\{\left|\ln\left(\delta + \sqrt{\delta^2 + b_2^2}\right)\right|, \left|\ln\left(\hat{a}_0 - \hat{a}_4 + \sqrt{(\hat{a}_0 - \hat{a}_4)^2 + b_2^2}\right)\right|\right\} \tag{3.5}$$

where

$$F(\zeta) = \ln\left(\zeta - a_2 + \sqrt{(\zeta - a_2)^2 + b_2^2}\right) - \ln b_2. \tag{3.6}$$

If $\alpha_2 = \alpha_4$, *that is,* $a_2 = a_4 = a$ *and* $b_2 = b_4 = b$, *and if additionally* $a < -T$, $a_0 > T$, *and* $a_0 - a > \delta$, *then*

$$2x + 8at = F(T) + F(-T) - 2F(a_0) \tag{3.7}$$

where

$$F(\zeta) = \frac{1}{2}\ln((\zeta - a)^2 + b^2) + O(\max\{b_0, b_0^2 \ln b\}). \tag{3.8}$$

PROOF:

Inequality (1). We begin with (2.9) with $\alpha = \alpha_{2j}$.

$$\frac{1}{2}\operatorname{Im}\int_{a_{2j}}^{\alpha_{2j}} \int\limits_{|\zeta|\geq T} \frac{R(z)\operatorname{sign}(\zeta)d\zeta\, dz}{(\zeta - z)|R(\zeta)|} = \frac{\pi}{2}\left(\frac{\mu}{2} - |\hat{a}_{2j}|\right)\operatorname{sign}\left(\frac{\mu}{2} - a_{2j}\right). \tag{3.9}$$

The key idea in the proof is iterating the obvious identity

$$\frac{1}{\zeta - z} = \frac{1}{z - u}\left[-1 + \frac{\zeta - u}{\zeta - z}\right]$$

twice, choosing $u \in \mathbb{C}$ to be $\bar{\alpha}_{2k}$ and $\bar{\alpha}_{2i}$, respectively, and using the moment conditions M_0 and M_1:

$$\int\limits_{|\zeta|\geq T} \frac{\operatorname{sign}(\zeta)d\zeta}{|R(\zeta)|} = 0, \qquad \int\limits_{|\zeta|\geq T} \frac{(\zeta - u)\operatorname{sign}(\zeta)d\zeta}{|R(\zeta)|} = 8t. \tag{3.10}$$

We obtain

$$\begin{aligned}
&\left(\frac{\mu}{2} - |\hat{a}_{2j}|\right)\operatorname{sign}\left(\frac{\mu}{2} - a_{2j}\right) \\
&\quad = -\frac{8t}{\pi}\operatorname{Im}\int_{a_{2j}}^{a_{2j}+ib_{2j}} \frac{R(z)dz}{(z - \bar{\alpha}_{2i})(z - \bar{\alpha}_{2k})} \\
&\qquad + \frac{1}{\pi}\operatorname{Im}\int_{a_{2j}}^{a_{2j}+ib_{2j}} \int\limits_{|\zeta|\geq T} \frac{R(z)}{(z - \bar{\alpha}_{2i})(z - \bar{\alpha}_{2k})} \\
&\qquad\qquad \cdot \frac{(\zeta - \bar{\alpha}_{2i})(\zeta - \bar{\alpha}_{2k})\operatorname{sign}(\zeta)}{(\zeta - z)|R(\zeta)|}\, d\zeta\, dz.
\end{aligned} \tag{3.11}$$

(It still holds if constants α_{2k} and α_{2i} are replaced by any complex constants.) We estimate the simple and the double integrals in (3.11), denoted, respectively, by I_1 and I_2. We begin with

$$I_1 = \int_{a_{2j}}^{a_{2j}+ib_{2j}} \sqrt{(z - a_{2j})^2 + b_{2j}^2}\sqrt{\frac{(z - \alpha_{2i})(z - \alpha_{2k})}{(z - \bar{\alpha}_{2i})(z - \bar{\alpha}_{2k})}}\, dz. \tag{3.12}$$

Since $\operatorname{Im} z \geq 0$, we have

$$\left|\frac{(z - \alpha_{2i})(z - \alpha_{2k})}{(z - \bar{\alpha}_{2i})(z - \bar{\alpha}_{2k})}\right| \leq 1,$$

so that, introducing $z = a_{2j} + iy$,

$$|I_1| \le \int_0^{b_{2j}} \sqrt{b_{2j}^2 - y^2}\, dy = b_{2j}^2 \int_0^1 \sqrt{1 - y^2}\, dy = \frac{1}{4}\pi b_{2j}^2. \tag{3.13}$$

In order to estimate I_2 we first notice that

$$\begin{aligned} I_3 &= \int\limits_{|\zeta|\ge T} \frac{(\zeta - \bar{\alpha}_{2i})(\zeta - \bar{\alpha}_{2k})\,\mathrm{sign}(\zeta)}{(\zeta - z)|R(\zeta)|}\, d\zeta \\ &= \int\limits_{|\zeta|\ge T} \sqrt{\frac{(\zeta - \bar{\alpha}_{2i})(\zeta - \bar{\alpha}_{2k})}{(\zeta - \alpha_{2i})(\zeta - \alpha_{2k})}} \cdot \frac{\mathrm{sign}(\zeta) d\zeta}{(\zeta - z)\sqrt{(\zeta - a_{2j})^2 + b_{2j}^2}}. \end{aligned} \tag{3.14}$$

Since $\zeta \in \mathbb{R}$, the absolute value of the square root is 1.

We now need to evaluate the integral

$$\int\limits_{|\zeta|\ge T} \frac{d\zeta}{|\zeta - z|\sqrt{(z - a_{2j})^2 + b_{2j}^2}} = K_1 + K_2,$$

where in K_1 we integrate over $[a_{2j} - b_{2j}, a_{2j} + b_{2j}] \cap \{z : |\zeta| \ge T\}$ and in K_2 over the rest of $\{z : |\zeta| \ge T\}$. Since $z = a_{2j} + y$, $0 \le y \le b_{2j}$, we obtain

$$\begin{aligned} K_2 &\le 2 \int_{a_{2j}+b_{2j}}^{\infty} \frac{d\zeta}{(\zeta - a_{2j})\sqrt{(\zeta - a_{2j})^2 + b_{2j}^2}} \\ &= 2 \int_{b_{2j}}^{\infty} \frac{du}{u\sqrt{u^2 + b_{2j}^2}} = \frac{2}{b_{2j}} \ln(1 + \sqrt{2}), \\ K_1 &\le 2 \int_{a_{2j}}^{a_{2j}+b_{2j}} \frac{d\zeta}{\sqrt{(\zeta - a_{2j})^2 + y^2}\sqrt{(\zeta - a_{2j})^2 + b_{2j}^2}} \\ &\le \frac{2}{b_{2j}} \int_0^{b_{2j}} \frac{du}{\sqrt{u^2 + y^2}} \\ &= \frac{2}{b_{2j}} \ln(u + \sqrt{u^2 + y^2})\Big|_0^{b_{2j}} \le \frac{2}{b_{2j}} [\ln b_{2j} + \ln(1 + \sqrt{2}) - \ln y]. \end{aligned} \tag{3.15}$$

Thus, $|I_3| \le 2/(b_{2j})[\ln b_{2j} + 2\ln(1+\sqrt{2}) - \ln y]$. Therefore

$$
\begin{aligned}
|I_2| &\le \int_{a_{2j}}^{a_{2j}+ib_{2j}} \sqrt{(z-a_{2j})^2 + b_{2j}^2}\sqrt{\frac{(z-\alpha_{2i})(z-\alpha_{2k})}{(z-\bar{\alpha}_{2i})((z-\bar{\alpha}_{2k})}}|I_3|\,|dz| \\
&\le \frac{2}{b_{2j}}\int_0^{b_{2j}} [\ln b_{2j} + \ln(1+\sqrt{2}) - \ln y]\sqrt{b_{2j}^2 - y^2}\,dy \\
&= 2b_{2j}\int_0^1 [2\ln(1+\sqrt{2}) - \ln t]\sqrt{1-t^2}\,dt \\
&\le 2b_{2j}[1 + 2\ln(1+\sqrt{2})].
\end{aligned} \tag{3.16}
$$

The proof of the lemma follows from (3.11) and inequalities (3.13) and (3.16).

Inequality (2). According to $\hat{M}_1$,

$$
I = \int\limits_{|\zeta|\ge T} \frac{d\zeta}{|\zeta-\alpha_{2i}|\,|\zeta-\alpha_{2k}|} \ge \int\limits_{|\zeta|\ge T} \frac{(\zeta - a_{2j})\,\mathrm{sign}(\zeta)d\zeta}{\prod_{j=0}^{2}|\zeta-\alpha_{2j}|} = 8t \tag{3.17}
$$

for any $j = 0, 1, 2$ and $i, k \ne j$. By the Cauchy-Schwartz inequality, that yields

$$
I^2 \le \int\limits_{|\zeta|\ge T} \frac{d\zeta}{|\zeta-\alpha_{2i}|^2} \cdot \int\limits_{|\zeta|\ge T} \frac{d\zeta}{|\zeta-\alpha_{2k}|^2} = \frac{\pi^2}{b_{2i}b_{2k}}. \tag{3.18}
$$

Thus

$$
\min_{i\ne k} \frac{1}{b_{2i}b_{2k}} \ge \frac{64t^2}{\pi^2}, \tag{3.19}
$$

and the proof of the first part is completed. The second part follows from the first trivially.

Inequality (3). Inequality (i) is obtained from relation $\hat{M}_3$, which, using (2.3), can be rewritten as

$$
\begin{aligned}
&\Bigg| \int\limits_{|\zeta|>T} [1 - p_2(\zeta)\,\mathrm{sign}(\zeta - a_4)\,\mathrm{sign}\,\zeta\,\mathrm{sign}(\zeta - a_0)]d\zeta \\
&\qquad - 4t\sum_{j=0}^{2} b_{2j}^2 - \mu + 2T \Bigg| \\
&\le \pi(b_0 + b_4).
\end{aligned} \tag{3.20}
$$

Note that replacing a_0 and a_4 by $\hat{a}_0$ and $\hat{a}_4$, respectively, does not change the integral in (3.20). It now remains to apply Proposition 2.1:

$$
\begin{aligned}
&\int\limits_{|\zeta|>T} (1 - f(\zeta))\,\mathrm{sign}(\zeta - \hat{a}_4)\,\mathrm{sign}\,\zeta\,\mathrm{sign}(\zeta - \hat{a}_0)d\zeta \\
&\quad = (\zeta - F(\zeta))\Big|_{\zeta=\hat{a}_0}^{\infty} + (\zeta + F(\zeta))\Big|_{T}^{\hat{a}_0} + (\zeta - F(\zeta))\Big|_{\hat{a}_4}^{-T} + (\zeta + F(\zeta))\Big|_{-\infty}^{\hat{a}_4}
\end{aligned}
$$

where $f(\zeta)$ and its antiderivative $F(\zeta)$ are given by

$$f(\zeta) = \frac{\zeta - a_2}{\sqrt{(\zeta - a_2)^2 + b_2^2}}, \qquad F(\zeta) = \sqrt{(\zeta - a_2)^2 + b_2^2}. \tag{3.21}$$

Inequality (ii) follows from applying the estimate $|a| \le \sqrt{a^2 + b^2} \le |a| + |b|$ to inequality (i).

Inequality (iii) follows similarly to inequality (i). Now, when $a_4 = a_2 = a$, equation (3.20) becomes

$$\left| \int_{|\zeta|>T} [1 - f(\zeta) \operatorname{sign}\zeta \operatorname{sign}(\zeta - a_0)] d\zeta - 4t \sum_{j=0}^{2} b_{2j}^2 - \mu + 2T \right| \le \pi b_0. \tag{3.22}$$

The integral takes the form

$$\begin{aligned} &\int_{|\zeta|>T} [1 - f(\zeta)) \operatorname{sign}\zeta \operatorname{sign}(\zeta - a_0)] d\zeta \\ &\quad = (\zeta - F(\zeta))\Big|_{\zeta=a_0}^{\infty} + (\zeta + F(\zeta))\Big|_{T}^{a_0} + (\zeta - F(\zeta))\Big|_{-\infty}^{-T} \end{aligned}$$

where f and its antiderivative F are

$$\begin{aligned} f(\zeta) &= \frac{(\zeta - a)^2}{(\zeta - a)^2 + b^2} = 1 - \frac{b^2}{(\zeta - a)^2 + b^2}, \\ F(\zeta) &= \zeta - b \tan^{-1} \frac{\zeta - a}{b}. \end{aligned}$$

Inserting these in ($\widehat{\mathrm{M}}_3$), we obtain (3).

Inequality (4). Inequality (3.3) is obtained directly from relation $\hat{M}_2$ with integrand $s_0 p_2 p_4$. The proof of the other two inequalities (3.4) and (3.5) requires the integrands $p_0 p_2 s_4$ and $p_0 s_2 p_4$, respectively.

Replacing p_2 and p_4 with $\operatorname{sign}(\zeta - a_2)$ and $\operatorname{sign}(\zeta - a_4)$, respectively, and using (2.7) with $D = (a_0 - \delta, a_0 + \delta)$, we reduce $\hat{M}_2$ to

$$\begin{aligned} &\left| 2x + 8a_0 t - \int_{|\zeta|>T} \operatorname{sign}(\zeta - a_2) \operatorname{sign}(\zeta - a_4) \frac{\operatorname{sign}\zeta \, d\zeta}{\sqrt{(\zeta - a_0)^2 + b_0^2}} \right| \\ &\quad \le \frac{\pi(b_2 + b_4)}{\sqrt{\delta^2 + b_0^2}} + \frac{2(b_2^2 + b_4^2)}{\delta^2} \tanh^{-1} \frac{\delta}{\sqrt{\delta^2 + b_0^2}}. \end{aligned} \tag{3.23}$$

We now use Proposition 2.1 with

$$
\begin{aligned}
f(\zeta) &= \frac{1}{\sqrt{(\zeta - a_0)^2 + b_0^2}}, \\
F(\zeta) &= \ln \frac{\zeta - a_0 + \sqrt{(\zeta - a_0)^2 + b_0^2}}{b_0^2}.
\end{aligned}
\tag{3.24}
$$

Inequality (3.3) follows from

$$
\begin{aligned}
&F(\infty) + F(-\infty) + 4 \ln b_0 = \\
&\lim_{M \to \infty} \ln \left\{ \left[\sqrt{(M - a_0)^2 + b_0^2} + (M - a_0) \right] \right. \\
&\qquad \left. \cdot \left[\sqrt{(M + a_0)^2 + b_0^2} - (M + a_0) \right] \right\} = 2 \ln b_0
\end{aligned}
\tag{3.25}
$$

and from

$$
\begin{aligned}
&2(|F(\hat{a}_2)| + |F(\hat{a}_4)|) + |F(T)| + |F(-T)| \\
&\quad \le 6 \max \left\{ \left| \ln\left(\delta + \sqrt{\delta^2 + b_0^2}\right) \right|, \left| \ln\left(a_0 - \hat{a}_4 + \sqrt{(a_0 - \hat{a}_4)^2 + b_0^2}\right) \right| \right\}.
\end{aligned}
\tag{3.26}
$$

To obtain the latter inequality, we observe that $u = \xi - a_0$ is negative for $\xi = \hat{a}_2, \hat{a}_4, \pm T$ and that

$$
\left| \ln \frac{\sqrt{u^2 + b^2} - u}{b^2} \right| = \left| \ln\left(\sqrt{u^2 + b^2} + u \right) \right| \tag{3.27}
$$

for any positive b and u.

For inequality (3.4), we have $f(\zeta) = 1/\sqrt{(\zeta - a_4)^2 + b_4^2}$. We choose

$$
F(\zeta) = \ln\left[\zeta - a_4 + \sqrt{(\zeta - a_4)^2 + b_4^2} \right] \tag{3.28}
$$

instead of (3.24). Otherwise, the proof is similar to that of (3.3).

For inequality (3.5), we have $f(\zeta) = 1/\sqrt{(\zeta - a_2)^2 + b_2^2}$. We choose

$$
F(\zeta) = \ln \frac{\zeta - a_2 + \sqrt{(\zeta - a_2)^2 + b_2^2}}{b_2} \tag{3.29}
$$

instead of (3.24). Then

$$
\begin{aligned}
&\int_{|\zeta| > T} \operatorname{sign}(\zeta - a_0) \operatorname{sign}(\zeta - a_4) \frac{\operatorname{sign} \zeta \, d\zeta}{\sqrt{(\zeta - a_2)^2 + b_2^2}} \\
&\quad = F(\infty) - 2F(\hat{a}_0) + F(T) + F(-T) - 2F(\hat{a}_4) + F(-\infty).
\end{aligned}
\tag{3.30}
$$

It is now clear that $\lim_{M\to\infty}[F(M)+F(-M)]=0$ and

$$2F(\hat{a}_0)+2F(\hat{a}_4)=2\ln\frac{\sqrt{(\hat{a}_0-a_2)^2+b_2^2}+\hat{a}_0-a_2}{\sqrt{(\hat{a}_4-a_2)^2+b_2^2}+a_2-\hat{a}_4}, \tag{3.31}$$

which completes the proof of (3.5).

To prove the last inequality, (3.7), we notice that, taking into account the estimate (2.6), the moment condition $\hat{M}_2$ becomes

$$2x+8at=\int\limits_{|\zeta|>T}\frac{(\zeta-a)\operatorname{sign}(\zeta-a_0)\operatorname{sign}\zeta\,d\zeta}{(\zeta-a)^2+b^2}+O(\max\{b_0,b_0^2\ln b\}). \tag{3.32}$$

Calculation of the elementary integral together with Proposition 2.1 completes the proof. □

The a priori inequalities obtained above allow us to prove the following important theorem.

