# ON THE LOWER BOUND FOR THE INJECTIVITY RADIUS OF 1/4-PINCHED RIEMANNIAN MANIFOLDS 

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The purpose of this paper is to prove that if $M$ is a compact simply connected riemannian manifold whose sectional curvature $K$ satisfies $1 / 4 \leqslant$ $K \leqslant 1$, then every closed geodesic has length $\geqslant 2 \pi$, or equivalently, the injectivity radius of the exponential map is $\geqslant \pi$.

This result, which for example, is necessary for the proof of Berger's rigidity theorem [1], [2], is well known in the even dimensional case (assuming only $0<K \leqslant 1$ ), and in the odd dimensional case [6] provided one assumes $1 / 4<K \leqslant 1$. The case $1 / 4 \leqslant K \leqslant 1$ is treated in [7], but the argument seems to be somewhat unclear at several points. Our proof is in spirit a modification of the ideas in [7].

We will use some fairly general standard facts about lifting curves by the exponential map exp: $T M \rightarrow M$. Our basic references are [2] and [4]. For $m \in M, \exp _{m}$ is the restriction of exp to the tangent space $M_{m}$ of $M$ at $m$. All curves under consideration will be continuous and parametrized on [ 0,1$]$. Let $P$ denote the set of all curves in $M$, and $\tilde{P}$ the set of vertical curves $\tilde{c}$ in $T M$ emanating from the zero section, i.e., $c=\pi \circ \tilde{c}$ is constant, $\tilde{c}(0)=0 \in M_{c(0)}$, where $\pi: T M \rightarrow M$ is the natural projection. Throughout this paper, we will use the topology of uniform convergence for $P, \tilde{P}$, and any subset.

Let Exp: $\tilde{P} \rightarrow P$ denote the continuous map induced by $\exp , \operatorname{Exp}(\tilde{c})=$ $\exp \circ \tilde{c}$. The image of $\operatorname{Exp}$ is the set of liftable curves; any $\tilde{c} \in \tilde{P}$ is a lift of $c=\operatorname{Exp}(\tilde{c})$. We call $\tilde{c}$ a regular lift if $\exp _{m}$ has maximal $\operatorname{rank} n=\operatorname{dim} M$ at all points $\tilde{c}(t), 0 \leqslant t \leqslant 1, m=\pi \circ \tilde{c}(0)$, or equivalently, $\pi \times \exp$ has maximal rank $2 n$ at all $\tilde{c}(t)$. Clearly, any $c \in P$ has a unique lift if it has a regular lift, but in general need not have a regular lift or any lift at all. All regular lifts form an open subset $\tilde{Q} \subset \tilde{P}$, and Exp imbeds $\tilde{Q}$ homeomorphically onto an open subset $Q \subset P$. We are particularly interested in the set of closed curves $P_{0} \subset P$. Lifts of closed curves, if they exist, need not be closed. $P_{0}$ certainly

[^0]contains the Exp-image of the closed curves $\tilde{P}_{0}$ in $\tilde{P}$. Finally, let $\tilde{Q}_{0}=\tilde{P}_{0} \cap$ $\tilde{Q}$ denote the set of closed regular lifts, which is open in $\tilde{P}_{0}$. It follows from the above that Exp maps $\tilde{Q}_{0}$ homeomorphically onto an open subset $Q_{0}$ of $P_{0}$. We conclude this discussion with two simple

Remarks. (1) Any geodesic $c \in P$ always has a canonical (radial) lift $\tilde{c}$, regular or not, which is never closed.
(2) Suppose for some $m \in M$ and $r>0, \exp _{m}$ is nonsingular on the open ball $U_{r}(m)=\{v\|v\|<r\} \subset M_{m}$. Let $c \in P$ be piecewise differentiable, $c(0)$ $=m$, of length $L(c)<r$. Then $c \in Q$, so $c$ has a unique regular lift $\tilde{c}=$ $\operatorname{Exp}^{-1}(c)$, and $\tilde{c}(t) \in U_{r}, 0 \leqslant t \leqslant 1$. In essence, this is a consequence of the Gauss lemma.

