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ON THE LOW-BURNETT-KROLL THEOREM FOR SOFT PHOTON EMISSION

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ABSTRACT

Burnett and Kroll's extension of Low's theorem is proved for particles of arbitrary spin.

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1. INTRODUCTION

It has long been known [1] that the amplitude for radiation of a soft photon is determined by the amplitude of the corresponding non-radiative process. Indeed the first two terms in an expansion in powers of the photon momentum are so determined. In the same approximation according to Burnett and Kroll [2], the radiative intensity is determined by the non-radiative intensity in the case of unpolarized initial particles and unobserved final spins. They derived this result for the case of particles of spin 0 and $\frac{1}{2}$, and conjectured its general validity. That conjecture is verified in this note.

2. AMPLITUDE

We work in the space of physical spin states $|\vec{p},s\rangle$, identified for example [3] by putting them in correspondence with rest states $|0,s\rangle$ via rotation-free Lorentz transformations. For the rest states, s is just the component of angular momentum along some chosen axis. The norm

$$\langle \vec{p}\,' s' | \vec{p}\, s \rangle = (2\pi)^3\, \delta(\vec{p}-\vec{p}\,')\, \delta_{ss'}$$

will be employed. Actually spin indices will usually be suppressed in what follows: the various transition amplitudes are then matrices in a spin space of appropriate dimension, $(2S+1)^2$ for each particle of spin S.

Let $\epsilon_\mu M_\mu$ be the radiative amplitude, where ϵ_μ is the photon polarization. We write

$$M_\mu = P_\mu + R_\mu \qquad (1)$$

where P_μ gives correctly the one-particle poles. It is assumed that these are the only relevant singularities so that the remainder R_μ is not singular for small photon momentum k_μ. Let the initial particle momenta be $\vec{q}_1, \vec{q}_2, \ldots,$ and the final momenta $\vec{p}_1, \vec{p}_2, \ldots,$ Then we take

$$P_\mu = \sum_n \frac{\langle \vec{p}_n | J_\mu | \vec{p}_n + \vec{k} \rangle \langle \vec{p}_1 \cdots \vec{p}_n + \vec{k} \cdots | M | \vec{q}_1 \vec{q}_2 \cdots \rangle}{\Delta_n} \tag{2}$$

$$+ \; - - - -$$

The terms given explicitly correspond to radiation from a final line; the dots indicate similar terms from initial lines. The quantity $\langle \vec{p}_n | J_\mu | \vec{p}_n + \vec{k} \rangle$ is the matrix element of the electromagnetic current between the indicated states of particle n, and $\langle \; | M | \; \rangle$ is the matrix element for the non-radiative process. The energy denominator is

$$\Delta_n = k_0 + E_n - E'_n \tag{3}$$

with

$$E_n = \left(m_n^2 + \vec{p}_n^{\,2} \right)^{1/2}$$
$$E'_n = \left(m_n^2 + (\vec{p}_n + \vec{k})^2 \right)^{1/2} \tag{4}$$

Now the gauge invariance requirement $k_\mu M_\mu = 0$ gives

$$k_\mu R_\mu = - k_\mu P_\mu$$

Calculating the right-hand side from (2), using the consequence of current continuity

$$\vec{k} \langle \vec{p}_n | \vec{J} | \vec{p}_n + \vec{k} \rangle = (E'_n - E_n) \langle \vec{p}_n | J_0 | \vec{p}_n + \vec{k} \rangle \tag{5}$$

gives

$$k_\mu R_\mu = \sum_n \langle \vec{p}_n | J_0 | \vec{p}_n + \vec{k} \rangle \langle \vec{p}_1 \cdots \vec{p}_n + \vec{k} \cdots | M | \vec{q}_1 \vec{q}_2 \cdots \rangle + \cdots\cdots \qquad (6)$$

If M is continued in some smooth way off the momentum conservation shell (we come back to this point) we can write

$$\langle \vec{p}_1 \cdots \vec{p}_n + \vec{k} \cdots | M | \vec{q}_1 \vec{q}_2 \cdots \rangle = \left(1 + \vec{k} \frac{\partial}{\partial \vec{p}_n}\right) \langle \vec{p}_1 \cdots \vec{p}_n \cdots | M | \vec{q}_1 \vec{q}_2 \cdots \rangle \qquad (7)$$

to first order in $\vec{k}$. We also have to first order

$$\langle \vec{p}_n | J_0 | \vec{p}_n + \vec{k} \rangle = Q_n - i \vec{k} \vec{\Lambda}_n (2\vec{p}_n + \vec{k}) \qquad (8)$$