THEOREM 3.2 *If $\mu\geq 2$, then for any $\xi,\sigma>0$ the values of $\alpha_j(x,t)$ satisfying the MI conditions* (A.6) *are uniformly bounded on the set $S_{\xi,\sigma}=\{(x,t):|x|\leq t\xi,\ t\geq\sigma\}$. Moreover, $\alpha_j(x,t)$ are uniformly separated from $\mathbb{R}$ on any bounded subset S of $S_{\xi,\sigma}$.*

PROOF: By inequality (2), two of the three $b_{2i}(x,t)$'s are bounded by $\pi/(8t)$ in $S_{\xi,\sigma}$. By inequality 1, the two corresponding a_{2i}'s are also bounded in $S_{\xi,\sigma}$. We denote the third branch point $\alpha=a+ib$ and assume that $a(x,t)$ approaches either plus or minus infinity over a sequence $s=\{(x_n,t_n)\}_1^\infty\subset S_{\xi,\sigma}$. Hence, a is either a_0 or a_4, respectively. In that case the estimate

$$|a|=2b^2t(1+o(1)) \tag{3.33}$$

along s follows from inequality (3).

Let us first consider the case $a=a_0$. Taking into account $\ln b=\frac{1}{2}(\ln a-\ln t)+o(1)$, we see that the right-hand side of inequality (3.3) is at most logarithmic in a and in t, while the smaller left-hand side is equal to $2t(x/t+4a-(\ln b)/t)$. Since $|x/t|\leq\xi$, the left-hand side behaves like $8ta(1+o(1))$ along s, which leads to a contradiction. So, $a_0(x,t)$ is bounded along s. The same argument will work in the case $a=a_4$. Thus, by inequality (3), b is also globally bounded.

We now exclude the possibility that some branch point α, say α_0, approaches the real axis as $(x,t)\to(x^*,t^*)\in S_{\xi,\sigma}$. Indeed, then, according to inequality 2, $\alpha_0\to\mu/2$. Since all the branch points α_{2j} are bounded, the left-hand side of the moment condition M_1 approaches $+\infty$, whereas its right-hand side is bounded. The obtained contradiction completes the proof. □

Note that $S_{\xi,0}$ contains the whole genus 2 region when ξ is sufficiently large.

4 Long-Term Behavior of the α's

In this section we study behavior of the branch points α in the long-time limit along the rays $x/t = \xi$, where $0 \le \xi < 2\mu$. As follows from [13, sec. 4], these rays lie in the genus 2 region. As in the previous section, we consider the pure radiation case only, i.e., the case $\mu \ge 2$.

THEOREM 4.1 *If $\mu \ge 2$ and $\xi \in [0, 2\mu)$, then the branch points α_0, α_2, and α_4 converge to $\mu/2$, $-\xi/4$, and $-\mu/2$, respectively, as $t \to \infty$, $x = \xi t$. The convergence of a_0 and a_4 to $\mu/2$ and $-\mu/2$, respectively, is exponentially fast. Moreover,*

$$\ln b_4 = -4\left(\frac{\mu}{2} - \frac{\xi}{4}\right)t + O(1), \qquad \ln b_0 = -4\left(\frac{\mu}{2} + \frac{\xi}{4}\right)t + O(1),$$

$$b_2 = \frac{\sqrt{\mu/2 - \max\{\xi/4, T\}}}{\sqrt{2t}}(1 + O(t^{-1/2})), \tag{4.1}$$

$$a_2 = -\frac{\xi}{4} + \kappa\left(\frac{\xi}{4}\right)\frac{\ln t}{8t} + O\left(\frac{1}{t}\right)$$

where $\kappa(s) = 0, \frac{1}{2}, 1$ if $s < T$, $s = T$, or $s > T$, respectively, in the case $T > 0$; $\kappa(0) = 0$ and $\kappa(s) = 1$ if $s > 0$ in the case $T = 0$.

PROOF: As $t \to +\infty$, the boundedness of the three a_{2i} and of the three b_{2i} (Theorem 3.2) and a priori inequality (3) from Theorem 3.1 imply that $b_{2i} \sim O(t^{-1/2})$, $i = 1, 2, 3$. Inequalities (1) and (2) from Theorem 3.1 imply that $b_{2j} \sim o(t^{-1/2})$ is equivalent to $|a_{2j}| \to \mu/2$ and that at least two of the branch points α approach points $\pm\mu/2$. To prove that all α_{2i}, $i = 1, 2, 3$, are separated from each other along the ray $x = \xi t$, $0 \le \xi < 2\mu$, as $t \to +\infty$, we need the following statement:

PROPOSITION 4.2 *If $a_{2k} \le a_{2i}$ and $d > 0$, then there exists $a_* \in [a_{2k}, a_{2i}]$ such that*

$$H(a_*) = \int_{\bar{a}-d}^{\bar{a}+d} \frac{(\zeta - a_*)d\zeta}{\sqrt{(\zeta - a_{2k})^2 + b_{2k}^2}\sqrt{(\zeta - a_{2i})^2 + b_{2i}^2}} = 0 \tag{4.2}$$

where $\bar{a} = \frac{1}{2}(a_{2k} + a_{2i})$.

PROOF: Since $H(a_*)$ is a continuous function, it is sufficient to show that $H(a_{2k}) > 0$ and $H(a_{2i}) < 0$. Indeed, $(\zeta - a)/\sqrt{(\zeta - a)^2 + b^2}$ is an odd and increasing function, whereas $1/\sqrt{(\zeta - a)^2 + b^2}$ is an even function with the sole maximum at $\zeta = a$. Thus,

$$0 < \int_{\bar{a}-d}^{\bar{a}+d} \frac{(\zeta - a_{2k})d\zeta}{\sqrt{(\zeta - a_{2k})^2 + b_{2k}^2}} < H(a_{2k}). \tag{4.3}$$

The second inequality $H(a_{2i}) < 0$ can be proved similarly. □

Suppose now that all $|a_{2i}| \to \mu/2$, $i = 1, 2, 3$, as $t \to \infty$. According to M_0, they should approach both $\pm\mu/2$. Let as assume, for example, that both $a_4, a_2 \to -\mu/2$, whereas the remaining $a_0 \to \mu/2$. The other case can be considered similarly. Taking a linear combination of M_0, M_1, and M_2, we obtain

$$\int_{|\zeta|>|T|} \frac{(\zeta - a_*) p_0(\zeta) \operatorname{sign} \zeta \, d\zeta}{\sqrt{(\zeta - a_4)^2 + b_4^2}\sqrt{(\zeta - a_2)^2 + b_2^2}} = 2x + 8t(a_4 + a_2 - a_*). \tag{4.4}$$

Using (2.6), we obtain

$$\left| 2x + 8t(a_4 + a_2 - a_*) - \int_{|\zeta|>|T|} \frac{(\zeta - a_*) \operatorname{sign}(\zeta - a_0) \operatorname{sign} \zeta \, d\zeta}{\sqrt{(\zeta - a_4)^2 + b_4^2}\sqrt{(\zeta - a_2)^2 + b_2^2}} \right| < C \tag{4.5}$$

where the constant $C > 0$ does not depend on t. By shrinking the region of integration in (4.5) to $[\bar{a} - d, \bar{a} + d]$, where $\bar{a} = \frac{1}{2}(a_2 + a_4)$ and $d \in (0, |\bar{a}| - T)$, we only change the value of C in this formula. Then (4.5) becomes

$$|2x + 8t(a_4 + a_2 - a_*) - H(a_*)| < C.$$

Using Proposition 4.2, we obtain $\xi - 2\mu \to 0$ as $t \to \infty$, which contradicts the condition $\xi < 2\mu$. Thus, one of the branch points, say α_{2j}, stays away from vicinities of both $\pm\mu/2$. According to inequalities (1) and (2), Theorem 3.1, the corresponding $b_{2j} = O(t^{-1/2})$ (whereas the remaining b's are of the order $o(t^{-1/2})$). We call this α_{2j} a "slow" branch point. Each of the remaining two "fast" branch points α should approach $\pm\mu/2$, respectively, in order to comply with M_0. So, for large t all the branch points α are separated and two of them approach $\pm\mu/2$, respectively. Then we can use all inequalities (4) from Theorem 3.1.

For example, for large t inequality (3.3) implies

$$-(\xi + 4a_0)t + \frac{C}{t} \ln b_0 \le \ln b_0 \le -(\xi + 4a_0)t + \frac{C}{t} \ln b_0 \tag{4.6}$$

where $C > 0$ is independent of t. Thus, $b_0 = e^{-(\xi+4a_0)(t+O(1))}$, which means α_0 is a fast branch point. Similarly, we get $b_4 = e^{(\xi+4a_4)(t+O(1))}$. As $\limsup_{t\to\infty} a_4 \le -\mu/2$, we see that $\xi < 2\mu$ implies that α_4 is a fast branch point. Therefore, the slow branch point is α_2. Notice also that, according to inequality (1), a_0 and a_4 approach $\pm\mu/2$ exponentially fast.

The first two equations in (4.1) follow from (4.6) and the corresponding inequality for α_4. To obtain the remaining equations, we start by considering inequality (3.5), Theorem 3.1, in the limit $t \to \infty$. As follows from (3.6), there is $K > 0$ such that $|F(T) + F(-T)| \le 2|\ln b_2| + K$ for all sufficiently large t and any $a_2 \in (a_4 + \delta, a_0 - \delta)$. Thus, we obtain

$$\frac{\xi}{4} + a_2 = O\left(\frac{\ln t}{t}\right) \tag{4.7}$$

i.e., $a_2 \to -\xi/4$ as $t \to \infty$. Substituting (4.7) into inequality (3)(ii), we obtain the third equation (4.1). To obtain the last equation, we calculate

$$\lim_{t\to\infty}[F(T)+F(-T)]$$

$$\begin{cases} \ln \frac{\sqrt{(T+\frac{\xi}{4})^2+b_2^2}+T+\frac{\xi}{4}}{\sqrt{(T-\frac{\xi}{4})^2+b_2^2}+T-\frac{\xi}{4}} & \text{if } 0 \le \frac{\xi}{4} < T, \\ \ln \frac{\sqrt{4T^2+b_2^2}+2T}{b_2} & \text{if } \frac{\xi}{4} = T, \\ \ln \frac{\left[\sqrt{(T+\frac{\xi}{4})^2+b_2^2}+T+\frac{\xi}{4}\right]\left[\sqrt{(\frac{\xi}{4}-T)^2+b_2^2}+\frac{\xi}{4}-T\right]}{b_2^2} & \text{if } T < \frac{\xi}{4} < \frac{\mu}{2} \end{cases} \tag{4.8}$$

along the ray $x = \xi t$. Substitution of (4.8) into inequality (3.5) completes the proof. □

Theorem 4.1 establishes that three branch points α_{2j}, $j = 0, 1, 2$, on the contour γ approach three different points $\mu/2$, $-\xi/4$, and $-\mu/2$ on the real axis as $t \to \infty$, $x = \xi t$, where $0 \le \xi < 2\mu$. The points α_{2j} are also naturally ordered by the oriented contour γ connecting $\mu/2$ with $-\mu/2$. A priori, it is not clear that this order coincides with another order we used above, where $a_0 = \max_{j=0,1,2}\{a_{2j}\}$ and $a_4 = \min_{j=0,1,2}\{a_{2j}\}$. In order to show that the two orders do indeed coincide, we start with the following statement:

PROPOSITION 4.3 *Suppose that the branch points $\alpha_{2k} = a_{2k} + ib_{2k}$, $k = 0, 1, 2$, are approaching the different points a_{2k} on the real axis, respectively, and a continuously differentiable-on-$\{z : |\zeta| \ge T\}$ function $\rho(\zeta)$ is such that the integral below exists. We also assume that $\Omega a_{2k} = |T|$ for any $k = 0, 1, 2$ implies that α_{2k} approaches a_{2k} transversally. Then*

$$\int_{|\zeta|\ge T} \frac{\rho(\zeta)}{|R(\zeta)|}\,d\zeta = -2\sum_{k=0}^{2} \frac{\operatorname{sign}(a_{2k})\kappa(|a_{2k}|)\rho(a_{2k})\ln b_{2k}}{|a_{2k}-a_{2i}|\,|a_{2k}-a_{2j}|} + O(1) \tag{4.9}$$

where $\{i, j, k\}$ is a permutation of $\{0, 1, 2\}$, $R(\zeta) = \prod_{j=0}^{2}|\zeta - a_{2j}|$. (With a slight abuse of notation, we use here a_{2k} to denote both the real part of α_{2k} and the limiting value of α_{2k}.)

PROOF: Let us choose a small but fixed $\delta > 0$ such that the intervals $I_j = (a_{2j} - \delta, a_{2j} + \delta)$ do not intersect each other for $j = 0, 1, 2$ and that I_i and I_k do not contain $\pm T$. Then

$$\int_{I_j} \frac{\rho(\zeta)}{|R(\zeta)|}\,d\zeta = \int_{I_j} \frac{\rho(a_{2j})s_{2j}(\zeta)d\zeta}{|a_{2j}-\alpha_{2k}||a_{2j}-\alpha_{2i}|} + \int_{I_j} u_j(\zeta)p_{2j}(\zeta)d\zeta \tag{4.10}$$

where u_j is a continuous-on-I_j function. The second integral is obviously bounded as α approaches $\mathbb{R}$. Direct calculation of the first integral yields

$$-2\frac{\rho(a_{2j})\ln b_{2j}^2}{|a_{2j}-\alpha_{2k}||a_{2j}-\alpha_{2i}|}\left[1+O\left(\frac{1}{\ln b_{2j}}\right)\right]. \tag{4.11}$$

The observation that the integral outside the union of I_j, $j = 0, 1, 2$, is uniformly bounded completes the proof. □

Suppose now that the two orders do not coincide. Then there should exist a zero level curve of $\operatorname{Im} h(z)$ crossing the vertical interval $I_k = (a_{2k}, \alpha_{2k})$ for at least one k, $k = 0, 1, 2$, when t is large enough. Since $\operatorname{Im} h(\alpha_{2k}) = 0$, it sufficient to show that $\operatorname{Re} h'(z) \neq 0$ on I_k to obtain a contradiction.

Using the same arguments as for calculating the moment conditions ((M_0)), we see that

$$h'(z) = \frac{1}{2} R(z) \int\limits_{|\zeta| \geq T} \frac{\operatorname{sign} \zeta \, d\zeta}{(\zeta - z)|R(\zeta)|}. \tag{4.12}$$

Representing $R(z) = \sqrt{(z - \alpha_{2k})(z - \bar{\alpha}_{2k})}\,\tilde{R}(z)$, we obtain

$$\begin{aligned} \tilde{R}(z) &= \tilde{R}(a_{2k})(1 + [\ln R(z)]'_{z=a_{2k}}(z - a_{2k}) + \cdots) \\ &= |a_{2i} - a_{2k}|\,|a_{2j} - a_{2k}| \\ &\quad \cdot \left(1 + \left[\frac{1}{a_{2k} - a_{2i}} + \frac{1}{a_{2k} - a_{2j}} + O(b_{2i}^2 + b_{2j}^2)\right](z - a_{2k}) + \cdots\right) \end{aligned} \tag{4.13}$$

uniformly on I_k. Therefore, for $z \in I_k$, i.e., for $z = a_{2k} + i\eta$, $0 \leq \eta \leq b_{2k}$, we have

$$R(z) = b_{2k}\sqrt{1 - y^2}|a_{2i} - a_{2k}|\,|a_{2j} - a_{2k}|(1 + i\eta[K + O(b_{2i}^2 + b_{2j}^2)] + \cdots) \tag{4.14}$$

where $y = \eta/b_{2k}$ and $K = 1/(a_{2k} - a_{2i}) + 1/(a_{2k} - a_{2j})$.

Let us now calculate the contribution to $\operatorname{Re} h'(z)$ coming from the point a_{2k}; see Proposition 4.3. Indeed, as $t \to \infty$, the leading-order term of

$$\int_{a_2 - \delta}^{a_2 + \delta} \frac{\operatorname{sign} a_{2k} \, d\zeta}{(\zeta - z) R(\zeta)},$$

where $\delta > 0$ is a small fixed number, is

$$\begin{aligned} &\frac{\operatorname{sign} a_{2k}}{|a_{2i} - a_{2k}|\,|a_{2j} - a_{2k}|} \int_{-\delta}^{\delta} \frac{(1 - Ku)du}{(u - i\eta)\sqrt{u^2 + b_2^2}} \\ &= \frac{2 \operatorname{sign} a_{2k}}{|a_{2i} - a_{2k}||a_{2j} - a_{2k}|} \int_0^{\delta} \frac{(i\eta - Ku^2)du}{(u^2 + \eta^2)\sqrt{u^2 + b_2^2}} \end{aligned} \tag{4.15}$$

where $u = \zeta - a_2$. Thus, the contribution to $\operatorname{Re} h'(z)$, according to (4.14) and (4.15), is

$$-2K \frac{\operatorname{sign} a_{2k}}{|a_{2i} - a_{2k}|\,|a_{2j} - a_{2k}|} b_{2k}^2 \sqrt{1 - y^2}\, |a_{2i} - a_{2k}|\,|a_{2j} - a_{2k}| \cdot \int_0^{\delta} \frac{du}{\sqrt{u^2 + b_2^2}}. \tag{4.16}$$

In the case when α_{2k} is a slow branch point, i.e., when $2k = 2$, we see that the contribution from a_{2k} (the integral in (4.16)) is of smaller order than the contribution from the fast branch points a_0 and a_4. Using this observation together with Theorem 4.2 and Proposition 4.3, we obtain

$$\begin{aligned}
&\lim_{t\to\infty} \Re h'(a_2 + iyb_2)\\
&\quad = -\frac{1}{2} b_2\sqrt{1-y^2}(a_0-a_2)(a_2-a_4)\left[\frac{-2\ln b_0}{(a_0-a_2)^2(a_0-a_4)}\right.\\
&\qquad\qquad \left. + \frac{-2\ln b_4}{(a_4-a_2)^2(a_0-a_4)}\right]\\
&\quad = 4tb_2\sqrt{1-y^2},
\end{aligned} \tag{4.17}$$

where $y \in [0,1]$. That equation shows that $\Re h'(z) > 0$ for all $z \in [a_2, \alpha_2)$. Thus, a zero level curve of $\operatorname{Im} h(z)$ cannot cross the interval I. Similarly, in the case when α_{2k} is a fast point, i.e., when $k = 0$ or $k = 2$, the leading contributions to $\operatorname{Re} h'(z)$ also come from points a_0 and a_4. Direct calculations for this case, using (4.16), Theorem 4.2, and Proposition 4.3, yield

$$\lim_{t\to\infty} \operatorname{Re} h'(a_{2k} + iyb_{2k}) = -4tb_{2k}\sqrt{1-y^2}. \tag{4.18}$$

Thus, we have established the following fact:

THEOREM 4.4 *If $\mu \ge 2$ and $\xi \in [0, 2\mu)$, then the branch points α_0, α_2, and α_4, taken in the order they appear on the oriented contour γ^+ connecting points $\mu/2$ and $-\mu/2$ in the upper half-plane, converge to $\mu/2$, $-\xi/4$, and $-\mu/2$, respectively, as $t \to \infty$, $x = \xi t$.*

5 Long-Term Behavior on the Breaking Curve

In Section 4 we considered the behavior of α's along the lines $x/t = \xi$, where the constant $\xi \in [0, 2\mu)$, i.e., in the genus 2 region. The subject of this section is the long-term behavior of α's along the breaking curve l, which is defined as a graph of the function $t = t_0(x)$ such that $\alpha_2(x, t_0(x)) = \alpha_4(x, t_0(x))$. It was proven in [13] that the curve l has asymptotics $x/t = 2\mu$ as $t \to \infty$. The aim of this section is to study the asymptotics of $\alpha = \alpha_2 = \alpha_4$ along l, as well as to compute the refined asymptotics of l. Note that the results of Section 4 do not apply to the breaking curve since in this case we cannot assume that all α's are distinct.