Since we have to employ some arguments from Morse theory, we will also work with the standard approximations of $P_{0}$ by the finite dimensional subspaces $\Omega^{<a}$ of all closed broken geodesics $c$ of energy $E(c) \leqslant a$, with break point parameters chosen as usual, sufficiently fine and fixed for an energy level $b \geqslant a$ large enough in a given situation. Let $\Omega^{<a}$ denote the open subset of all curves in $\Omega^{\leqslant a}$ whose energy is strictly less than $a$. Observe that $\Omega^{<a}$ need not be the closure of $\Omega^{<a}$. The critical points of the energy function $E$ on these free loop spaces are precisely smoothly closed geodesics in $M$.

We now prove a crucial fact about limits of closed lifts which is quite general. For any curve $c$ and $0 \leqslant s \leqslant 1$, let $c_{s}$ be the curve given by $c_{s}(t)=c(s t)$, and let $c^{-}$denote the reversed curve, $c^{-}(t)=c(1-t)$. By $c_{s}^{-}$we will mean $\left(c_{s}\right)^{-}$.

Lemma 1 (Lifting lemma). Suppose, for some $r>0$ and all $m \in M, \exp _{m}$ is nonsingular on the open ball $U_{r}(m) \subset M_{m}$. Let $c \in P_{0}$ be in the closure of $\operatorname{Exp}\left(\tilde{P}_{0}\right) \cap \Omega^{<4 r^{2}}$. Then either $c \in Q_{0}$ (and thus is not a closed geodesic if nonconstant), or both $c_{1 / 2}$ and $c_{1 / 2}^{-}$are geodesics of length $r$ with conjugate end points. Furthermore, $C=Q_{0} \cap \Omega^{<4 r^{2}}$ is a connected component of $\Omega^{<4 r^{2}}$.

Remarks. (3) $\Omega^{<4 r^{2}}$ is a manifold, so its connected components are precisely its pathwise connected components. But $\Omega^{\leqslant 4 r^{2}}$ need not be a manifold (with boundary) if $4 r^{2}$ is a critical value of the energy $E$. To avoid difficulties, we will mostly work with connected, rather than pathwise connected components.
(4) If we replace the assumption $\exp _{m} \mid U_{r}(m)$ nonsingular for all $m$ by the assumption $\exp _{c(0)} \mid U_{r}(c(0))$ is nonsingular and leave the rest of the hypothesis unchanged, then the conclusions of Lemma 1 continue to hold for the loop space at $c(0)$. But this is not sufficient for our purposes.
Proof. We can assume that at least one of the two branches $c_{1 / 2}, c_{1 / 2}^{-}$(say $c_{1 / 2}$ ) is not a geodesic, or if so, is free of conjugate points. Note that
$L\left(c_{1 / 2}\right) \leqslant \sqrt{E\left(c_{1 / 2}\right)} \leqslant r$, with equality iff $c_{1 / 2}$ is a geodesic of length $r$. Therefore, by Remarks 1 and $2, c_{1 / 2} \in Q$. But $Q$ is open, so in fact $c_{s} \in Q$ for some $1 / 2<s \leqslant 1$. Then, again by Remark $2, c_{1-s}^{-} \in Q$. By hypothesis, there exists a sequence $\tilde{c}_{i} \in \tilde{P}_{0}$ with $\operatorname{Exp}\left(\tilde{c}_{i}\right)=c_{i}$ converging to $c$. Now $(\operatorname{Exp} \mid \tilde{Q})^{-1}$ is well-defined and continuous, so it follows that $\operatorname{Exp}^{-1}\left(c_{i, s}\right)=\tilde{c}_{i, s}$ $\rightarrow \tilde{c}_{s}$, and $\operatorname{Exp}^{-1}\left(c_{i, 1-s}^{-}\right)=\tilde{c}_{i, 1-s}^{-} \rightarrow \tilde{c}_{1-s}^{-}$.