where $\Lambda_n(2\vec{p}_n + \vec{k})$ is some Hermitian matrix with respect to spin indices, a function of the indicated argument, and Q_n is just the particle charge multiplied by the unit matrix. Thus to first order

$$k_\mu R_\mu = \sum_n \left(Q_n \vec{k} \cdot \frac{\partial}{\partial \vec{p}_n} - i \vec{k} \Lambda_n \right) M + \cdots\cdots \qquad (9)$$

where the arguments of M are the undisplaced $\vec{p}_1$ and $\vec{q}_1$. The possible term of zero order has cancelled in (9) because of charge conservation

$$\sum_n Q_n + \cdots = 0$$

The equation (9) determines R_μ in zero order, to be the coefficient of k_μ on the right-hand side : $R_0 = 0$, and

$$\vec{R} = \sum_n \left(Q_n \frac{\partial}{\partial \vec{p}_n} - i \vec{\Lambda}_n \right) M + \cdots \tag{10}$$

To expand P_μ also to zero order requires, as well as (7) and (8), the expansion

$$(E_n + E_{n'}) \langle \vec{p}_n | \vec{J} | \vec{p}_n + \vec{k} \rangle = (2\vec{p}_n + \vec{k}) Q_n - i \vec{k} \overleftrightarrow{\Sigma}_n \tag{11}$$

where the dyadic $\overleftrightarrow{\Sigma}$, a function of $2\vec{p}+\vec{k}$, is a Hermitian matrix with respect to spin indices. Finally, to zero order, in a gauge with $\epsilon_0 = 0$,

$$\epsilon_\mu M_\mu = \sum_n \Big\{ \frac{Q_n}{\Delta_n} \vec{\epsilon} \cdot \frac{2\vec{p}_n + \vec{k}}{E_n + E_n'} \left(1 + \vec{k} \cdot \frac{\partial}{\partial \vec{p}_n} \right) - i \frac{\vec{k} \cdot \overleftrightarrow{\Sigma}_n \cdot \vec{\epsilon}}{\Delta_n (E_n + E_n')} + Q_n \vec{\epsilon} \frac{\partial}{\partial \vec{p}_n} - i \vec{\epsilon} \vec{\Lambda}_n \Big\} M + \cdots \tag{12}$$

As already remarked, this formula requires an extension of M off the momentum conservation shell $\sum \vec{p} = \sum \vec{q}$. To check that the result does not depend on the precise way this is done, note that the difference in two such extrapolations will be of the form

$$\vec{F} \cdot (\Sigma \vec{p} - \Sigma \vec{q})$$

where $\vec{F}$ is some non-singular function of the $\vec{p}$'s and $\vec{q}$'s. The additional contribution to (12) is, up to zero order terms,

$$\sum_n \Big\{ \frac{Q_n}{\Delta_n} \vec{\epsilon} \cdot \frac{2\vec{p}_n + \vec{k}}{E_n + E_n'} \left(\vec{F} (\Sigma \vec{p} - \Sigma \vec{q}) + \vec{k} \cdot \vec{F} \right) + Q_n \vec{\epsilon} \cdot \vec{F} \Big\} + \cdots$$

This is seen to vanish on recalling

$$\vec{k} + \Sigma \vec{p} - \Sigma \vec{q} = 0$$

$$\Sigma\, Q_n + \ldots\ldots = 0$$

The quantities $\overleftrightarrow{\Lambda}$ and $\overleftrightarrow{\Sigma}$ in (12), defined by (8) and (11), can of course be expressed in terms of the particle charges and magnetic moments. Although we do not need them in what follows, we exhibit here the explicit expressions. Dropping the index n, for a particle of mass M, charge Q, and gyromagnetic ratio g :

$$\vec{k}\cdot\overleftrightarrow{\Sigma} = \frac{g}{2}\left(\vec{\sigma}\times\vec{k} + \frac{E-E'}{E+M}\,\vec{\sigma}\times\vec{\bar{p}}\right) + \left(\frac{g}{2}-Q\right)\frac{\vec{\bar{p}}}{M}\;\frac{\vec{\sigma}\cdot\vec{k}\times\vec{\bar{p}}}{\bar{E}+M}$$

$$\overleftrightarrow{\Lambda} = \left(\frac{Q}{2M(\bar{E}+M)} - \frac{g}{2M\bar{E}}\right)\vec{\sigma}\times\vec{\bar{p}}$$

where

$$\vec{\bar{p}} = \vec{p} + \tfrac{1}{2}\vec{k} \qquad\qquad \bar{E} = \tfrac{1}{2}(E+E')$$

and $\vec{\sigma}$ is the generalization of appropriate dimension of the Pauli spin matrices (i.e., twice the angular momentum matrix of the rest system). These expressions are obtained by boosting from the Breit system, $\vec{\bar{p}} = 0$; the terms in Q arise from the Thomas precession, or Wigner rotation.