We start with an observation that the radical $R(z) \to (z-\alpha_2)(z-\bar\alpha_2)R_0(z)$, where $R_0(z) = \sqrt{(z-a_0)^2 + b_0^2}$, in the limit $|\alpha_2 - \alpha_4| \to 0$. It was shown in [13, sec. 3] that in this limit the function h corresponding to the genus 2 region coincides with the function h corresponding to the genus 0 region, i.e., $h(z; R) \equiv h(z; R_0)$. It is also easy to see that on the breaking curve l, integral condition I_m becomes trivial, whereas the second integral condition I_c (which will be denoted by I) coincides with $\operatorname{Im} h(\alpha_2) = 0$. Regarding the moment conditions, it is easy to show that the conditions M_2 and M_3 are identical to two genus 0 moment conditions that define α_0 (see [13, sec. 4.1], whereas the remaining moment conditions M_0 and M_1 are equivalent to the condition $h'(\alpha_2) = 0$.

THEOREM 5.1 *The function $t = t_0(x)$, defining the breaking curve l, has asymptotics*

$$t_0(x) = \frac{x}{2\mu} - \frac{1}{2\mu}\ln\frac{2\mu}{\mu+2T} - \frac{T/\mu}{\mu+2T} + O\left(\frac{1}{x}\right) \tag{5.1}$$

as $x \to \infty$. Moreover, along this curve,

$$\begin{aligned} b &= \frac{\pi}{8t}\left(1 - \frac{\mu^2/2-1}{4\mu t} + O(t^{-2})\right), \\ a &= -\frac{\mu}{2} - \frac{1}{4t}\left[\ln\frac{2\mu^2}{\mu+2T} + \frac{2T}{\mu+2T}\right] + O(t^{-2}), \end{aligned} \tag{5.2}$$

as $t \to \infty$, where $\alpha = a + ib$ is the double branch point.

PROOF: As we know from [13, sec. 4.6], the branch point $\alpha(t)$ is located in the left half-plane ($t = t_0(x)$, $x > 0$), whereas the remaining branch point $\alpha_0(t)$ is located in the right half-plane. According to inequalities (1) and (2) of Theorem 3.1, we obtain $b(t) \le \pi/(8t)$ and, as $t \to \infty$, $a(t) \to -\mu/2$. By the same argument as in the proof of Theorem 4.3, we see that on the breaking curve $\xi = x/t$ approaches 2μ as $t \to \infty$. Then inequality (3)(ii) from Theorem 3.1 shows that $a_0 - \mu/2 - 2tb_0^2 \to 0$, i.e., $a_0 \ge \mu/2$ as $t \to \infty$. Now we can apply (3.3) to obtain the fact that the behavior of α_0 described by Theorem 4.3 for the genus 2 region is also valid on the breaking curve.

To obtain the first equation (5.2), we consider $b \to 0$ in (3). Replacing the inverse tangent with its expansion at infinity, we obtain

$$\begin{aligned} &\left|8tb^2 - \pi b - (2a_0 - \mu) - b^2\left(\frac{2}{a_0 - a} - \frac{1}{T-a} + \frac{1}{T+a}\right)\right| \\ &\qquad\qquad \le \pi b_0 + 4tb_0^2 + \frac{4b^4}{3(\mu/2 - T)}. \end{aligned} \tag{5.3}$$

The expression for b in (5.2) follows from (5.3).

To complete the proof, it remains to calculate smaller-order terms for $t_0(x)$ and a. Our next step is to prove (5.1) using the integral condition $I_c : \operatorname{Im} h(\alpha; R_0) = 0$, which can be written as

$$\operatorname{Im}\frac{R_0(\alpha)}{2\pi i}\left[-2\pi i \operatorname{Res}\frac{f(\zeta)}{(\zeta-\alpha)R_0(\zeta)}\bigg|_{\zeta=\infty} - 2i\int_{|\zeta|\ge T}\frac{\operatorname{Im} f(\zeta)d\zeta}{(\zeta-\alpha)R_0(\zeta)}\right] = 0. \tag{5.4}$$

By elementary residue calculus, we can reduce (5.4) to

$$\operatorname{Im} R_0(\alpha)\left[x + 2t(a_0 + \alpha) - \frac{1}{\pi}\int_{|\zeta|\ge T}\frac{\operatorname{Im} f(\zeta)d\zeta}{(\zeta-\alpha)R_0(\zeta)}\right] = 0. \tag{5.5}$$

Since we know that $\operatorname{Im} f(a) = (\pi/2)(\mu/2 - |\tilde{a}|)\operatorname{sign}(\mu/2 - a)$ (see (2.11)), the last term of (5.5) becomes

$$-\frac{1}{2}\int\limits_{|\zeta|\geq T} \frac{\mu/2 - \zeta \operatorname{sign}\zeta}{(\zeta-\alpha)|R_0(\zeta)|}\,d\zeta = -\frac{1}{2}(m+in) \tag{5.6}$$

where m and n are the real and imaginary parts of the integral.

Using the expression (4.1) for b_0 and (2.2), we can represent

$$(m+in) = \int\limits_{\zeta\geq T} \frac{\operatorname{sign}(\mu/2-\zeta)d\zeta}{(\zeta-\alpha)} - \int\limits_{\zeta\leq -T} \frac{(\mu/2+\zeta)d\zeta}{(\zeta-\alpha)(\zeta-a_0)} + O(e^{-4\mu t}). \tag{5.7}$$

Computing the integrals in (5.7), we obtain

$$m+in = -\ln(\zeta-\alpha)\big|_{\mu/2}^{\infty} + \ln(\zeta-\alpha)\big|_{T}^{\mu/2} - [A\ln(\zeta-\alpha) + B\ln(\zeta-a_0)]\big|_{-\infty}^{-T} \tag{5.8}$$

where

$$A = -\frac{\alpha+\mu/2}{a_0-\alpha}, \qquad B = \frac{a_0+\mu/2}{a_0-\alpha}. \tag{5.9}$$

Calculations at $\pm\infty$ yield

$$\begin{aligned}A\ln(-M-\alpha) + B\ln(-M-a_0) - \ln(M-\alpha)&\\ = \ln\frac{-M-a_0}{M-\alpha} + A\ln\frac{-M-\alpha}{-M-a_0} \to i\pi&\end{aligned}$$

as $M \to \infty$. Thus

$$m+in = \ln\frac{(\mu/2-\alpha)^2}{(T-\alpha)(T+a_0)} - A\ln\frac{-T-\alpha}{T+a_0} + iA\pi. \tag{5.10}$$

Since $R_0(\alpha) = (a_0 - a) - ib + \rho$, where ρ is exponentially small in t, the integral condition (5.5) can be written as

$$\operatorname{Im}[(a_0-a)-ib]\left[x + 2t(a_0+a) + 2itb - \frac{1}{2}(m+in)\right] = 0$$

if we ignore exponentially small terms, or

$$-x - 4ta = \frac{1}{2}\left[\frac{(a_0-a)n}{b} - m\right]. \tag{5.11}$$

It follows from (5.9) and (5.10) that

$$\begin{aligned} m &= \ln\frac{(\mu/2-a)^2}{(T-a)(T+a_0)} + O(b),\\ \frac{n}{b} &= \frac{1}{T-a} - \frac{2}{\mu/2-a} - \frac{\pi(\mu/2+a)}{b(\mu/2-a)} + O(b). \end{aligned} \tag{5.12}$$

Substituting now (5.12) into (5.11), we obtain after some algebra

$$x = 2\mu t + \frac{1}{2}\Bigg[\ln\frac{(\mu/2-a)^2}{(T-a)(T+a_0)} + 2\frac{a_0-a}{\mu/2-a} - \frac{a_0-a}{T-a} + \frac{2(\mu/2+a)(T^2-a_0 a)}{(T-a)(T+a_0)}\Bigg] + O(t^{-1}). \tag{5.13}$$

This is the refined equation of the breaking curve. Taking into account $a_0 \to \mu/2$ exponentially fast and that $|\mu/2 + a| = O(t^{-1})$, the latter equation becomes

$$x = 2\mu t + \ln\frac{2\mu}{2T+\mu} + \frac{2T}{2T+\mu} + O(t^{-1}), \tag{5.14}$$

which implies (5.1).

We now use (3.7) to calculate the correction term K/t to $a = -\mu/2 + K/t + \cdots$. Rewriting (5.14) as $x = 2\mu t + C + O(t^{-1})$ and substituting into (3.7), we obtain $K = -\frac{1}{4}(C + \ln\mu) + O(t^{-1})$, which leads to the second equation (5.2). The proof is completed. □

Note that we can obtain the asymptotics of $\alpha(t)$ and of the breaking curve $x(t)$ with any accuracy t^{-k}, $k \in \mathbb{N}$.

6 Solution of the RHP and Error Estimates

Here we derive the leading-order term $\tilde{q}(x,t,\varepsilon)$ of the solution $q(x,t,\varepsilon)$ of (1.1)–(1.2) along the rays $x/t = \xi$ as $t \to \infty$, $\varepsilon \to 0$, where $0 \le \xi < 2\mu$, and estimate the error $|q(x,t,\varepsilon) - \tilde{q}(x,t,\varepsilon)|$. The relationship between parameters ε and t is described in Theorem 6.6 below. We first recall the result about the leading-order term $q_0(x,t,\varepsilon)$ of the solutin $q(x,t,\varepsilon)$ (as $\varepsilon \to \Omega$), obtained in [13, sec. 8]. Introduce

$$\lambda(z) = \left(\prod_{j=0}^{2}\frac{z-\tilde{\beta}_j}{z-\tilde{\alpha}_j}\right)^{1/4} \tag{6.1}$$

with vertical branch cuts along the segments $\tilde{\nu}_j = [\alpha_j, \beta_j]$. Here $\tilde{\alpha}_j$ and $\tilde{\beta}_j$ denote the beginning and the end points of a vertical segment $\tilde{\nu}_j$ (so that $\tilde{\alpha}_0 = \alpha_1$, $\tilde{\beta}_0 = \alpha_0$, $\tilde{\alpha}_1 = \alpha_2, \ldots, \tilde{\beta}_{2N} = \alpha_{4N+1}$); see Figure 6.1.

We choose the branch of $\lambda(z)$ such that $\lim_{z\to\infty}\lambda(z) = 1$ and $\lambda_+ = i\lambda_-$ on $\tilde{\nu}_j$.

As in [13, sec. 8], we introduce the canonical homology basis $\boldsymbol{A}_j, \boldsymbol{B}_j$, $j = 1,\ldots,N$, of the hyperelliptic surface $\tilde{\mathcal{R}}(x,t)$, determined by the cuts $\tilde{\nu}_j$ of RHP $\tilde{P}^{(\mathrm{mod})}$; see Figure 6.2. The dotted curves in Figure 6.2 are passing through the second sheet. For a given point (x,t) in the genus 2 region, the following theorem gives the leading-order term q_0 of the solution q in the limit $\varepsilon \to 0$ together with the error estimate.

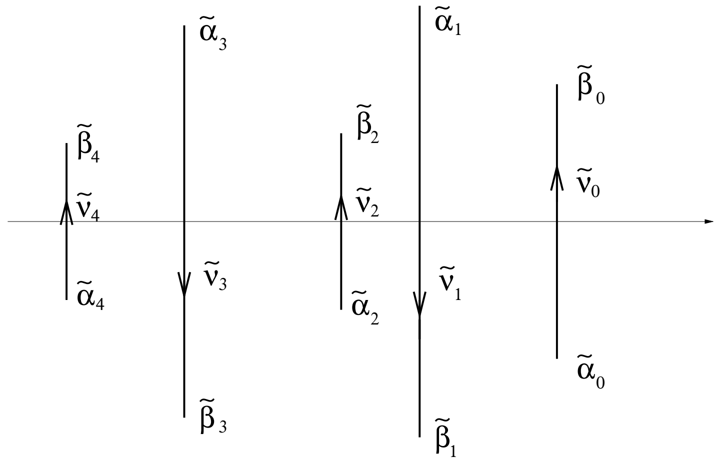

FIGURE 6.1. Contour $\tilde\Sigma^{(\mathrm{mod})}$, genus 4 case.

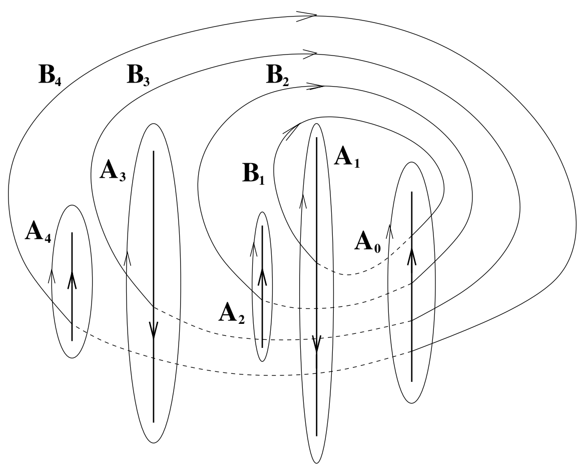

FIGURE 6.2. Basic cycles $\boldsymbol{A}_j$ and $\boldsymbol{B}_j$, genus 4 case.

THEOREM 6.1 [13] *If* $\lambda(z)-\lambda^{-1}(z)$ *has two simple zeroes, then the leading-order term of the solution* (*as* $\varepsilon \to 0$) *to* (1.1)–(1.2) *has the form*

(6.2)

$$q_0(x,t,\varepsilon) = \frac{\theta(u(\infty)+\hat\Omega/2\pi - d)\theta(u(\infty)+d)}{\theta(u(\infty)-\hat\Omega/2\pi + d)\theta(u(\infty)-d)} e^{(2i/\varepsilon)[2g(\infty)+\Omega_1]} \sum_{j=0}^{2} \operatorname{Im}\tilde\beta_j$$

where $\operatorname{Im}\tilde\beta_j = (-1)^j b_{2j},\ j = 0, 1, 2,$ *and*

$$g(\infty) = \frac{1}{2\pi i}\int_{\gamma_m} \frac{f(\zeta)+W}{R_+(\zeta)}\zeta^2\, d\zeta + \frac{1}{2\pi i}\int_{\gamma_c} \frac{\Omega}{R(\zeta)}\zeta^2\, d\zeta \tag{6.3}$$

in the region of genus 2. *Here γ_m and γ_c denote the union of all main and all complementary arcs, respectively*; *the theta functions and the basic holomorphic differentials ω, dual to $\boldsymbol{\alpha}$-cycles $\boldsymbol{A}$, are associated with the hyperelliptic Riemann surface $\tilde{\mathcal{R}}(x,t)$, and the vector $\omega^0 \in \mathbb{C}^2$ is the leading coefficient of ω*; $u(z) = \int_{\alpha_1}^{z}\omega$; $\hat{\Omega}_1 = -(2/\varepsilon)W_1$, $\hat{\Omega}_2 = -(2/\varepsilon)(W_1+\Omega_1)$; $f_0^{(0)}(z)$ *is the leading-order term of* $(i/2e)\ln r^{(0)}(z)$ *as* $\varepsilon \to 0$; $R(z) = \prod_{j=0}^{5}\sqrt{(z-\alpha_j)}$ *and the branch* $R_+(z) \to -z^3$ *as* $z \to \infty$; *and* $d = -\int_{\alpha_2}^{X_2(z_1)}\omega_1 - \int_{\alpha_5}^{X_2(z_2)}\omega_2$, *where $X_2(z)$ is the preimage of z on the second sheet of the hyperelliptic surface $\tilde{\mathcal{R}}(x,t)$. The real constant vectors W and Ω are determined through equations* (3.8) *in* [13]. *The error*

$$|q(x,t,\varepsilon) - q_0(x,t,\varepsilon)| = O(\varepsilon) \tag{6.4}$$

uniformly on compact subsets of the genus 2 region of the (x,t)*–plane.*

In our case $N = 1$ and equations (3.8) in [13] become

$$\Omega_1 = \int_{\alpha_4}^{\alpha_2} h'_-(z)dz, \qquad W_1 = \int_{\alpha_0}^{\alpha_2} h'(z)dz, \tag{6.5}$$

where $h_-(z)$ is the value of $h(z)$ on the negative side of $\gamma_{m,1}^{+}$.