Since $\tilde{c}_{i, s}(1)=\tilde{c}_{i, 1-s}^{-}(1)$ for all $i$, we have $\tilde{c}_{s}(1)=\tilde{c}_{1-s}^{-}(1)$. Thus $\tilde{c}_{i}$ converges to some $\tilde{c} \in \tilde{Q}_{0}$, and $\operatorname{Exp}(\tilde{c})=c \in Q_{0} . C=Q_{0} \cap \Omega^{<4 r^{2}}$ is open, and we have just shown it is also closed in $\Omega^{<4 r^{2}}$. Therefore $C$ is a union of connected components of $\Omega^{<4 r^{2}}$. If $C_{1}$ is any component of $C$, then $C_{1}$ contains a closed geodesic on which $E$ takes its minimum value. Since $C$ contains no nontrival closed geodesics, it follows that $C=C_{1}$ is connected.

We need the following result from Morse theory.
Lemma 2. Let $f$ be a smooth function on the differentiable manifold $X$ of dimension $k$, and $p$ a possibly degenerate critical point of index $\geqslant 2$ (or a regular point) with $f(p)=a$. Then there exists a neighborhood $N$ of $p$ such that $N \cap X^{<a}$ is (pathwise) connected and dense in $N \cap X^{<a}$.

Proof. We may assume $X=\mathbf{R}^{k}, p=0, f(0)=a=0$. If 0 if a regular point, our claim is trivial. Otherwise, according to the generalized Morse Lemma in [5], we have, after a change of coordinates near the origin in $\mathbf{R}^{k}=\mathbf{R}^{2} \times \mathbf{R}^{k-2}$,

$$
f(x, y)=-\|x\|^{2}+g(y)
$$

on some neighborhood $U$, where $g$ is a smooth function in $\mathbf{R}^{k-2}$. Now choose $d, r>0$ such that $(x, y) \in U$ and $g(y)<d^{2}$ on

$$
N=\left\{(x, y) \mid\|x\|^{2} \leqslant d^{2},\|y\|^{2} \leqslant r^{2}\right\} .
$$

Let $q_{0}=\left(x_{0}, y_{0}\right), q_{1}=\left(x_{1}, y_{1}\right) \in N \cap X^{<0}$. It suffices to construct a continuous path $\tau:[0,1] \rightarrow N \cap X^{\leqslant 0}$ from $q_{0}$ to $q_{1}$ so that $\tau(t) \in N \cap X^{<0}$ whenever $0<t<1$. For $x \in \mathbf{R}^{2}$, let $h(x)=d x /\|x\|$ if $x \neq 0, h(0)=(d, 0)$. Notice that $f(h(x), y)<0$ for all $(x, y) \in N$. The path $\tau$ can be chosen as the composition of the following four simple curves: First, move ( $x_{0}, y_{0}$ ) linearly to $\left(h\left(x_{0}\right), y_{0}\right)$, then $\left(h\left(x_{0}\right), y_{0}\right)$ to $\left(h\left(x_{1}\right), y_{0}\right)$ through a rotation on the circle $\|x\|=d$ keeping $y_{0}$ fixed (here we are using index $\geqslant 2$ ), then ( $\left.h\left(x_{1}\right), y_{0}\right)$ linearly into $\left(h\left(x_{1}\right), y_{1}\right)$, and finally $\left(h\left(x_{1}\right), y_{1}\right)$ linearly to $\left(x_{1}, y_{1}\right)$.

Lemmas 1 and 2 have the following consequence.
Lemma 3. Assume the hypothesis of Lemma 1, and furthermore that any smoothly closed geodesic $c \in Q_{0} \cap \Omega^{<4 r^{2}}$ (necessarily of length $2 r$ ) has index $\geqslant$ 2. Then $\bar{Q}_{0} \cap \Omega^{\leqslant 4 r^{2}}$ is the closure of $Q_{0} \cap \Omega^{<4 r^{2}}$, and a connected component of $\Omega^{<4 r^{2}}$.