3. TRANSITION PROBABILITY

If final particle spins are not observed and if initial particles are unpolarized, the transition probability is proportional to

$$\mathrm{Trace}\quad \epsilon^*_\mu\, M^+_\mu\, \epsilon_\nu\, M_\nu \tag{13}$$

where the Trace is with respect to spin indices and M^+ is the Hermitian conjugate of M with respect to those indices. Then from (12), to the leading and next highest order in k, (13) is equal to

$$\left(\sum_n \sum_m \frac{Q_n}{\Delta_n}\frac{Q_m}{\Delta_m}\ \vec{\epsilon}^{\,*}.\frac{2\vec{p}_n+\vec{K}}{E_n+E'_n}\ \vec{\epsilon}.\frac{2\vec{p}_m+\vec{K}}{E_m+E'_m}\right)\mathrm{Trace}\ M^{+}M$$

$$+\ 2\ \mathrm{Real\ Trace}\ M^{+}\sum_n \frac{Q_n}{\Delta_n}\ \vec{\epsilon}^{\,*}.\frac{2\vec{p}_n+\vec{K}}{E_n+E'_n}$$

$$\sum_m\left(\frac{Q_m}{\Delta_m}\ \vec{\epsilon}.\frac{2\vec{p}_m+\vec{K}}{E_m+E'_m}\ \vec{K}.\frac{\partial}{\partial\vec{p}_m} + Q_m\ \vec{\epsilon}.\frac{\partial}{\partial\vec{p}_m} - i\,\frac{\vec{K}.\overleftrightarrow{\Sigma}_m.\vec{\epsilon}}{\Delta_m(E_m+E'_m)}\right.$$

$$\left.-\,i\,\vec{\epsilon}.\vec{\Lambda}_m\right)M \qquad + \ -----$$

If we now take $\vec{\epsilon}$ real, the expression simplifies considerably. Firstly the $\overleftrightarrow{\Sigma}$ and $\vec{\Lambda}$ contributions disappear, for because of the Hermiticity of these matrices

$$\mathrm{Real\ Trace}\ M^{+}\,i\,\Sigma_n M = 0$$

$$\mathrm{Real\ Trace}\ M^{+}\,i\,\Lambda_n M = 0$$

Secondly we can use

$$2\ \mathrm{Real\ Trace}\ M^{+}\frac{\partial}{\partial\vec{p}_n}M = \frac{\partial}{\partial\vec{p}_n}M^{+}M$$

to obtain the final form

$$\left\{\sum_n \frac{Q_n}{\Delta_n}\ \vec{\epsilon}.\frac{2\vec{p}_n+\vec{K}}{E_n+E'_n}\sum_m\left[\frac{Q_m}{\Delta_m}\ \vec{\epsilon}.\frac{2\vec{p}_m+\vec{K}}{E_m+E'_m}\left(1+\vec{K}.\frac{\partial}{\partial\vec{p}_m}\right)\right.\right.$$

$$\left.\left.+\ Q_m\ \vec{\epsilon}.\frac{\partial}{\partial\vec{p}_m}\right]\right\}P \qquad + \ ----- \qquad (14)$$

where

$$P = \mathrm{Trace}\ M^{+}M$$

is essentially the non-radiative transition probability. In these formulae we require P outside the momentum conserving shell $\sum\vec{p} = \sum\vec{q}$. By a reasoning similar to that at the end of Section 2, different smooth continuations do not give significantly different results.

Formula (14) is the desired result, expressing the radiative probability in terms of the non-radiative probability and derivatives thereof. The restriction to real $\vec{\epsilon}$ means that we can deal with transverse photon polarization, or of course with summation over photon polarizations. The result applies not only to real photons, but also to appropriately polarized soft virtual photons, for we did not use either $k^2 = 0$ or $\epsilon_\mu k_\mu = 0$.