Assertions of Theorem 6.1 are valid for sufficiently large t because in this case the assumption about simple zeroes of $\lambda(z) - \lambda^{-1}(z) = (1-\lambda^2(z))/\lambda(z)$ is satisfied. Indeed, note that the condition $\lambda^2(z) = 1$, where

$$\lambda^2(z) = \left(\frac{z-\alpha_4}{z-\bar{\alpha}_4}\cdot\frac{z-\bar{\alpha}_2}{z-\alpha_2}\cdot\frac{z-\alpha_0}{z-\bar{\alpha}_0}\right)^{1/2} \tag{6.6}$$

is equivalent to the equation $\lambda^4 = 1$ together with $\arg\lambda(z) = \pi k$ for some $k \in \mathbb{Z}$. Writing the former equation as

$$\frac{z-\alpha_4}{z-\bar{\alpha}_4}\cdot\frac{z-\bar{\alpha}_2}{z-\alpha_2}\cdot\frac{z-\alpha_0}{z-\bar{\alpha}_0} = 1, \tag{6.7}$$

it is easy to see that this is a quadratic equation with real coefficients. Therefore, multiple roots are possible only when $z \in \mathbb{R}$. It is easy to see that

$$-4\pi < \phi(z) < 2\pi \tag{6.8}$$

where $z \in \mathbb{R}$ and the function $\phi(z)$ is defined by

$$\phi(z) = \arg\frac{z-\alpha_4}{z-\bar{\alpha}_4} + \arg\frac{z-\bar{\alpha}_2}{z-\alpha_2} + \arg\frac{z-\alpha_0}{z-\bar{\alpha}_0}. \tag{6.9}$$

We have $|\lambda(z)| = 1$ for all $z \in \mathbb{R}$; hence $\phi(z) = 0$ is the necessary and sufficient condition for $z \in \mathbb{R}$ being a root of $\lambda^2(z) = 1$.

LEMMA 6.2 *If $b_2 > b_0 + b_4$, then equation $\lambda^2(z) = 1$ has a simple real root.*

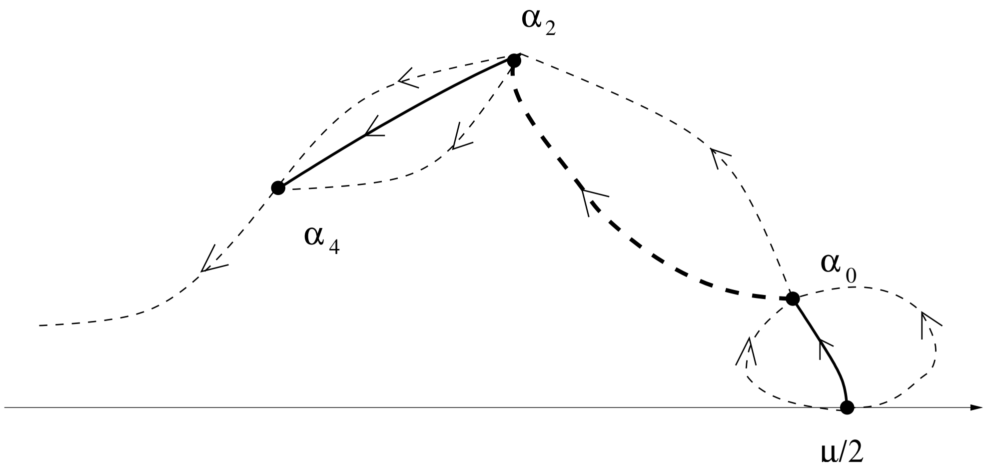

FIGURE 6.3. The upper half-plane part $\Sigma^{(4)+}$ of the contour $\Sigma^{(4)}$. The lower half-plane part $\Sigma^{(4)-}$ is symmetrical to $\Sigma^{(4)+}$. The thinner parts have jump matrices approaching I, whereas the bold parts have jump matrices approaching some constant limits.

PROOF: It is easy to see that for any fixed values of branch points α_0, α_2, and α_4 the function $\phi(z)$ is continuous on $\mathbb{R}$ with asymptotic behavior

$$\phi(z) \sim \frac{b_2 - b_0 - b_4}{z} \quad \text{and} \quad \phi(z) \sim \frac{b_2 - b_0 - b_4}{z} - 2\pi \tag{6.10}$$

as $z \to \pm\infty$, respectively. Thus, the graph of $\phi(z)$ should intersect two horizontal lines $\phi = 0$ and $\phi = -2\pi$ at two different points, which correspond to two different solutions of the quadratic equation (6.7). □

In the case $b_2 = b_4 + b_0$ one can deduce some further conditions that guarantee simple roots of (6.7). Lemma 6.2 together with Theorem 4.1 justifies the validity of (6.2) from Theorem 6.1 in the genus 2 region for $\mu \geq 2$ and for a sufficiently large (but finite) t.

The expression (6.2) together with Theorem 4.1 indicates that the leading contribution to q_0 comes from the branch point α_2, since $b_0, b_4 = o(b_2)$ as $t \to \infty$. However, our error estimates from section 2.8 in [13] are not valid in the case $t \to \infty$, since branch points α_{2j}, $j = 0, 1, 2$, approach their complex conjugates as $t \to \infty$. Therefore, a different error estimate approach is taken below.

We start with the RHP $P^{(4)} : m_+^{(4)} = m_-^{(4)} V^{(4)}$ on $\Sigma^{(4)}$ from section 2.7 in [13], which approaches the constant matrix $e^{2(i/\varepsilon)g(\infty)\sigma_3}$ as $z \to \infty$. It is also denoted as the RHP $(V^{(4)}, \Sigma^{(4)}, m^{(4)})$; see Figure 6.3.

An RHP (V, Σ, m) is called normalized at infinity if $m \to I$ as $z \to \infty$. Please note that local (away from ∞) deformations (equivalent to a local change of variables) *do not change* the normalization (or a constant value) at ∞. In fact, unless specified otherwise, *we assume all RHPs in this section are normalized*

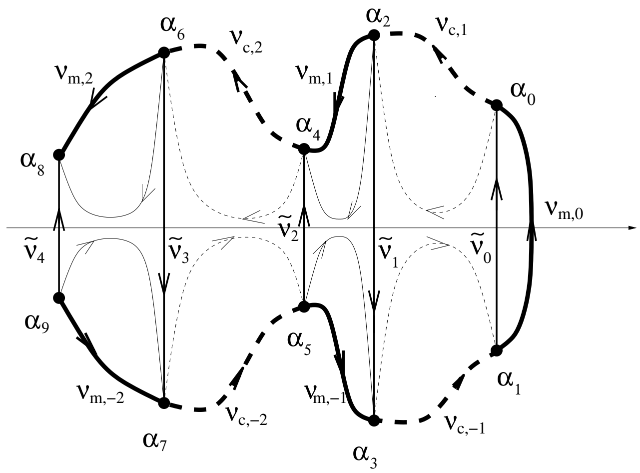

FIGURE 6.4. Deformation of the bold parts of contour $\Sigma^{(4)}$, genus 4 case.

at ∞. Also, by the nature of our problem, only the solution near ∞ is material in the reconstruction of the potential. We therefore omit most details of these deformations.

Deforming the bold parts of contour $\Sigma^{(4)}$ as described in section 8 of [13] (see also Figure 6.4), we arrive at the equivalent RHP $(V, \Gamma, m^{(4)})$, where the contour Γ^+ is shown on Figure 6.5.

On the vertical segments $\tilde{\nu}_j$, $j = 0, 1, 2$, the jump matrix $V = V_j$, where the V_j are

$$\begin{pmatrix} 0 & e^{\frac{2i}{\varepsilon}\Omega_1} \\ -e^{-\frac{2i}{\varepsilon}\Omega_1} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & e^{-\frac{2i}{\varepsilon}(W_1-\Omega_1)} \\ -e^{\frac{2i}{\varepsilon}(W_1-\Omega_1)} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & e^{-\frac{2i}{\varepsilon}W_1} \\ -e^{\frac{2i}{\varepsilon}W_1} & 0 \end{pmatrix}, \tag{6.11}$$

respectively. On the remaining dashed parts of Γ the jump matrix V approaches I exponentially fast as $\varepsilon \to 0$, i.e., like $e^{-\kappa/\varepsilon}$, where the positive constant κ depends on z. It is known (see [13]) that for a given (x, t) κ is separated from zero everywhere on the dashed parts of Γ except vicinities of the branch points.

Our analysis is based on the following lemma, first proven in [6]. The proof is repeated below for reader's convenience.

LEMMA 6.3 *Let* (V, Γ, m) *and* $(\hat{V}, \Gamma, \hat{m})$ *be two RHPs with*

$$m = I + \frac{2\pi i Q}{z} + o(z^{-1}), \qquad \hat{m} = I + \frac{2\pi i \hat{Q}}{z} + o(z^{-1}), \tag{6.12}$$

as $z \to \infty$. *Assume the following*:

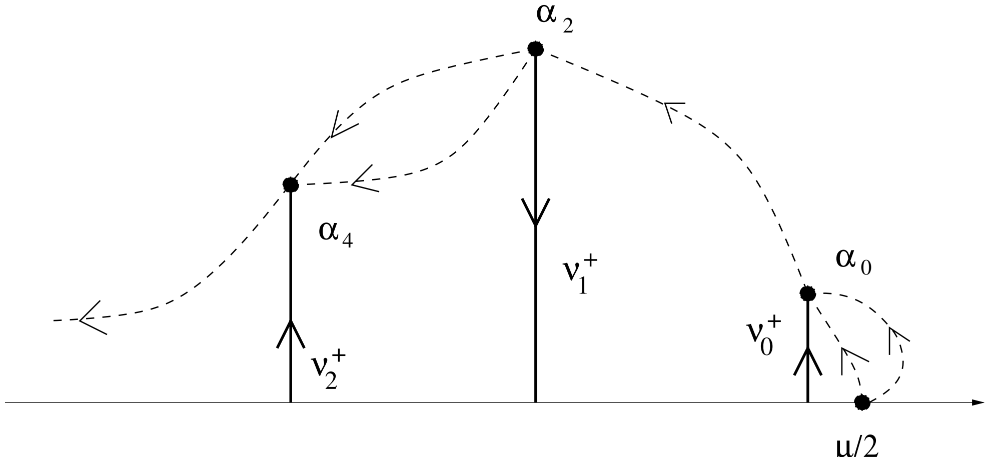

FIGURE 6.5. Contour Γ^+.

(1) *Norms* $\|(1-C_{\hat{V}})^{-1}\|_{L^2 \circlearrowleft}$, $\|V-I\|_{L^2\cap L^\infty}$, *and* $\|\hat{V}-I\|_{L^2\cap L^\infty}$ *are bounded by a constant* $M>0$.

(2) *The norm* $\|\hat{V}-V\|_{L^\infty}$ *is sufficiently small.*

Then there exists some $c>0$ *such that*

$$|Q-\hat{Q}| \le c\|V-\hat{V}\|_{L^1\cap L^\infty}. \tag{6.13}$$

Here $|\cdot|$ *denotes some matrix norm.*

PROOF: The fact that $\|\hat{V}-V\|_{L^\infty}$ is sufficiently small implies that $\|C_V - C_{\hat{V}}\|_{L^2 \circlearrowleft}$ is sufficiently small. Therefore, $\|(1-C_V)^{-1}\|_{L^2 \circlearrowleft}$ is bounded by a constant. We also have

$$\|\hat{m}-I\|_{L^2} = \|(1-C_{\hat{V}})^{-1}C_{\hat{V}}I\|_{L^2} \le c\|\hat{V}-I\|_{L^2} \le c \tag{6.14}$$

and

$$\begin{aligned}\|m_- - \hat{m}_-\|_{L^2} &= \|(1-C_{\hat{V}})_{-1}(C_V - C_{\hat{V}})(1-C_V)_{-1}I\|_{L^2} \\ &\le \|(1-C_{\hat{V}})_{-1}(C_V-C_{\hat{V}})I\|_{L^2} \\ &\quad + \|(1-C_{\hat{V}})^{-1}(C_V-C_{\hat{V}})(1-C_V)^{-1}C_V I\|_{L^2} \\ &\le c\|V-\hat{V}\|_{L^2} + c\|V-\hat{V}\|_{L^\infty} \\ &\le c\|V-\hat{V}\|_{L^2\cap L^\infty}.\end{aligned} \tag{6.15}$$

In these formulae, as well as in (6.17) below, c denotes a "generic" positive constant.

The Cauchy representation gives

$$Q = \int_\Gamma (m_+ - m_-) = \int_\Gamma m_-(V-I) \quad \text{and} \quad \hat{Q} = \int_\Gamma \hat{m}_-(\hat{V}-I). \tag{6.16}$$

Therefore, using (6.14)–(6.15), we obtain

$$
\begin{aligned}
|\hat{Q} - Q| &= \left| \int_\Gamma m_-(V - I) - \hat{m}_-(\hat{V} - I) \right| \\
&\le \left| \int_\Gamma (m_- - \hat{m}_-)(V - I) \right| + \left| \int_\Gamma (\hat{m}_- - I)(\hat{V} - V) \right| \\
&\quad + \left| \int_\Gamma (V - \hat{V}) \right| \\
&\le \|m_- - \hat{m}_-\|_{L^2} \|V - I\|_{L^2} + \|\hat{m}_- - I\|_{L^2} \|\hat{V} - V\|_{L^2} \\
&\quad + \|\hat{V} - V\|_{L^1} \\
&\le c\|\hat{V} - V\|_{L^2 \cap L^\infty} + c\|\hat{V} - V\|_{L^2} + c\|\hat{V} - V\|_{L^1} \\
&\le c\|\hat{V} - V\|_{L^1 \cap L^\infty}.
\end{aligned}
\tag{6.17}
$$

The proof is completed. □

To apply Lemma 6.3, we need to introduce a number of RHPs. Let m_{α_k}, $k = 0, 2, 4$, denote the standard Airy parametrices used in [13] in the vicinities of points α_k, respectively. The exact form of m_{α_k} is not essential for our purposes. Similarly, $m_{\mu/2}$ denotes the parametrix in the vicinity of $z = \mu/2$ that was implicitly obtained in [13] through the RHP approach. We remark that these parametrices are local solutions to the RHP and are defined uniquely up to nonsingular analytic left multipliers that give correct matching. These parametrices will be used inside the cicles S_k, $k = 0, 2, 4$, and $S_{\mu/2}$, as indicated in Figure 6.6. Since all $\operatorname{Im}\alpha_k \to 0$ as $t \to \infty$ (see Theorem 4.1), the radii of these circles must shrink proportionally. In particular, we choose the radius of S_k to be proportional to $b_k = \operatorname{Im}\alpha_k$ and the radius of $S_{\mu/2}$ to be proportional to b_0 in such a way that these circles do not intersect each other and, except for $S_{\mu/2}$, do not intersect the real axis. The contour $\tilde{\nu}_0$ is bent to the left, if necessary, so that it does not intersect $S_{\mu/2}$. The circles S_k around the complex-conjugated branch point α_k, $k = 1, 3, 5$, in the lower half-plane, are defined as $S_k = \bar{S}_{k-1}$.

Let m_j denotes the solution of the RHP $(V_j \tilde{\nu}_j, m_j)$, $j = 0, 1, 2$, normalized at infinity. Since b_{2j}, $j = 0, 1, 2$, are shrinking as $t \to \infty$, we choose to localize each vertical slit $\tilde{\nu}_j$, together with the parametrix regions S_{2j}, $\bar{S}_{2j}$, and $S_{\mu/2}$, by fixed-size (t-independent) ellipses E_j, as shown in Figure 6.7.

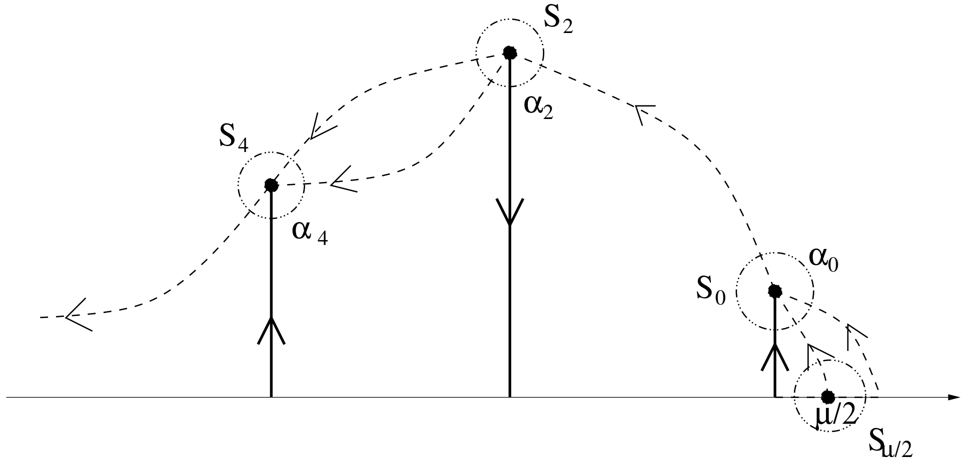

FIGURE 6.6. Small circles around branch points and the point $z = \mu/2$.

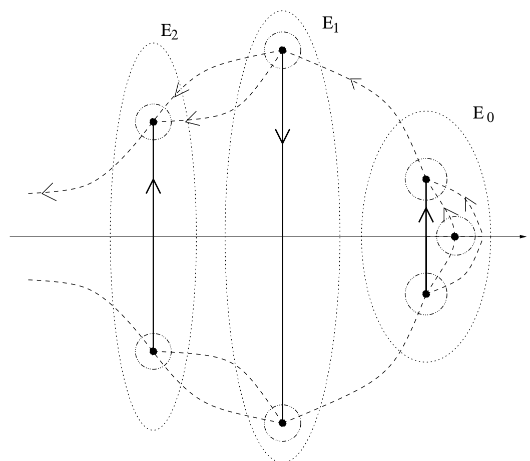

FIGURE 6.7. Fixed-size ellipses $E_j,\ j = 0, 1, 2$.