Proof. Let $p \in \bar{Q}_{0} \cap \Omega^{<4 r^{2}}$, and let $N$ be a neighborhood of $p$ in $X=\Omega^{<b}$ for some $b>4 r^{2}=a$ as in Lemma 2. We have $N \cap Q_{0} \cap \Omega^{<4 r^{2}} \neq \varnothing$. But $N \cap \Omega^{<4 r^{2}}$ is dense in $N \cap \Omega^{\leqslant 4 r^{2}}$ and $Q_{0}$ open, so that also $N \cap Q_{0} \cap \Omega^{<4 r^{2}}$ $\neq \varnothing$. Now $N \cap \Omega^{<4 r^{2}}$ is connected, and by the lifting lemma, $Q \cap \Omega^{<4 r^{2}}$ is a connected component of $\Omega^{<4 r^{2}}$. Therefore $N \cap \Omega^{<4 r^{2}} \subset Q_{0}$, and thus $N \cap$ $\Omega^{<4 r^{2}}$ is contained in the closure of $Q_{0} \cap \Omega^{<4 r^{2}}$. This implies that $\bar{Q}_{0} \cap \Omega^{<4 r^{2}}$ is the closure of the (connected) set $Q_{0} \cap \Omega^{<4 r^{2}}$, and is relatively open (and closed) in $\Omega^{\leqslant 4 r^{2}}$, which completes the argument.

The following fact is basically standard.
Lemma 4 (Connectedness lemma). Let $f$ be a smooth proper function on a finite dimensional manifold $X$. Suppose, for some regular value $b$, all critical points of $f$ in $X^{<b}-X^{\leqslant a}$ have index $\geqslant 2$ (but are possibly degenerate). Let $C_{1}, \ldots, C_{N}$ be the connected components of $X^{\leqslant b}$. Then $C_{1} \cap X^{<a}, \ldots, C_{N} \cap$ $X^{<a}$ are the connected components of $X^{<a}$. In particular, if $X^{\leqslant b}$ is connected, so is $X^{\leqslant a}$.

Remark. (5) If in addition, all critical points in $f^{-1}(a)$ have index $>2$, then $X^{<b}$ (path) connected implies that $X^{<a}$ is also path connected, since by Lemma $2, X^{\leqslant a}$ is locally path connected.

Proof. Choose a decreasing sequence $b>a_{k}>a, \lim _{k \rightarrow \infty} a_{k}=a$, of regular values of $f$. By standard Morse theory, we can approximate $f$ on $X^{<b}$ by a nondegenerate Morse function $f_{k}$ which agrees with $f$ on $X^{<a_{k}} \cup f^{-1}(b)$. If $f_{k}$ is sufficiently close to $f$, all critical points of $f_{k}$ in $X^{<b}-X^{<a_{k}}$ will have index $\geqslant 2$. Then it follows from the Morse inequalities that $H_{i}\left(X^{<b}, X^{<a_{k}}\right)$ $=0$ for $i=0$, 1 . Thus $C_{1} \cap X^{<a_{k}}, \cdots, C_{N} \cap X^{<a_{k}}$ are the path connected components of $X^{\leqslant a_{k}}$. Since the intersection of a decreasing sequence of compact connected sets is connected, the lemma is proved.

Finally, we need
Lemma 5 (Index lemma). Let $M$ be odd dimensional with sectional curvature $0<K \leqslant 1$. Then any nonconstant smoothly closed geodesic $c \in \bar{Q}_{0} \cap$ $\Omega^{\leqslant 4 \pi^{2}}$ (necessarily of length $2 \pi$ ) has ${ }^{1}$ index $\geqslant 2$.

Proof. Recall that by standard index form comparison, $K \leqslant 1$ implies that for any geodesic in $M$ of length $\pi$, at most the end points can be conjugate, and if this is the case, any Jacobi field vanishing at the end points looks like a Jacobi field on the Euclidean sphere of curvature 1, i.e., is a multiple of a parallel field. Therefore, in our situation, the hypothesis of Lemma 1 is satisfied for $r=\pi$, and the geodesic $c$ has precisely two conjugate points for $t=1 / 2$ and $t=1$, at length $\pi$ and $2 \pi$ respectively.