4. COVARIANT NORMALIZATION

Although it is not essential for our development, already complete, we will rewrite (12) in a form closer to that usually given for scalar particles. Firstly, we introduce "covariantly" normalized amplitudes $\mathcal{M}$ and $\mathcal{M}_\mu$ such that

$$\frac{\mathcal{M}}{M} = \frac{\mathcal{M}_\mu}{M_\mu} = \prod_n 2^{1/2}\left(M_n^2 + \vec{p}_n^{\,2}\right)^{1/4} \prod_m 2^{1/2}\left(M_m^2 + \vec{q}_m^{\,2}\right)^{1/4}$$

Secondly, we introduce new denominators

$$\begin{aligned}\delta_n &= -\left[M_n^2 + (\vec{p}_n + \vec{k})^2\right] \\ &= (E_n + k_0)^2 - E_n'^2 \\ &= (k_0 + E_n - E_n')(k_0 + E_n + E_n') \\ &= \Delta_n (E_n + E_n')\left(1 + \frac{k_0}{E_n + E_n'}\right)\end{aligned} \qquad (15)$$

The extra terms generated by commuting the normalization factors through the differential operators cancel neatly against those arising from the last term in (13), leaving in zero order

$$\epsilon_\mu \mathcal{M}_\mu = \sum_n \left\{ \frac{Q_n}{\delta_n} \vec{\epsilon}.(2\vec{p}_n + \vec{k})\left(1 + k_\mu \frac{\partial}{\partial p_\mu}\right) - i\,\frac{\vec{k}\,\overleftrightarrow{\Sigma}_n\,\vec{\epsilon}}{\delta_n} + Q_n \vec{\epsilon}.\frac{\partial}{\partial \vec{p}_n} - i\,\vec{\epsilon}.\vec{\Lambda}_n \right\} \mathcal{M} \qquad (16)$$

$$+ \;-\;-\;-\;-$$

In the first instance one obtains in (14) $\vec{k}.\partial/\partial\vec{p}$ rather than $k_\mu\,\partial/\partial p_\mu$ in the first term; these forms are equivalent when $\mathcal{M}$ is a function only of $\vec{p}$ as implied hitherto. With the form written, one can in fact continue $\mathcal{M}$ off the mass shell in any other smooth way without significant change. This is because if one makes an addition to $\mathcal{M}$ of the form

$$\sum_n \lambda_n \left(M_n^2 + p_n^2 \right)$$

where the λ_n are some smooth functions, the extra zero order terms are

$$\sum_n \frac{Q_n}{\delta_n} \vec{\epsilon}.2\vec{p}_n\;\lambda_n 2 k_\mu p_{n\mu} + \sum Q_n 2\vec{\epsilon}.\vec{p}_n$$

which cancel because $\delta_n \approx -2k_\mu p_{n\mu}$.

In the same way, (14) can be rewritten

$$\mathrm{Trace}\;\epsilon_\mu M_\mu^+ \epsilon_\nu M_\nu =$$

$$\left\{ \sum_n \frac{Q_n}{\delta_n} \epsilon_\mu (2p_n + k)_\mu \sum_m \frac{Q_m}{\delta_m} \epsilon_\nu (2p_n + k)_\nu \left(1 + k_\mu \frac{\partial}{\partial p_\mu}\right) + Q_m \epsilon_\mu \frac{\partial}{\partial p_\mu} \right\} \mathcal{P} \qquad (17)$$

$$+ \;-\;-\;-$$

where

$$\mathcal{P} = \mathrm{Trace}\, \mathcal{M}^{+} \mathcal{M}$$

In the first instance we obtain this in the gauge $\epsilon_0 = 0$. However, this restriction can be dropped, for the form written, is actually invariant under the gauge transformation $\epsilon_\mu \to \epsilon_\mu + \lambda k_\mu$. This is readily verified remembering

$$k_\mu (2p_n + k)_\mu = -\delta_n$$

$$\sum Q_n + \cdots = 0$$

One also verifies in the usual way the insensitivity of (17) to how $\mathcal{P}$ is continued off the mass and energy momentum conservation shells. As usual the dots in (17) indicate extra terms arising from the particles in the initial state; they can be omitted by formally considering initial particles as final antiparticles with reversed charge and four-momentum.

Note finally the manifest covariance of (17). We did not bother with manifest covariance in the derivation, and especially in the decomposition (1)-(2); of course this does not prevent us reaching right answers if the subsequent reasoning is correct.

We are indebted to R. Stora for useful discussion.

REFERENCES

1) F.E. Low, Phys.Rev. 110, 974 (1958).
For a more compact treatment, see :
S.L. Adler and Y. Dothan, Phys.Rev. 151, 1267 (1966).

2) T.H. Burnett and N.M. Kroll, Phys.Rev.Letters 20, 86 (1968).

3) If one wishes to include some hard photons among the "particles" of the "non-radiative" process, one can use, say, a helicity labelling. In the lowest order of α for the given process R_{μ} should still be non-singular and the demonstration goes through.