We then define the global parametrix M_p by

$$M_p = \begin{cases} I \text{ outside ellipses } E_j,\ j = 0, 1, 2, \\ m_{\alpha_k} \text{ within } S_k,\ k = 0, 2, 4, \\ m^{\dagger}_{\alpha_k} \text{ within } S_{k+1} = \overline{S_k},\ k = 0, 2, 4, \text{ where } f^{\dagger}(z) = f^{*}(\bar{z}), \\ m_{\mu/2} \text{ within } S_{\mu/2}, \\ m_j \text{ within } E_j \text{ but outside } S_{2j} \cup \overline{S_{2j}} \\ \quad \text{(and outside } S_{\mu/2} \text{ if } j = 0). \end{cases} \tag{6.18}$$

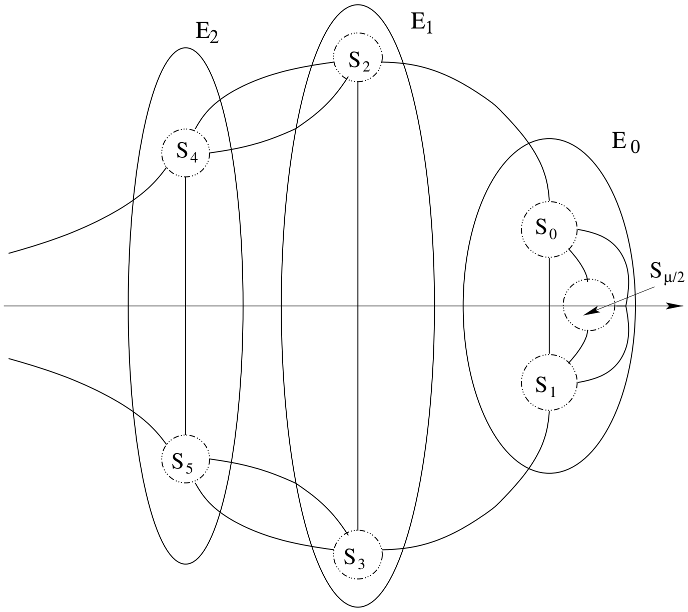

FIGURE 6.8. Contour Γ.

Define $M = mM_p^{-1}$, where $m^{(4)} = e^{2(i/\varepsilon)g(\infty)\sigma_3}m$ and $\sigma_3 = \text{diag}(1, -1)$. Then M is the solution of the RHP $(\hat{V}, \hat{\Gamma}, M)$ with the contour $\hat{\Gamma}$ shown on Figure 6.8. The jump matrix $\hat{V}$ for this problem can be calculated directly from the jump matrix of the RHP (V, Γ, m) and from (6.18).

Remark 6.4. By choosing to localize the problem within the ellipses E_j, $j = 0, 1, 2$, the parametrix matching of the circles S_{2j} and $\overline{S_{2j}}$ (and of $S_{\mu/2}$ when $j = 0$) becomes easier, because it is similar to the genus 0 case: no Riemann theta functions are needed.

We now want to estimate the $\|\cdot\|_{L^\infty}$–norm of $\hat{V} - I$. On the ellipses E_j we have $\hat{V} = M_p = m_j$, $j = 0, 1, 2$. Then

$$\|\hat{V} - I\|_{L^\infty} = O(b_k) \tag{6.19}$$

follows from the explicit expression for m_k (see, e.g., [13, sec. 4.4]).

To estimate $\|\hat{V} - I\|_{L^\infty}$ on the shrinking circles S_k (as $t \to \infty$), we choose the local scaling $z \mapsto b_k z$ to bring each S_k, $k = 0, 2, 4$, to a fixed size.

PROPOSITION 6.5 *In the limit $t \to \infty$, $x = \xi t$, the function $h'(z)$ has behavior*

$$h'(z) = \left(\mu - 2\left(\frac{\hat{\xi}}{4}\right)\right)\sqrt{\frac{2i}{b_2^3}}\sqrt{z - \alpha_2}\left(1 + O(\sqrt{z - \alpha_2})\right)\left(1 + O\left(\frac{\ln t}{t}\right)\right) \tag{6.20}$$

near the slow branch point α_2*. Near the fast branch point* α_0

$$h'(z) = \frac{\ln b_0}{\mu}\sqrt{2ib_0}\sqrt{z-\alpha_0}(1+O(\sqrt{z-\alpha_0}))\left(1+O\left(\frac{\ln t}{t}\right)\right). \tag{6.21}$$

An expression similar to (6.21) *is valid near the remaining branch point* α_4*.*

PROOF: We start with the observation that the analytic function $h'(z)$ has a square root branch point at $z = \alpha_k$, $k = 0, 2, 4$. To find the coefficient in front of $\sqrt{z-\alpha_2}$, we apply Proposition 4.3 to the integral representation (4.12) of $h'(z)$. Note that in this case $\rho(\zeta) = \operatorname{sign}\zeta/(\zeta - z)$. Since $\ln b_2 = o(\ln b_0)$ in the limit $t \to \infty$, we retain only the two terms with $\ln b_0$ and $\ln b_4$ in the right-hand side of (4.9), which, after some algebra, yield

$$h'(z) \sim 4t\sqrt{2ib_2}\sqrt{z-\alpha_2}. \tag{6.22}$$

The corresponding formula for α_0 is

$$h'(z) \sim -4t\,\frac{\mu/2+\xi/4}{\mu}\sqrt{2ib_0}\sqrt{z-\alpha_0}. \tag{6.23}$$

Speaking technically, we cannot use Proposition 4.3 to calculate the contribution of a vicinity of the point α_2 to the integral in (4.12), since $\rho(\zeta)$ is not uniformly bounded near $\zeta = a_2$ as $t \to \infty$. However, we can apply Proposition 7.1, which is proved independently of the present section, to estimate the contribution of a vicinity of the point α_2 to the integral in (4.12) as $O(\ln t)$. Then (6.20) follows from (6.22) and Theorem 4.1. Similar considerations lead to (6.21) and the corresponding formula for α_4. □

Now we know that, according to Proposition 6.5,

$$\operatorname{Im} h(z) \sim c_2\sqrt{\frac{1}{b_2^3}}(z-\alpha_2)^{3/2}, \qquad \operatorname{Im} h(z) \sim c_j \ln b_j \sqrt{b_j}(z-\alpha_j)^{3/2} \tag{6.24}$$

in the vicinities of the branch points α_2 and α_j, $j = 0, 4$, respectively, where

$$c_2 = \left(\mu - 2\left(\frac{\hat{\xi}}{4}\right)\right)\sqrt{2i}, \quad c_j = \frac{\sqrt{2i}}{\mu}. \tag{6.25}$$

After the scalings $z \mapsto b_k z$, the expressions (6.24) are transformed into

$$\operatorname{Im} h(z) \sim c_2\left(z - \frac{\alpha_2}{b_2}\right)^{3/2}, \qquad \operatorname{Im} h(z) \sim c_j \ln b_j b_j^2\left(z - \frac{\alpha_j}{b_j}\right)^{3/2}, \tag{6.26}$$

respectively, where $j = 0, 4$.

We can choose m_{α_2} so that (as in [13]) on S_2 (and on $\overline{S_2}$) we have

$$\|\hat{V} - I\|_{L^\infty} = O(\varepsilon). \tag{6.27}$$

Similarly, on S_j, $j = 0, 4$, (and on $\overline{S_j}$) we have

$$\|\hat{V} - I\|_{L^\infty} = O\left(\frac{\varepsilon}{|\ln b_j| b_j^2}\right). \tag{6.28}$$

To estimate the behavior of $\|\hat{V} - I\|_{L^\infty}$ on $S_{\mu/2}$ we use (2.11) to obtain

$$\operatorname{Im} h(z) = -\frac{\pi}{2}\left|\frac{\mu}{2} - z\right|, \quad z \in \mathbb{R},$$

in a vicinity of $z = \mu/2$. Repeating the scaling $z \mapsto b_0 z$, we arrive at

$$\|\hat{V} - I\|_{L^\infty} = O\left(\frac{\varepsilon}{b_0}\right) \tag{6.29}$$

on $S_{\mu/2}$.

So far, we have shown that

$$\varepsilon = o(b_0^2 \ln b_0) \tag{6.30}$$

is a sufficient condition for

$$\|\hat{V} - I\|_{L^\infty} \to 0 \tag{6.31}$$

on the circles S_k, $k = 0, 2, 4$, and $S_{\mu/2}$. We now show that (6.31) also holds on the remaining part of the contour $\hat{\Gamma}$, i.e., on the curves from the contour Γ that connect different circles S_k, $k = 0, 2, 4$, $S_{\mu/2}$. The points of intersection of these curves with the circles are called endpoints.

Let us show that a curve l, connecting circles, say, S_2 and S_0, can be deformed in such a way that the minimum of $\operatorname{Im} h$ along l is attained at one of its endpoints. Indeed, by construction, $\operatorname{Im} h > 0$ along l. Let us consider the region E_0^+ inside the ellipse E_0, where $\operatorname{Im} h > 0$. Since the distance between the boundary ∂E_0 and α_0 is greater than some $\delta > 0$ for all t, we can use the asymptotic formula (8.2) for $\operatorname{Im} h$ on ∂E_0. According to (8.2), there exists a level curve $\operatorname{Im} h(z) = \operatorname{Im} h(\lambda_0)$ in E_0^+, where the endpoint $\lambda_0 = l \cap S_0$, that connects the point λ_0 with some point $e_0 \in \partial E_0$; see Figure 6.9. We can deform l inside E_0 to coincide with the piece of the level curve connecting λ_0 and e_0 and lying outside S_0. (A part of this level curve that may be inside S_0 can be replaced by the corresponding arc of the circle S_0.) A similar construction allows us to define $e_2 \in \partial E_2$ such that $\operatorname{Im} h(z) = \operatorname{Im} h(\lambda_2)$, where the endpoint $\lambda_2 = l \cap S_2$. Now, according to (8.2), we can deform, if necessary, contour l outside E_2 and E_0 so that $\min \operatorname{Im} h(z)$ on the deformed contour $\tilde{l}$ is attained at one of the endpoints. Similar arguments can be applied to each piece of the contour $\hat{\Gamma}$ that lies outside the ellipses E_k, $k = 0, 2, 4$. In the case of pieces of $\hat{\Gamma}$ connecting the circle $S_{\mu/2}$ with S_0 and $\overline{S}_0$, the required statement follows from the topology of the level curves of $\operatorname{Im} h$ in $E_0^- = E_0 \setminus E_0^+$. (Indeed, we can control $\operatorname{Im} h$ on ∂E_0 through (8.2), whereas on $\mathbb{R}$ we have $\operatorname{Im} h = -\operatorname{Im} f$.)

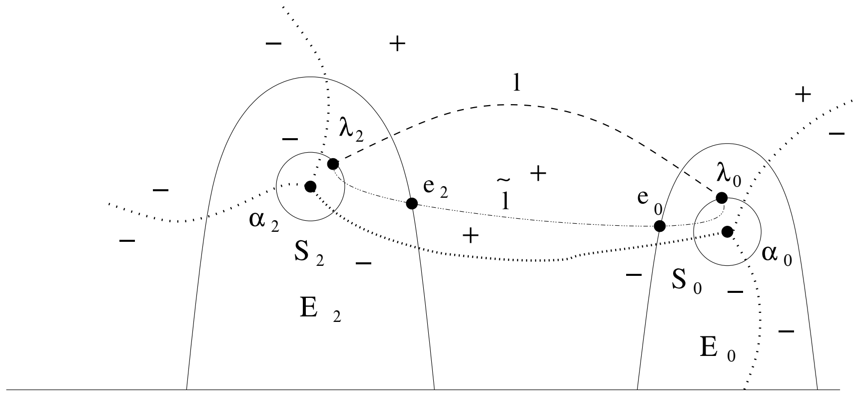

FIGURE 6.9. Control of $\operatorname{Im} h$ on the contour Γ outside circles S_k, $k = 0, 2, 4$. The figure contains branch points α_0 and α_2 together with zero level curves of $\operatorname{Im} h$ emanating from them.

Using (6.26) to estimate $\hat{V} - I$ at the endpoints, we find out that there exists a constant $c > 0$ such that

$$\|\hat{V} - I\|_{L^\infty} = O\left(\exp \frac{-cb_0^2|\ln b_0|}{\varepsilon}\right). \tag{6.32}$$

Thus, we have proved that the RHP $(\hat{V}, \hat{G}, M)$ is a small norm RHP under condition (6.30).

Our plan is to compare this RHP with the RHP $(V_{\text{mod}}, E_2, M_{\text{mod}})$, where $V_{\text{mod}} = M_p\big|_{E_2}$. We make the trivial extension of the latter RHP to the entire contour $\hat{G}$ and denote it by $(V_{\text{mod}}, \hat{G}, M_{\text{mod}})$. Then our previous analysis shows that

$$\|\hat{V} - V_{\text{mod}}\|_{L^\infty} = O\left(\max\left\{\frac{\varepsilon}{b_0^2|\ln b_0|}, b_4\right\}\right). \tag{6.33}$$

Note that $\|\hat{V} - V_{\text{mod}}\|_{L^1}$ is of the same order or smaller than $\|\hat{V} - V_{\text{mod}}\|_{L^\infty}$, since infinite parts of the contour $\hat{\Gamma}$ can be deformed to lie on the real axis, where the explicit expression for $\operatorname{Im} h$ is given by (2.11).

Applying now Lemma 6.3 to the RHPs $(\hat{V}, \hat{G}, M)$ and $(V_{\text{mod}}, \hat{G}, M_{\text{mod}})$, we obtain

$$|\hat{Q} - Q_{\text{mod}}\|_{L^\infty} = O\left(\max\left\{\frac{\varepsilon}{b_0^2|\ln b_0|}, b_4\right\}\right) \tag{6.34}$$

where $\hat{Q} = \lim_{z\to\infty} z(M - I)/2\pi i$, $Q_{\text{mod}} = \lim_{z\to\infty} \frac{z(M_{\text{mod}}-I)}{2\pi i}$. Since the RHP $(V_{\text{mod}}, \hat{G}, M_{\text{mod}})$, i.e., the matrix m_2, can be found explicitly (e.g., [13, sec. 4.4]), we obtain the following theorem:

THEOREM 6.6 *Assuming that* $\varepsilon \to 0$, $t \to \infty$, *and* $b_0^2\sqrt{|\ln b_0|}/\varepsilon \to \infty$, *we have*

$$q(x,t,\varepsilon) = -b_2 e^{(2i/\varepsilon)(2g(\infty)+\Omega_1-W_1)} + O\left(\frac{\varepsilon}{b_0^2|\ln b_0|}\right) + O(b_4), \tag{6.35}$$

The latter limit guarantees that the error terms in (6.35) are of a smaller order than the leading term. The exact expression of the leading-order behavior of solutions to (1.1)–(1.2) is given in Theorem 7.5.

7 Long-Time Behavior of the Wave Form

In this section we translate the information obtained about the behavior of the branch points $\alpha(x,t)$ along the rays $x/t = \xi$ as $t \to \infty$, where $0 \le \xi < 2\mu$, into the leading-order behavior of $q(x,t,\varepsilon)$ described in Theorem 6.6.

PROPOSITION 7.1 *Suppose that the branch points* $\alpha_{2k} = a_{2k} + ib_{2k}$ *depend on a parameter* t *in such a way that they approach, respectively, three different points* a_{2k} *on the real axis such that* $|a_{2k}| > T$, *as* $t \to \infty$, $k = 0, 1, 2$. *Suppose also that* $\rho(\zeta)$ *is a bounded function on* $\mathbb{R}$, *that it is twice continuously differentiable in vicinities of* a_{2k}, *and that the integral below exists. Let us choose some* $\delta_1 > 0$ *and* $\phi \in (0, \pi/2)$, *and denote by* Z *a compact subset of the set of all* z *such that either* $\operatorname{Im} z > \delta_1$ *or* $|\arg(z-\alpha_{2k}) - \pi/2| \le \phi$ *for some* α_{2k} *and for all* t. *Then for any* $z \in Z$

$$\begin{aligned}\int\limits_{|\zeta|\ge T} \frac{\rho(\zeta)}{(\zeta - z)|R(\zeta)|}\,d\zeta \\ = 2\sum_{k=0}^{2} \frac{1}{\sqrt{(z-a_{2k})^2 + b_{2k}^2}} \left[\frac{\rho(a_{2k})[\ln b_{2k} - \ln(z-a_{2k})]}{|a_{2k}-\alpha_{2i}||a_{2k}-\alpha_{2j}|} + O(1)\right]\end{aligned} \tag{7.1}$$

uniformly in Z *in the limit* $t \to \infty$, *where* $\{i, j, k\}$ *is a permutation of* $\{0, 1, 2\}$, $R(\zeta) = \prod_{j=0}^{2} |\zeta - a_{2j}|$. *If some* $|a_{2k}| < T$, *its contribution to the sum* (7.1) *is zero. If some* $|a_{2k}| = T$ *and* α_{2k} *approaches* a_{2k} *along an orthogonal-to-*$\mathbb{R}$ *trajectory, then the contribution of* a_{2k} *to the sum* (7.1) *should be multiplied by* $\frac{1}{2}$.

PROOF: Let us choose some α_{2k} and compute its contribution to the sum (7.1). In order to do that, we consider a small but fixed $\delta > 0$ such that the intervals $I_j = (a_{2j} - \delta, a_{2j} + \delta)$ do not intersect each other for $j = 0, 1, 2$ and that I_i and I_k do not contain $\pm T$. It is clear that the integral in the left-hand side of (7.1), restricted to $\{|z| > T\} \setminus \bigcup_{j=0}^{2} I_j$, is uniformly bounded for all $z \in Z$ as $t \to \infty$.