[^1]We can find broken Jacobi fields $J_{ \pm}$along $c$, not identically zero, where $J_{+}, J_{-}$are Jacobi on $[0,1 / 2],[1 / 2,1]$ respectively, and vanish otherwise. Now $J_{+}$and $J_{-}$span a 2-dimensional space $V$ on which the index form $I$ is zero. If $V$ does not intersect the null space $N$ of $I$ nontrivially, then the orthogonal projection of $V$ on the negative eigenspace of $I$ must be an injection which implies index $(c) \geqslant 2$. Otherwise, there is $0 \neq J \in V \cap N$. We are working in the free loop space, so $N$ consists of periodic Jacobi fields. By the above comparison argument, we have $J(t)=\sin 2 \pi t \cdot E(t)$, where $E$ is a closed parallel field along $c$. Using the Synge argument, since $M$ is odd dimensional and $K>0, M$ is orientable, and we conclude that there exists a second closed parallel field $F$ normal to $c$. Then $E$ and $F$ span a 2-dimensional space on which $I$ is negative definite.

We can now derive our main result.
Theorem 6. Let $M$ be a simply connected compact riemannian manifold of odd dimension $n$. Suppose the sectional curvature satisfies $1 / 4 \leqslant K \leqslant 1$. Then
(a) $\Omega^{<4 \pi^{2}}$ is the closure of $\Omega^{<4 \pi^{2}}$, and is connected;
(b) $\Omega^{<4 \pi^{2}}$ is contained in $Q_{0}$, and is (pathwise) connected;
(c) $c \in \Omega^{\leq 4 \pi^{2}}$ implies either $c \in Q_{0}$, or both $c_{1 / 2}$ and $c_{1 / 2}^{-}$are geodesics of length $\pi$ with conjugate end points;
(d) any nonconstant smoothly closed geodesic in $M$ has length $\geqslant 2 \pi$ and index $\geqslant 2$.

Proof. As described in the proof of Lemma $5, K \leqslant 1$ implies that the hypothesis of Lemma 1 is satisfied for $r=\pi$. By the same comparison technique, since $1 / 4 \leqslant K$, any geodesic in $M$ of length $>2 \pi$ has index $n-1 \geqslant 2$.

We argue first that for any $a>4 \pi^{2}, \Omega^{<a}$ is connected. If not, let $c_{1}, c_{2}$ be curves in different connected components of $\Omega^{<a}$. Since $M$ is simply connected, $c_{1}$ and $c_{2}$ are homotopic. After suitable refinement of the break point subdivision of the parameter interval [ 0,1 ], we can therefore join $c_{1}$ and $c_{2}$ by a continuous path in $\Omega^{\leqslant b}$ for some $b \geqslant a$. So $c_{1}$ and $c_{2}$ belong to the same connected component of $\Omega^{\leqslant b}$ for some regular value $b \geqslant a$, contradicting Lemma 4.

Using Lemma 4 again, we conclude that $\Omega^{\leqslant 4 \pi^{2}}$ is connected. Now we apply Lemmas 5 and 3 to obtain that $\bar{Q}_{0} \cap \Omega^{\leqslant 4 \pi^{2}}=\Omega^{<4 \pi^{2}}$ is the closure of $Q_{0} \cap$ $\Omega^{<4 \pi^{2}}$, which completes the proof of (a). In particular, $Q_{0} \cap \Omega^{<4 \pi^{2}}$ is dense, and by Lemma 1 , connected and closed in $\Omega^{<4 \pi^{2}}$. Thus $Q_{0} \cap \Omega^{<4 \pi^{2}}=\Omega^{<4 \pi^{2}}$ is connected, which proves (b). The last two statements (c) and (d) are an immediate consequence of (a), (b), and Lemmas 1 and 5.

We conclude our discussion with some remarks. The result in (d) that all
nonconstant smoothly closed geodesics have index $\geqslant 2$ is rather surprising. One has as immediate consequence (which is not the strongest conclusion that can be drawn) that for any $a \geqslant 0$, the loop spaces $\Omega^{<a}$ (and $\Omega^{<a}$ as well) are pathwise connected; cf. also Remark 5. The last statement holds also in even dimensions under the much weaker assumption $0<K \leqslant 1$, since by the Synge Lemma, every nonconstant smoothly closed geodesic has index $\geqslant 1$. Clearly, Theorem 6 holds in that case except that in (d), the index is only $\geqslant 1$. Using the preceding remark, just Lemma 1 is needed for the proof. Finally, both in odd and even dimensions, Theorem 6(b) provides an obvious direct argument for the fact that the injectivity radius of the exponential map is $\geqslant \pi$.

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