Then

$$\int_{I_k} \frac{\rho(\zeta)}{(\zeta - z)|R(\zeta)|} d\zeta \tag{7.2}$$
$$= \int_{I_k} \frac{\psi(a_{2k})}{(\zeta - z)\sqrt{(\zeta - a_{2k})^2 + b_{2k}^2}} d\zeta + \int_{I_k} \frac{(\zeta - a_{2k})\tilde{\psi}(\zeta)}{(\zeta - z)\sqrt{(\zeta - a_{2k})^2 + b_{2k}^2}} d\zeta$$

where

$$\psi(\zeta) = \frac{\rho(\zeta)}{|\zeta - a_{2i}||\zeta - a_{2j}|}$$

and

$$\tilde{\psi}(\zeta) = \frac{\psi(\zeta) - \psi(a_{2k})}{\zeta - a_{2k}}.$$

Let us introduce $s = \zeta - a_{2k}$ and $y = z - a_{2k}$. To estimate the first term in (7.2), we have

$$\begin{aligned} \int_{I_k} \frac{\psi(a_{2k})}{(\zeta - z)\sqrt{(\zeta - a_{2k})^2 + b_{2k}^2}} d\zeta &= \psi(a_{2k}) \int_{-\delta}^{\delta} \frac{(s + y)ds}{(s^2 - y^2)\sqrt{s^2 + b_{2k}^2}} \\ &= 2y\psi(a_{2k}) \int_0^{\delta} \frac{ds}{(s^2 - y^2)\sqrt{s^2 + b_{2k}^2}}. \end{aligned} \tag{7.3}$$

According to [9, sec. 2.284], the antiderivative of the latter integrand is

$$\begin{aligned} &\frac{-1}{2y\sqrt{y^2 + b_{2k}^2}} \ln \frac{y\sqrt{s^2 + b_{2k}^2} + s\sqrt{y^2 + b_{2k}^2}}{y\sqrt{s^2 + b_{2k}^2} - s\sqrt{y^2 + b_{2k}^2}} \\ &\qquad = \frac{-1}{y\sqrt{y^2 + b_{2k}^2}} \tanh^{-1} \frac{s\sqrt{y^2 + b_{2k}^2}}{y\sqrt{s^2 + b_{2k}^2}}. \end{aligned} \tag{7.4}$$

Multiplying the ratio inside the logarithm by the conjugate, we obtain

$$\begin{aligned} &\psi(a_{2k}) \int_{-\delta}^{\delta} \frac{(s + y)ds}{(s^2 - y^2)\sqrt{s^2 + b_{2k}^2}} \\ &\qquad = \frac{-2\psi(a_{2k})}{\sqrt{y^2 + b_{2k}^2}} \ln \frac{y\delta\left(\sqrt{1 + b_{2k}^2/\delta^2} + \sqrt{1 + b_{2k}^2/y^2}\right)}{b_{2k}\sqrt{y^2 - \delta^2}} \end{aligned} \tag{7.5}$$

Note that the construction of the set Z implies that $y \pm \delta = O(1)$ as $t \to \infty$ uniformly in $z \in Z$. The logarithm of the square root in the numerator is also uniformly bounded, since $1 + b_{2k}^2/\delta^2 > 1$ and $\text{Re}[1 + b_{2k}^2/y^2] \geq 0$ for all $z \in Z$.

Thus, the right-hand side of (7.5) yields the contribution of α_{2k} to the right-hand side of (7.1).

According to (2.6) with $D = J$, the second term in (7.2) can be represented as

$$\int_{I_k} \frac{(\zeta - a_{2k})\tilde{\psi}(\zeta)}{(\zeta - z)\sqrt{(\zeta - a_{2k})^2 + b_{2k}^2}}\, d\zeta = \int_{I_k} \frac{\operatorname{sign}(\zeta - a_{2k})\tilde{\psi}(\zeta) d\zeta}{(\zeta - z)} + O(1). \tag{7.6}$$

The latter integral is uniformly bounded if $|z - a_{2k}| \geq \delta_1$. When this is not the case, we have $\tilde{\psi}(\zeta) = \tilde{\psi}(\operatorname{Re} z) + (\zeta - \operatorname{Re} z)l(\zeta)$, where $l(\zeta) \in C(I_k)$, since $\rho(\sigma) \in C^2(I_k)$ and $\operatorname{Re} z$ is close to a_{2k}. Substituting this expression into (7.6), we obtain

$$\int_{I_k} \frac{(\zeta - a_{2k})\tilde{\psi}(\zeta)}{(\zeta - z)\sqrt{(\zeta - a_{2k})^2 + b_{2k}^2}}\, d\zeta = \tilde{\psi}(\Re z) \int_{I_k} \frac{\operatorname{sign}(\zeta - a_{2k}) d\zeta}{(\zeta - z)} + O(1) \tag{7.7}$$

since $|\zeta - \operatorname{Re} z|/|\zeta - z| < 1$. The evaluation of the latter integral yields

$$\begin{aligned} &- 2\tilde{\psi}(\operatorname{Re} z) \ln(z - a_{2k}) + O(1) \\ &\qquad = -2\tilde{\psi}(\operatorname{Re} z) \frac{\sqrt{(z - \alpha_{2k})(z - \bar{\alpha}_{2k})} \ln(z - a_{2k})}{\sqrt{(z - a_{2k}^2) + b_{2k}^2}} + O(1). \end{aligned} \tag{7.8}$$

It is now clear that the numerator of the latter expression is uniformly bounded for all $z \in Z$ and for all large t. The proof is completed. □

THEOREM 7.2 *If $\mu \geq 2$ and $\xi \in [0, 2\mu)$ but $\mu + \xi \neq 2$, then*

$$\begin{aligned} \Omega_1 &= 2t\left(\frac{\mu}{2} - \frac{\xi}{4}\right)^2 + \frac{\ln t}{2}\left[\frac{\mu}{2} - \left(\frac{\hat{\xi}}{4}\right)\right] + O(1), \\ W_1 &= -2t\left(\frac{\mu}{2} + \frac{\xi}{4}\right)^2 - \frac{\ln t}{2}\left[\frac{\mu}{2} - \left(\frac{\hat{\xi}}{4}\right)\right] + O(1), \end{aligned} \tag{7.9}$$

as $t \to \infty$, $x = \xi t$. Here $(\hat{\xi}/4) = \xi/4$ if $\xi/4 \geq T$ or $(\hat{\xi}/4) = T$ otherwise. In the special case $\mu = 2$, $\xi = 0$, equations (7.9) *become*

$$\Omega_1 = 2t + \frac{1}{2}\ln t + O(1), \qquad W_1 = -2t - \frac{1}{2}\ln t + O(1). \tag{7.10}$$

PROOF: We will prove only the first of the expressions (7.9) based on (6.5) since the proof of the second one is similar. Indeed, the arguments of Section 3 together with (6.5) imply

$$\Omega_1 = \frac{1}{2}\int_{\alpha_4}^{\alpha_2} \int_{|\zeta| \geq T} \frac{R(z) \operatorname{sign} \zeta \, d\zeta \, dz}{(\zeta - z)|R(\zeta)|}, \tag{7.11}$$

where we integrate along the negative side of the main arc $\gamma_{m,1}^{+}$. We now observe that, according to Theorem 4.1, the inner integral (in ζ) in (7.11) satisfies the conditions of Proposition 7.1 (with $\rho(\zeta) = \operatorname{sign}(\zeta)$) for all possible values of $\mu \geq 2$,

$\xi \in [0, 2\mu)$, with the exception of the case $\mu = 2$, $\xi = 0$. (We can always deform the contour of integration $[\alpha_4, \alpha_2]$ in a z-variable to ensure that $z \in Z$.) The latter case will be considered separately.

Using Proposition 7.1, we obtain

$$\Omega_1 = \sum_{k=0}^{2} \frac{\operatorname{sign}(a_{2k}) \ln b_{2k}}{|a_{2k} - \alpha_{2i}|\,|a_{2k} - \alpha_{2j}|} \int_{\alpha_4}^{\alpha_2} \frac{R(z)dz}{\sqrt{(z - a_{2k})^2 + b_{2k}^2}} + O(1) \tag{7.12}$$

uniformly in z as $t \to \infty$. Note that on the contour $[\alpha_4, \alpha_2]$ for any $k = 0, 1, 2$ we have

$$\sqrt{(z - a_{2k})^2 + b_{2k}^2} = z - a_{2k} + \frac{b_{2k}^2}{\sqrt{(z - a_{2k})^2 + b_{2k}^2} + z - a_{2k}}. \tag{7.13}$$

Since $z \in Z$ and the arguments of two terms in the denominator differ by no more than $\pi/2$, the absolute value of the whole ratio is less than b_{2k}. Thus,

$$\sqrt{(z - a_{2k})^2 + b_{2k}^2} = z - a_{2k} + O(b_{2k}) \tag{7.14}$$

uniformly in z as $t \to \infty$. Substituting (7.14) in (7.13) and using the fact that b_0 and b_4 are exponentially small as $t \to \infty$, we obtain

$$\begin{aligned}
\Omega_1 &= \int_{\alpha_4}^{\alpha_2} \Bigg[\frac{\ln b_0 (z - a_4)\sqrt{(z - a_2)^2 + b_2^2}}{|a_0 - \alpha_2||a_0 - \alpha_4|} - \frac{\ln b_4 (z - a_0)\sqrt{(z - a_2)^2 + b_2^2}}{|a_4 - \alpha_2||a_4 - \alpha_0|} \\
&\qquad - \frac{\kappa(\xi/4) \ln b_2 (z - a_0)(z - a_4)}{|a_2 - \alpha_0||a_2 - \alpha_4|} \Bigg] dz + O(1) \\
&= \int_{\alpha_4}^{\alpha_2} \Bigg\{ \Bigg[\frac{\ln b_0}{|a_0 - \alpha_2||a_0 - \alpha_4|} - \frac{\ln b_4}{|a_4 - \alpha_2||a_0 - \alpha_4|} \Bigg] (z - a_2)\sqrt{(z - a_2)^2 + b_2^2} \\
&\qquad + \Bigg[\frac{(a_2 - a_4) \ln b_0}{|a_0 - \alpha_2||a_0 - \alpha_4|} + \frac{(a_0 - a_2) \ln b_4}{|a_4 - \alpha_2||a_0 - \alpha_4|} \Bigg] \sqrt{(z - a_2)^2 + b_2^2} \\
&\qquad - \frac{\kappa(\xi/4) \ln b_2 (z - a_0)(z - a_4)}{|a_0 - \alpha_2||a_2 - \alpha_4|} \Bigg] \Bigg\} dz + O(1)
\end{aligned} \tag{7.15}$$

Here $\kappa(s) = 0, \frac{1}{2}, 1$ if $s < T$, $s = T$ or $s > T$ respectively and the sign of $R(z)$ is choosen so that it is positive on $[a_4, a_2]$. Using Theorem 4.1, we obtain

$$\begin{aligned}
\frac{\ln b_0}{|a_0 - \alpha_2||a_0 - \alpha_4|} - \frac{\ln b_4}{|a_4 - \alpha_2||a_4 - \alpha_0|} &= -\frac{\kappa(\xi/4) \ln t}{2(\mu^2/4 - a_2^2)} + O(1), \\
\frac{(a_2 - a_4) \ln b_0}{|a_0 - \alpha_2||a_0 - \alpha_4|} + \frac{(a_0 - a_2) \ln b_4}{|a_4 - \alpha_2||a_4 - \alpha_0|} &= -\frac{4t(\mu^2/4 - \xi^2/16)}{(\mu^2/4 - a_2^2)} + O(1).
\end{aligned} \tag{7.16}$$

So,

$$\begin{aligned}\Omega_1 = {} & \frac{1}{\mu^2/4 - a_2^2} \\ & \cdot \int_{\alpha_2}^{\alpha_4} \left\{ \frac{\kappa(\xi/4)\ln t}{2} \left[(z-a_2)\sqrt{(z-a_2)^2+b_2^2} + \frac{\mu^2}{4} - z^2 \right] \right. \\ & \left. + 4t\left(\frac{\mu^2}{4} - \frac{\xi^2}{16}\right)\sqrt{(z-a_2)^2+b_2^2} \right\} dz + O(1).\end{aligned} \tag{7.17}$$

Direct calculations of the leading (linear in t) term, taking into account Theorem 4.1, yield

$$\begin{aligned}& \frac{4t(\mu^2/4 - \xi^2/16)}{\mu^2/4 - a_2^2} \left[\frac{1}{2}\left(\frac{\mu}{2} + a_2\right)^2 + \frac{(\mu/2 - (\hat{\xi/4})\ln t)}{8t} \right] + O(1) \\ & = 2t\left(\frac{\mu}{2} - \frac{\xi}{4}\right)^2 + \frac{\mu\kappa(\xi/4)\ln t}{4} \cdot \frac{\mu/2 - \xi/4}{\mu/2 + \xi/4} \\ & + \frac{1}{2}\ln t\left(\frac{\mu}{2} - \left(\hat{\frac{\xi}{4}}\right)\right) + O(1).\end{aligned} \tag{7.18}$$

To complete the proof for the generic case, we calculate the remaining term in (7.17), which yields

$$-\frac{\mu\kappa(\xi/4)\ln t}{4} \cdot \frac{\mu/2 - \xi/4}{\mu/2 + \xi/4}.$$

Considering the special case $\mu = 2$, $\xi = 0$, we notice that Proposition 7.1 is applicable to contributions from branch points α_0 and α_4 but not α_2, since $\rho(\zeta) = \operatorname{sign}(\zeta)$ is not smooth on $I_2 = (a_2 - \delta, a_2 + \delta)$ for any fixed $\delta > 0$. Notice that, according to Theorem 4.1, $b_2 = O(t^{-1/2})$ and $a_2 = O(t^{-1})$ as $t \to \infty$. We calculate the contribution of α_2 to the integral (7.1) for $\rho(\zeta) = \operatorname{sign}(\zeta)$ and show that its contribution to (7.11) is of order $O(1)$ if $a_2 = o(b_2)$. Without any loss of generality, we can assume that the contour of integration $[\alpha_4, \alpha_2]$ is deformed in such a way that for any z on the contour within distance 2δ from α_2 we have $\arg(z - \alpha_2) = \pi/2$.

Indeed, introducing $s = \zeta - a_2$, $y = z - a_2$, we obtain

$$\begin{aligned}& \int_{I_2} \frac{\operatorname{sign}(\zeta)d\zeta}{(\zeta - z)\sqrt{(\zeta-a_2)^2+b_2^2}} \\ & = 2\int_{-a_2}^{\delta} \frac{s\,ds}{(s^2-y^2)\sqrt{s^2+b_2^2}} - 2y\int_0^{-a_2} \frac{ds}{(s^2-y^2)\sqrt{s^2+b_2^2}}.\end{aligned} \tag{7.19}$$

The antiderivative

$$\frac{1}{\sqrt{y^2+b_2^2}}\ln\frac{\sqrt{s^2+b_2^2}-\sqrt{y^2+b_2^2}}{\sqrt{s^2+b_2^2}-\sqrt{y^2+b_2^2}} \tag{7.20}$$

of the first integral in (7.19) shows that, apart from the factor $1/\sqrt{(\zeta-a_2)^2+b_2^2}$, this integral is uniformly bounded for our choice of z. Indeed, it is obvious if $|z-\alpha_2|>2\delta$; whereas for $\arg(z-\alpha_2)=\pi/2$ the absolute value of expression (7.20) is bounded by π. We now use the antiderivative (7.4) to calculate the second integral in (7.19) as

$$\frac{2}{\sqrt{y^2+b_2^2}}\ln\frac{y\sqrt{a_2^2+b_2^2}-a_2\sqrt{y^2+b_2^2}}{b_2\sqrt{y^2-a_2^2}}. \tag{7.21}$$

In the case $a_2=o(b_2)$ (7.21) becomes

$$\frac{2}{\sqrt{y^2+b_2^2}}\left[\ln y+\ln\left(\sqrt{1+\frac{a_2^2}{b_2^2}}-\frac{a_2}{b_2}\sqrt{1+\frac{b_2^2}{y^2}}\right)-\frac{1}{2}\ln(y^2-a^2)\right]. \tag{7.22}$$

So it is clear that both integrals from the right-hand side of (7.19) yield an $O(1)$ contribution to (7.11). We can now repeat the arguments from Proposition 7.1 to show that the contribution of

$$\int_{I_2}\frac{\operatorname{sign}(\zeta)[(\zeta-a_2)\tilde{\psi}(\zeta)]d\zeta}{(\zeta-z)\sqrt{(\zeta-a_2)^2+b_2^2}} \tag{7.23}$$

where $\tilde{\psi}\in C^1(I_2)$, to (7.11) is $O(1)$ as $t\to\infty$. The proof is complete. □

Our next step toward computing $2g(\infty)+\Omega_1-W_1$ is given by the following lemma:

LEMMA 7.3 *If $\mu\ge 2$ and $\xi\in[0,2\mu)$, then*

$$\frac{1}{4\pi i}\int_{\hat{\gamma}}\frac{f(\zeta)\zeta^2\,d\zeta}{R(\zeta)}=t\left(\frac{\xi^2}{16}-\frac{\mu^2}{4}\right)-\frac{\xi^2\ln t}{64}\cdot\frac{\mu/2-(\xi\hat{/}4)}{\mu^2/4-\xi^2/16}+O(1) \tag{7.24}$$

as $t\to\infty$, $x=\xi t$, uniformly in μ and ξ.

PROOF: According to section 3.1 of [13], we can rewrite $f(z)$ as

$$\begin{aligned}f(z)=\frac{1}{2}z\ln\left(1-\frac{T^2}{z^2}\right)+\frac{\mu}{2}\ln z+\left(\frac{\mu}{2}-z\right)\left[\ln\left(1-\frac{\mu}{2z}\right)-\frac{i\pi}{2}\right]\\ -xz-2tz^2+\frac{\mu}{2}\ln 2+T\tanh^{-1}\frac{T}{z}-T\tanh^{-1}\frac{T}{\mu/2}+\frac{\pi}{2}\varepsilon.\end{aligned} \tag{7.25}$$

As in calculating equations ((M_0)), we deform the contour $\hat{\gamma}$ (see Figure A.2) so that

$$\begin{aligned}\frac{1}{4\pi i}\int_{\hat{\gamma}} \frac{f(\zeta)\zeta^2\,d\zeta}{R(\zeta)} &= -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\zeta^2 \operatorname{Im} f(\zeta)+R(\zeta)}{R(\zeta)}\,d\zeta - \frac{\mu}{2}\lim_{\rho\to\infty}\oint_{C_\rho}\frac{\ln\zeta\,d\zeta}{4\pi i\zeta} \\ &\quad + \lim_{\rho\to\infty}\oint_{C_\rho}\frac{-2t\zeta^2 - x\zeta + \frac{\mu}{2}(\ln 2+1) + T\tanh^{-1}\frac{T}{z} - T\tanh^{-1}\frac{T}{\mu/2}+\frac{\pi}{2}\varepsilon}{4\pi i R(\zeta)}\zeta^2\,d\zeta\end{aligned} \tag{7.26}$$

where C_ρ is a negatively oriented circle of radius ρ and $\rho\to\infty$.

Since

$$\begin{aligned}R^{-1}(\zeta) &= -\zeta^{-3}\left[1+\zeta^{-1}\sum_{j=0}^{2}a_{2j}+\frac{1}{2}\zeta^{-2}\left(\sum_{i\neq k}(a_{2i}+a_{2k})^2-\sum_{j=0}^{2}b_{2j}^2\right)+\cdots\right],\end{aligned} \tag{7.27}$$

the first integral in (7.26) is equal to

$$\begin{aligned}-\frac{1}{2}t\left[\sum_{i\neq k}(a_{2i}+a_{2k})^2-\sum_{j=0}^{2}b_{2j}^2\right]-\frac{1}{2}x\sum_{j=0}^{2}a_{2j} \\ +\frac{\mu}{4}(\ln 2+1)-\frac{1}{2}T\tanh\frac{T}{\mu/2}+\frac{\pi}{4}\varepsilon.\end{aligned} \tag{7.28}$$

The second integral in (7.26) is

$$-\frac{\mu}{2}\oint_{C_\rho}\frac{\ln z\,dz}{4\pi i z}=\frac{\mu}{4}\ln\rho. \tag{7.29}$$

We then notice that

$$\operatorname{Im} f(z)=\frac{\pi}{2}\left(\frac{\mu}{2}-|\tilde{\zeta}|\right)\operatorname{sign}\left(\frac{\mu}{2}-\zeta\right) \tag{7.30}$$

where $\tilde{\zeta}=\max\{\zeta,T\}$ if $\zeta\geq 0$ and $\tilde{\zeta}=\min\{\zeta,-T\}$ if $\zeta<0$. So, the integral along $\mathbb{R}$ in (7.26) becomes

$$-\frac{1}{4}\left[\int_{|\zeta|>T}\frac{(\mu/2)\zeta^2-\zeta^3\operatorname{sign}\zeta+|R(\zeta)|}{|R(\zeta)|}\,d\zeta+\int_{|\zeta|\leq T}\frac{\mu/2-T}{|R(\zeta)|}\zeta^2\,d\zeta\right]. \tag{7.31}$$

According to Proposition 4.3 and Theorem 4.1, the contribution of the branch point α_2 to (7.26) is

$$-\frac{1}{4}\left(\frac{\mu}{2}-|\tilde{a}_2|\right)\frac{-2a_2^2\ln b_2}{(a_0-a_2)(a_2-a_4)}=-\frac{\xi^2\ln t}{64}\cdot\frac{\mu/2-(\hat{\xi}/4)}{\mu^2/4-\xi^2/16}, \tag{7.32}$$

whereas contributions from the remaining branch points α_0 and α_4 are of order $O(1)$. There is also a contribution in (7.31) coming from

$$-\frac{1}{4}\int\limits_{\rho>|\zeta|>\mu/2+\delta}\frac{\mu/2\zeta^2}{|R(\zeta)|}\,d\zeta=-\frac{\mu}{4}\ln\rho+O(1)$$

that cancels (7.29). Now the statement of the lemma follows from (7.32), (7.28), and Theorem 4.1. □

LEMMA 7.4 *If $\mu\geq 2$ and $\xi\in[0,2\mu)$, then*

$$2g(\infty)+\Omega_1-W_1=\frac{t\xi^2}{8}+\frac{1}{2}\ln t\left(\frac{\mu}{2}-\left(\frac{\hat{\xi}}{4}\right)\right) \tag{7.33}$$

as $t\to\infty$, $x=\xi t$, uniformly in μ and ξ.

PROOF: It remains to calculate

$$2K=\frac{1}{\pi i}\left[\int\limits_{\gamma_m}\frac{W_1\zeta^2\,d\zeta}{R_+(\zeta)}+\int_{\gamma_c}\frac{\Omega_1\zeta^2\,d\zeta}{R(\zeta)}\right]$$

to obtain (7.33). Deforming contours of integration $\gamma_m^\pm$ and $\gamma_c^\pm$ as in Section 6, we see that

$$\int\limits_{\gamma_m}\frac{\zeta^2\,d\zeta}{R_+(\zeta)}=\left\{\int\limits_{\tilde{\nu}_2}+\int\limits_{\tilde{\nu}_1}\right\}\frac{\zeta^2\,d\zeta}{R_+(\zeta)},\quad\int\limits_{\gamma_c}\frac{\zeta^2\,d\zeta}{R_+(\zeta)}=-\left\{\int\limits_{\tilde{\nu}_1}+\int\limits_{\tilde{\nu}_0}\right\}\frac{\zeta^2\,d\zeta}{R_+(\zeta)}. \tag{7.34}$$

Since

$$R_+(\zeta)=\frac{1+M_j(\zeta)(\zeta-a_{2j})}{|a_{2j}-a_{2k}|\,|a_{2j}-a_{2i}|}$$

for any $j=0,1,2$, where the function $M_j(\zeta)$ is uniformly bounded on $\tilde{\nu}_j$, we have

$$\begin{aligned}\frac{(-1)^j}{2\pi i}\int\limits_{\tilde{\nu}_j}\frac{\zeta^2\,d\zeta}{R_+(\zeta)}&\\ &=\frac{1}{2\pi i|a_{2j}-a_{2k}|\,|a_{2j}-a_{2i}|}\int_{\alpha_{2j+1}}^{\alpha_{2j}}\frac{\zeta^2(1+M_j(\zeta)(\zeta-a_{2j}))d\zeta}{\sqrt{(\zeta-\alpha_{2j})(\zeta-\alpha_{2j+1})}}\end{aligned} \tag{7.35}$$

as $t\to\infty$, where (i,j,k) is a permutation of $(1,2,3)$. The change of variable $\zeta=a_{2j}+iy$ yields

$$\frac{1}{2\pi i}\int_{\alpha_{2j+1}}^{\alpha_{2j}}\frac{\zeta^2\,d\zeta}{\sqrt{(\zeta-\alpha_{2j})(\zeta-\alpha_{2j+1})}}=\frac{a_{2j}^2}{2}+O(b_{2j}^2). \tag{7.36}$$

Then

$$
\begin{aligned}
2K = {} & \frac{W_1}{(a_2 - a_4)}\left[\frac{a_4^2}{a_0 - a_4} - \frac{a_2^2}{a_0 - a_2}\right] \\
& - \frac{\Omega_1}{(a_0 - a_2)}\left[\frac{a_0^2}{a_0 - a_4} - \frac{a_2^2}{a_2 - a_4}\right] + O(1)
\end{aligned} \tag{7.37}
$$

as $t \to \infty$. Using the fact that a_0 and a_4 are exponentially close to $\pm\mu/2$, respectively, we obtain

$$
2K = (W_1 - \Omega_1) - \frac{\mu}{4}\left[\frac{W_1}{\mu/2 - a_2} - \frac{\Omega_1}{\mu/2 + a_2}\right] + O(1). \tag{7.38}
$$

Calculation of the expression in the square brackets yields

$$
-2t\mu - \frac{1}{2}\ln t\left(\frac{\mu}{2} - \left(\frac{\hat{\xi}}{4}\right)\right)\frac{\mu}{\mu^2/4 - \xi^2/16}. \tag{7.39}
$$

Now the statement of the lemma follows from (6.3), (7.24), (7.38), and (7.39). □

Theorem 6.6, together with Lemma 7.4, yields the following result:

THEOREM 7.5 *If $\mu \geq 2$ and $\xi \in [0, 2\mu)$, then the leading-order behavior of the solution $q(x,t,\varepsilon)$ to* (1.1)–(1.2) *as $\varepsilon \to 0$, $t \to \infty$, and $\varepsilon = o(b_0^2\sqrt{|\ln b_0|})$ along the ray $x = \xi t$ is given by*

$$
\begin{aligned}
& q(x,t,\varepsilon) \\
& = -\sqrt{\frac{\mu/2 - (\widehat{\xi/4})}{2t}}\, e^{(i/\varepsilon)(t\xi^2/4 + \ln t[\mu/2 - (\widetilde{\xi/4})])(1+O(t^{-1}))}(1 + O(t^{-1/2})) \\
& \quad + O\left(\frac{\varepsilon}{b_0^2|\ln b_0|}\right),
\end{aligned} \tag{7.40}
$$

where $\ln b_0 = -4t(\mu/2 + \xi/4) + O(1)$. Here $\hat{c}$ denotes c if $c > T$ or T otherwise for any $c \geq 0$.

In fact, the RHP $(M', V_1, \tilde{\nu}_1)$ from Section 6 can be viewed as corresponding to a genus 0 situation. Taking into account Theorem 4.1 and $\xi = x/t$, we obtain $\frac{d}{dx}[2g(\infty) + \Omega_1 - W_1] = -a_2(x,t)$. Thus, we recover the genus 0 region formula

$$
q(x,t,\varepsilon) \sim -b_2(x,t)e^{-2(i/\varepsilon)\int_0^x a_2(s,t)ds} \tag{7.41}
$$

for the leading-order behavior of the solution to (1.1)–(1.2) under the conditions of Theorem 7.5 (see theorem 1.1 in [13]).

Formula (7.40) is not valid on the breaking curve $t_0(x)$ since $t_0(x)$ is asymptotic to $x = 2\mu t$. Considered formally, (7.40) shows that the amplitude of $q(x,t,\varepsilon)$ becomes 0 for $\xi = 2\mu$. However, since the breaking curve separates genus 0 and genus 2 regions of the (x,t)–plane, we can use the genus 0 expression for $q_0(x,t,\varepsilon)$,

$$q_0(x,t,\varepsilon) = b_0(x,t)e^{4(i/\varepsilon)g(\infty)} \tag{7.42}$$

[13, sec. 4.4]) to calculate the long-time behavior of $q_0(x,t,\varepsilon)$ along the rays $x = \xi t$ in the genus 0 region (so that $\xi \geq 2\mu$) and along the breaking curve t_0. The corresponding error estimates are not included in this paper, so, speaking rigorously, the calculations in the rest of this section are formal.

In the genus 0 region (including the breaking curve), the formula

$$g(\infty) = \frac{1}{2}\left[\frac{\mu}{2}\ln b + t(2a^2 - b^2) - T\tanh^{-1}\frac{2Ttb^2}{T^2+\mu tb^2} + \frac{1}{2}\varepsilon\pi\right] \tag{7.43}$$

where $\alpha_0 = a + ib$, is valid for any x and t [13, sec. 4.4]). We recall from section 4.1 in [13] that

$$b^2\sinh^2 u = a^2 - T^2\tanh^2 u, \qquad a = \left(\frac{\mu}{2} + 2tb^2\right)\tanh u, \qquad \text{where } u = (x + 4at). \tag{7.44}$$

From these equations we derive that

$$b = 2e^{-u} + O(e^{-2u}), \qquad a = \frac{\mu}{2} + O(te^{-2u}), \tag{7.45}$$

as $t \to \infty$. Thus

$$g(\infty) = \frac{1}{4}\left[-\mu t(\xi+\mu) + \mu\ln 2 - \varepsilon\pi\right] + O(te^{-2u}),$$

where $x = \xi t$. Substituting this and (7.45) into q_0, we obtain

$$q_0(x,t,\varepsilon) = -\exp\left\{-\left(1 + \frac{i\mu}{2\varepsilon}\right)[t(\xi+\mu) - \ln 2](1 + O(e^{-u}))\right\} \tag{7.46}$$

in the genus 0 region as $t \to \infty$. In the particular case of the breaking curve, the expression

$$\xi = 2\mu + t^{-1}\left[\ln\frac{2\mu}{\mu+2T} + \frac{2T}{\mu+2T}\right] + O\left(\frac{1}{t^2}\right) \tag{7.47}$$

follows from (5.13). Therefore, along the breaking curve we have

$$\begin{aligned} &q_0(x,t,\varepsilon) \\ &= -\exp\left\{-\left(1+\frac{i\mu}{2\varepsilon}\right)\left[3\mu t + \ln\frac{\mu}{2T+\mu} + \frac{2T}{2T+\mu}\right](1+O(t^{-1}))\right\} \end{aligned} \tag{7.48}$$

in the limit $t \to \infty$. Note that (7.48) is consistent with (6.2) from Theorem 6.1, since, on the breaking curve, $\sum_{j=0}^{2}(-1)^j b_{2j} = b_0$ and the theta functions in (6.2) became degenerate. Making formal comparison of (7.40) and (7.47) for $\xi = 2\mu$, one can observe the abrupt change of phase near the breaking curve.

8 Long-Time Behavior of Zero Level Curves of *h*

In Section 4 we have shown that no zero level curve Γ of $h(z)$ can cross vertical segments $\tilde{\nu}_j = [\alpha_{2j}, \alpha_{2j+1}]$, where $j = 0, 1, 2$. Now we can actually calculate the asymptotic location of Γ for $t \to \infty$. In particular, we will show that, except for small neighborhoods of points α_{2j}, the level curve Γ lies within $O(t^{-1})$ distance from the real axis $\operatorname{Im}\zeta = 0$ and the vertical ray $\operatorname{Re}\zeta = -\xi/4$ and approaches these lines asymptotically as $\zeta \to \infty$. These statements are direct consequences of the following lemma, which is based on long-time behavior of constants W_1 and Ω_1 (Theorem 7.2).

LEMMA 8.1 *If $\mu \geq 2$ and $\xi \in [0, 2\mu)$, then for any $\delta > 0$*
(8.1)

$$h(z) = t\Bigg\{\left(z + \frac{\xi}{4}\right)^2 + \left[4\left(\frac{\mu}{2} - \widehat{\left(\frac{\xi}{4}\right)}\right) - \kappa\left(\frac{\xi}{4}\right)\left(\frac{\mu}{2} - \frac{\xi}{4}\right)\right]\frac{z^2 \ln t}{8t(\mu^2/4 - \xi^2/16)} - \frac{(\mu/2 - \widehat{(\xi/4)})(3\mu^2/4 + \xi^2/16)\ln t}{4t(\mu^2/4 - \xi^2/16)} + \frac{\mu^2\kappa(\xi/4)\ln t}{32t(\mu/2 + \xi/4)}\Bigg\} + O(1)$$

as $t \to \infty$ uniformly on compact subsets of $\operatorname{Im}\zeta \geq 0$ *without δ-neighborhoods of branch points α_{2j}, $j = 0, 1, 2$. In particular,*

$$\operatorname{Im} h(z) = 2t\sigma\left[\left(1 + O\left(\frac{\ln t}{t}\right)\right)s + \frac{\xi}{4}\right] + O(1) \tag{8.2}$$

where $z = s + i\sigma$.

PROOF: As in Lemma 7.3, we represent $h(z)$ as

$$\begin{aligned}\frac{R(z)}{4\pi i}\int_{\hat{\gamma}}\frac{f(\zeta)d\zeta}{R(\zeta)} &= -\frac{R(z)}{2\pi}\int_{\mathbb{R}}\frac{\operatorname{Im} f(\zeta)}{(\zeta - z)R(\zeta)}\,d\zeta \\ &\quad + \frac{R(z)}{2\pi i}\left[\int_{\gamma_m}\frac{W_1\,d\zeta}{(\zeta - z)R_+(\zeta)} + \int_{\gamma_c}\frac{\Omega_1\,d\zeta}{(\zeta - z)R(\zeta)}\right]\end{aligned} \tag{8.3}$$

since residue at $\zeta = \infty$ is 0. It is easy to see that, according to Theorem 4.1, Proposition 7.1, and expression (7.26) for $\operatorname{Im} f$, the contribution of the first integral in (8.3) is

$$-\frac{\kappa(\xi/4)(z^2 - \mu^2/4)\ln t}{8(\mu/2 + \xi/4)}. \tag{8.4}$$

Therefore, we focus on the remaining part of the right-hand side of (8.3), which we denote by $M(z)$. Repeating the arguments from Lemma 7.4, we obtain

$$M(z) = \frac{R(z)}{2\pi i}\left[W_1\left\{\int_{\tilde{\nu}_2} + \int_{\tilde{\nu}_1}\right\}\frac{d\zeta}{(\zeta - z)R_+(\zeta)} - \Omega_1\left\{\int_{\tilde{\nu}_1} + \int_{\tilde{\nu}_0}\right\}\frac{d\zeta}{(\zeta - z)R_+(\zeta)}\right]. \tag{8.5}$$

Using the antiderivative (7.12) we can, similarly to Lemma 7.4, evaluate

$$\begin{aligned}(8.6)\quad &\frac{R(z)}{2\pi i}\int_{\alpha_{2j+1}}^{\alpha_{2j}} \frac{d\zeta}{(\zeta - z)\sqrt{(\zeta-\alpha_{2j})^2 + b_{2j}^2}} \\ &\qquad = \frac{-R(z)}{2|a_{2j}-a_{2k}|\,|a_{2j}-a_{2i}|\sqrt{(z-\alpha_{2j})^2+b_{2j}^2}} + O(b_{2j}^2).\end{aligned}$$

Then

$$\begin{aligned}(8.7)\quad 2M(z) = {} & \frac{(\Omega_1 - W_1)(z^2-\mu^2/4)}{(a_0-a_2)(a_2-a_4)} + \frac{W_1(z-\mu/2)\sqrt{(z-a_2)^2+b_2^2}}{(a_0-a_4)(a_2-a_4)} \\ & - \frac{\Omega_1(z+\mu/2)\sqrt{(z-a_2)^2+b_2^2}}{(a_0-a_2)(a_0-a_4)} + O(1).\end{aligned}$$

Direct calculation of (8.7), together with Theorem 4.1, yields

$$\begin{aligned}(8.8)\quad M(z) = t\Bigg[& \left(1 + \frac{(\mu/2 - (\xi\hat{/}4))\ln t}{2t(\mu^2/4 - \xi^2/16)}\right) z^2 + \frac{1}{2}\xi z - \left(\frac{3\mu^2}{4} + \frac{\xi^2}{16}\right) \\ & - \frac{(\mu/2-(\xi\hat{/}4))(3\mu^2/4+\xi^2/16)\ln t}{4t(\mu^2/4-\xi^2/16)}\Bigg] + O(1).\end{aligned}$$

The statement of the lemma follows now from (8.4) and (8.8). □

It is clear that for $t \to \infty$ and bounded z (8.2) implies either $\operatorname{Im} z = O(t^{-1})$ or $\operatorname{Re} z = -\xi/4 + O((\ln t)/t)$. So the branches of zero level curve Γ either approach the real axis or the vertical line $\operatorname{Re} z = -\xi/4$.

Appendix: Some Results of [13]

The linear eigenvalue problem corresponding to the integration of the NLS

$$(A.1)\qquad i\varepsilon W' = \begin{pmatrix} z & q \\ \bar{q} & -z \end{pmatrix} W,$$

where $q = q(x, 0, \varepsilon)$ is referred to as a potential and $z \in \mathbb{C}$ is a spectral parameter, was studied in [12]. In this paper the scattering coefficients a and b (see [15]) and the reflection coefficient $r^{(0)}(z) = b(z)/a(z)$, corresponding to (A.1), were found as products of gamma functions:

$$\begin{aligned}(A.2)\quad a(z) &= \frac{\Gamma(w)\Gamma(w-w_+-w_-)}{\Gamma(w-w_+)\Gamma(w-w_-)}, \\ b(z) &= -i\varepsilon 2^{-i\mu/e}\frac{\Gamma(w)\Gamma(1-w+w_++w_-)}{\Gamma(w_+)\Gamma(w_-)},\end{aligned}$$

and

$$r^{(0)}(z) = \frac{b(z)}{a(z)} = -i\varepsilon 2^{-i\mu/e} \frac{\Gamma(1-w+w_+ + w_-)\Gamma(w-w_+)\Gamma(w-w_-)}{\Gamma(w_+)\Gamma(w_-)\Gamma(w-w_+-w_-)} \tag{A.3}$$

where

$$w_+ = -\frac{i}{\varepsilon}\left(T + \frac{\mu}{2}\right), \quad w_- = \frac{i}{\varepsilon}\left(T - \frac{\mu}{2}\right), \quad w = -z\,\frac{i}{\varepsilon} - \mu\,\frac{i}{2\varepsilon} + \frac{1}{2}, \quad \text{and} \quad T = \sqrt{\frac{\mu^2}{4} - 1}. \tag{A.4}$$

In the theory of inverse scattering, the coefficient $a(z)$ is defined in the upper z half-plane while $b(z)$ and the reflection coefficient are defined on the real-z axis. In the case $0 \le \mu < 2$ the eigenvalue problem (A.1) contains points of discrete spectrum (zeroes of $a(z)$) at $z_k = T - i\varepsilon(k - \frac{1}{2})$ with the corresponding norming constants

$$c_k^{(0)} = \frac{b(z_k)}{a'(z_k)} = \mathrm{Res}_{z=z_k}\, r^{(0)}(z). \tag{A.5}$$

Here $k \in \mathbb{N}$ and $k < \frac{1}{2} + \frac{|T|}{\varepsilon}$. Because of the Schwartz reflection symmetry of the problem, it is sufficient to specify the discrete spectrum in the upper half-plane only.

The time evolution of the scattering data [15] is very simple and explicit; thus, the calculation of the evolution of the initial value problem (1.1)–(1.2) essentially consists of solving the inverse scattering problem (ISP), i.e., reconstructing the potential $q = q(x, t, \varepsilon)$ in (A.1) from the explicitly available scattering data at time t. The leading-order term $q_0(x, t, \varepsilon)$ (with respect to ε) of $q(x, t, \varepsilon)$ was calculated in [13]. The tools developed in that paper are in many cases sufficient for the calculation of the higher-order terms; however, we have not included such calculations.

The inverse scattering problem is formulated as a (multiplicative) matrix Riemann-Hilbert problem (RHP) on the complex plane of the spectral variable z. For any given x, t, our procedure reduces the construction of $q_0(x, t, \varepsilon)$ to the solution of a model RHP on a contour that consists of $2N + 1$ arcs $\{\gamma_{m,j}\}_{j=-N}^{N}$ (we refer to them as "main arcs") interlaced with $2N$ "complementary arcs" $\{\gamma_{c,j}\}$, $j = \pm 1, \pm 2, \dots, \pm N$. The arcs, as well as their endpoints α_j, $j = 0, 1, \dots, 4N + 1$, depend on x and t but not on ε. On each of these arcs, whose determination is an important part of our procedure, the 2×2 jump matrix of the RHP is constant with respect to z but depends on the parameters x, t, and ε.

A major part of the solution to the model RHP is a scalar function $g(z; x, t)$, which is analytic on the two-sheeted Riemann surface $\mathcal{R}(x, t)$ determined by the

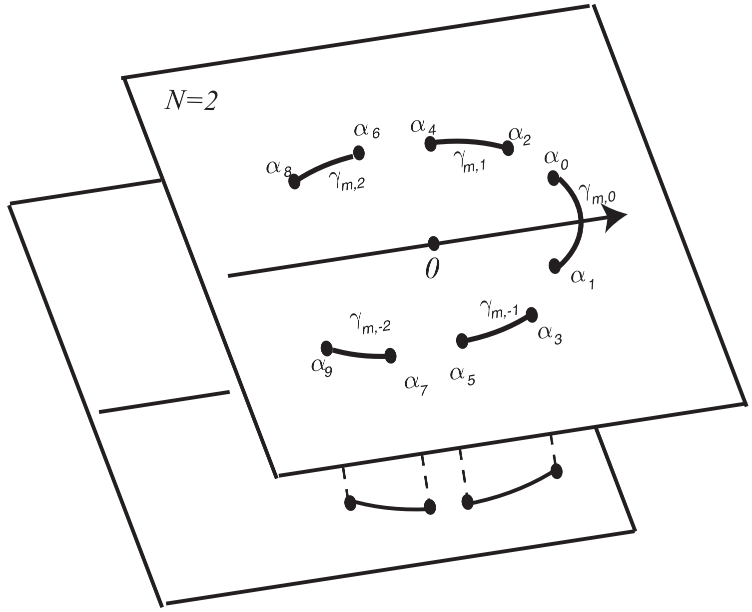

FIGURE A.1. Riemann surface $\mathcal{R}(x,t)$.

radical

$$R(z;x,t) = \prod_{j=0}^{4N+1} \sqrt{(z-\alpha_j)}$$

with branch cuts along the main arcs except at the points of nonanalyticity coming from the reflection coefficient $r^{(0)}(z)$. The solution of the model RHP is obtained explicitly through the dual basis of holomorphic differentials ω of $\mathcal{R}(x,t)$ and the corresponding Riemann theta function $\theta(u)$; see [13, sec. 7].

The expression for $q_0(x,t,\varepsilon)$ becomes somewhat simpler if we consider theta function θ and holomorphic differentials ω associated with the basic cycles of the Riemann surface $\tilde{\mathcal{R}}(x,t)$ (Figure 6.1) that consists of $2N+1$ vertical cuts $\tilde{\nu}_j$, connecting the corresponding (complex-conjugated) endpoints of the main arcs $\gamma_{m,j}$ and $\overline{\gamma}_{m,j}$, $j=0,1,\ldots,2N$; see section 8 in [13]. In a sense we study the evolution of $q_0(x,t,\varepsilon)$ through the evolution of $\mathcal{R}(x,t)$ (or of $\tilde{\mathcal{R}}(x,t)$), which identifies $q_0(x,t,\varepsilon)$ in a neighborhood of x,t as an N-phase NLS solution. The genus $2N$, $N=0,1,\ldots,$ of $\mathcal{R}(x,t)$ (or of $\tilde{\mathcal{R}}(x,t)$) is physically important because it specifies the number of oscillatory phases of the solution. By a mild abuse of terminology we call it the "genus of the solution $q_0(x,t,\varepsilon)$" or simply the "genus." A curve on the (x,t)–plane separating regions of different genera is called a "breaking curve."

The main result of [13] provides an explicit expression for $q_0(x,t,\varepsilon)$ in the cases of genera 0 and 2 when $\mu > 0$, $x \geq 0$, and $t \geq 0$ (due to a reflection symmetry with respect to the real axis, the genus must be even and $\overline{\gamma}_{m,j} = \gamma_{m,-j}$, $\overline{\gamma}_{c,j} = \gamma_{c,-j}$, $j = 1, 2, \dots, N$, $\overline{\gamma}_{m,0} = \gamma_{m,0}$). It follows from (1.1)–(1.2) that the solution $q(x, t, \varepsilon)$ is an even function in x for all t.

Certain important points concerning the results of the present paper are listed below:

- Evolution (with respect to t and x) of the branch points $\alpha_j(x, t)$ is governed by the system MI of moment and integral conditions, which contains the information about the initial condition (1.2) to (1.1) through the function $f^{(0)}(z) = (i/2e)r^{(0)}(z)$. In fact, $r^{(0)}(z)$ will later be replaced by its Stirling approximation $r(z)$ and, correspondingly, $f(z) = (i/2e)r(z)$.
- The branch points $\alpha_j(x, t)$ are crucial elements of the construction of the g-function $g(z; x, t)$, which is an essential element of the leading term $q_0(x, t, \varepsilon)$.
- In the region of genus 2 ($N = 1$, where the genus $g = 2N$) the system MI of $4N + 2$ real equations can be written as four moment conditions M_k and two integral conditions $I_{m,c}$,

$$\text{(A.6)} \quad \begin{aligned} &\frac{1}{\pi i}\int_{\hat{\gamma}} \frac{\zeta^k f'(\zeta)d\zeta}{R(\zeta)} = 0, \\ &\int_{\hat{\gamma}_{m,c}} h'(z)dz = \int_{\hat{\gamma}_{m,c}} R(z)dz \frac{1}{\pi i}\int_{\hat{\gamma}} \frac{f'(\zeta)d\zeta}{(\zeta - z)R(\zeta)} = 0, \end{aligned}$$

  where $k = 0, 1, 2, 3$, the contours $\hat{\gamma}_m = \hat{\gamma}_{m,1}$, $\hat{\gamma}_c = \hat{\gamma}_{c,1}$, and $R(z) = \sqrt{\prod_{j=0}^{5}(z - \alpha_j)}$.
- In general, for a given pair (x, t) we do not know a priori how to choose the correct N for the MI system, or even if such an N exists. Moreover, the system MI with a given N can have many solutions. Therefore, we took the evolutionary approach to solutions α of the MI system: we proved that for $t = 0$ the correct $N = 0$ and the correct solution $\alpha \in \mathbb{C}$ is the one satisfying $\operatorname{Re}\alpha \geq 0$, $0 < \operatorname{Im}\alpha \leq \sqrt{\mu + 2}$. The correct choice of N and α allows us to construct the g-function with the required properties.

  Using then the evolution and degeneracy theorems (see section 3.1 in [13]), we proved that the g-function with the required properties exists for any $x \geq 0$ and any $t \in [0, t_1(x)]$, where $t_1(x) \in (0, \infty]$. In the pure radiation case ($\mu \geq 2$) we proved that $t_1(x) = \infty$ for all x and that there exists a smooth, monotonically increasing curve $t = t_0(x)$, $x \geq 0$, that separates the genus 0 region (below the curve) from the genus 2 region (above the curve).

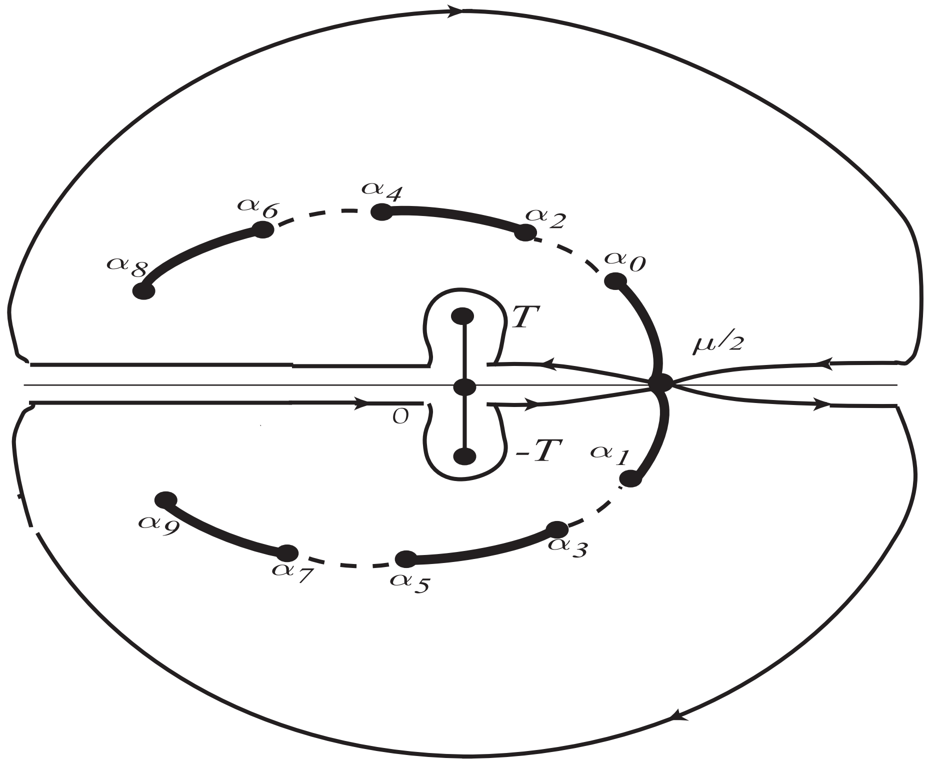

FIGURE A.2. Deformed contour $\hat{\gamma}$. In our (solitonless) case $T \in \mathbb{R}$, so there are no bypasses around $\pm T$.

To compute the integrals for moment conditions in (A.6), we use the following expression for $f'(z)$ [13, sec. 6.4]:

$$f'(z) = \frac{i\pi}{2} + \ln\frac{z}{z-\mu/2} + \frac{1}{2}\ln\left(1 - \frac{T^2}{z^2}\right) - x - 4tz = \frac{i\pi}{2} + \hat{f}(z), \tag{A.7}$$

where the logarithm terms have cuts along $[0, \mu/2]$ and $[-T, T]$, respectively. Note that $\hat{f}(z)$ is analytic at infinity, so for this term the contour of integration $\hat{\gamma}$ in (A.6) could be deformed into a union of integrals along $\mathbb{R}$ and a positively oriented circle of some large radius. The value of the latter integral could be computed through the residue at $z = \infty$; see Figure A.2. (Note that in the solitonless case $T \in \mathbb{R}$, so there are no bypasses around $\pm T$.) Because of the Schwarz reflection symmetry, the combined integrals over $\mathbb{R}$ yield

$$-\frac{1}{\pi}\int_{\mathbb{R}} \frac{\zeta^k \operatorname{Im} f'(\zeta)d\zeta}{R(\zeta)},$$

where $k = 0, 1, 2$ and

$$-\frac{1}{\pi}\int_{\mathbb{R}} \frac{[\zeta^3 \operatorname{Im} f'(\zeta) - R(\zeta)]d\zeta}{R(\zeta)}$$

for $k = 3$. Note that, according to (A.7), $\operatorname{Im} f'(\zeta) = \pi/2$ if $\zeta \le -T$ or $\zeta \ge \mu/2$, $\operatorname{Im} f'(\zeta) = -\pi/2$ if $T \le \zeta < \mu/2$, and $\operatorname{Im} f'(\zeta) = 0$ on $(-T, T)$. Thus, denoting $|R(\zeta)|$ by $R(\zeta)$ along $\mathbb{R}$ (i.e., replacing $R(\zeta)$ by $-R(z)$ for $\zeta > \mu/2$), we get $\operatorname{Im} f'(\zeta) = -(\pi/2)\operatorname{sign}\zeta$ if $|\zeta| \ge T$ and 0 otherwise. Therefore, for the solitonless case $\mu \ge 2$, the moment conditions (A.6) become

$$(\mathrm{M}_0) \qquad \int_{|\zeta|\ge T} \frac{\operatorname{sign}\zeta \, d\zeta}{|R(\zeta)|} = 0,$$

$$(\mathrm{M}_1) \qquad \int_{|\zeta|\ge T} \frac{\zeta \operatorname{sign}\zeta \, d\zeta}{|R(\zeta)|} = 8t,$$

$$(\mathrm{M}_2) \qquad \int_{|\zeta|\ge T} \frac{\zeta^2 \operatorname{sign}\zeta \, d\zeta}{|R(\zeta)|} = 2x + 8t\sum_{j=0}^{2} a_{2j},$$

$$(\mathrm{M}_3) \qquad \int_{|\zeta|\ge T} \frac{[\zeta^3 \operatorname{sign}\zeta - |R(\zeta)|]d\zeta}{|R(\zeta)|} = 2x\sum_{j=0}^{2} a_{2j} + 8t\,Q(\alpha) - \mu + 2T,$$

where $\alpha_j = a_j + ib_j$ and the quadratic form $Q(\alpha) = \frac{1}{2}\sum_{j<k}(a_{2j} + a_{2k})^2 - \frac{1}{2}\sum_{j=0}^{2} b_{2j}^2$. The integral in M_2 is a principal value integral.

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