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On the Material Invariant Formulation of Maxwell's Displacement Current

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Maxwell accounted for the apparent elastic behavior of the electromagnetic field by augmenting Ampere's law with the so-called displacement current, in much the same way that he treated the viscoelasticity of gases. Maxwell's original constitutive relations for both electrodynamics and fluid dynamics were not material invariant. In the theory of viscoelastic fluids, the situation was later corrected by Oldroyd, who introduced the upper-convective derivative. Assuming that the electromagnetic field should follow the general requirements for a material field, we show that if the upper convected derivative is used in place of the partial time derivative in the displacement current term, Maxwell's electrodynamics becomes material invariant. Note, that the material invariance of Faraday's law is automatically established if the Lorentz force is admitted as an integral part of the model. The new formulation ensures that the equation for conservation of charge is also material invariant in vacuo. The viscoelastic medium whose apparent manifestation are the known phenomena of electrodynamics is called here the metacontinuum.

KEY WORDS: Maxwell's electrodynamics; Hertz's equations; material invariance; Oldroyd's upper-convected derivative.

1. INTRODUCTION

The first attempt to explain the propagation of light as a field phenomenon was by Cauchy around 1827 (see the account in Ref. 39) who postulated the existence of an elastic continuum, through which light propagates as shear waves. Unfortunately, Cauchy's model of elastic aether contradicted the natural perception of a particle moving *through* the field. As a result, it did not receive much attention because the notion

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of an elastic liquid was not available at that time. Subsequently came the contributions of Faraday and Ampere which eventually led to the formulation of the electromagnetic model. The crucial advance was achieved, however, when Maxwell[26] added the term, $\frac{\partial \boldsymbol{E}}{\partial t}$, which he named the "displacement current," in Ampere's law. It was very similar to the non-local term in his constitutive relation for elastic gases[27] (see, also Ref. 19 for an insightful review on viscoelastic models). We observe that the electric field vector is a clear analog of the stress vector in continuum mechanics. One can say that Maxwell postulated an elastic constitutive relation by adding the displacement current to Ampere's law. Indeed, the new term transformed the system of equations, already established in electrostatics, into a hyperbolic system with a characteristic speed of wave propagation similar to the speed of sound in gases. Maxwell identified the characteristic speed with the speed of light, and thus paved the way in understanding electromagnetic wave phenomena.

The advantage of Maxwell's system over the model proposed by Cauchy was its success in incorporating the empirically observed laws, such as Faraday's, Ampere's, and Biot–Savart's, at the time when Cauchy's approach seemed unrelated to those. However, the most puzzling aspect of Maxwell's model was its apparent lack of Galilean invariance. This was an indication that the linear form of Maxwell's electrodynamics was somehow divorced from the basic principles of Newtonian mechanics, and continuum mechanics, in particular.

The difficulties in establishing Galilean invariance lie in the fact that the constitutive relations proposed by Maxwell are not material invariant neither in the theory of gases, nor in electrodynamics. Invariance in fluids was remedied by Oldroyd[29] who enunciated the principle of invariance of a constitutive law under deformational motions of the coordinate frame, which is currently known as "material indifference."[37] Instead of following the same path as in the theory of viscoelastic fluids, and reformulating the model to what is now called "material invariant," the theory of electrodynamics took another path—postulating that the electromagnetic field should be *exempt* from the requirement of material invariance. Then the natural step was to assume that it is Lorentz invariant as suggested by the structure of the linear Maxwell's equations. This brought into view the notion of invariance in four-dimensional (4D) space–time (see the account in Ref. 38) which is a different concept than material invariance in three dimensions. This tenet is the currently accepted one and it amounts conceptually to the assumption that the electromagnetic field is not a material continuum *per se*.

Consequently, it seems important to reexamine the issue of material invariance of electromagnetism (EM) beginning from the first principles.

To this end, we derive a material invariant formulation of the Maxwell–Ampere law, assuming that the proper description of a *material* field must be material invariant. It should be noted that a similar situation arises regarding the Maxwell–Cattaneo model of thermal waves. The importance of material derivatives there is discussed in Ref. 10. Concerning EM, if the material invariant formulation is able to explain the known phenomenology in a more consistent manner, it should be given the proper attention as an alternative to the dominant point of view that one need not assume material properties for the vacuum.

2. INVARIANCE OF ELECTRODYNAMICS IN MOVING FRAMES

Maxwell's equation are not invariant in moving frames if the time remains absolute. This kind of situation is well understood in continuum mechanics and it happens when the acceleration of a material point lacks the convective part. If the so-called material (aka convective or substantial) derivative is used, then the models are shown to remain invariant under a general change of coordinate system, which is not restricted to rigid (non-deformable) frames in rectilinear motion. The most natural way out of the perceived non-invariance is to replace the partial time derivatives in Maxwell's equations by convective time derivatives, as was done at end of the 19th century by Hertz, who regarded his formulation as an explanation of electromagnetic phenomena *inside* material bodies (see Ref. 16, ch. 14). The fact that the Maxwell–Hertz equations are Galilean invariant (and in fact, material invariant) is usually overlooked in the literature. Apparently, this point was originally raised in Ref. 30, as reported in Ref. 5, where the case for Galilean invariance is forcefully argued. In modern notations, the Maxwell–Hertz equations read[5,30]

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{B} = -\nabla \times \boldsymbol{E}, \tag{1}$$

$$\frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{E} = c^2 \nabla \times \boldsymbol{B} - \frac{1}{\epsilon_0} \boldsymbol{j}, \tag{2}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{3}$$

$$\epsilon_0 \nabla \cdot \boldsymbol{E} = \rho, \tag{4}$$

where ρ is the charge density.

The Galilean invariance of Eqs. (1)–(4) is a relatively minor point. The more important fact about them is that they are also material invariant as any system that contains convective derivatives for the acceleration must be. It is important to realize, however, that Maxwell's displacement current has an analog in the constitutive law for Maxwell's fluid, and using a mere convective derivative may not be enough to make the law material frame indifferent. (The correct expression of the displacement current that makes it material invariant *together* with the continuity of charge is the object of Section 4.)

Now, the Maxwell–Hertz equations (called also "progressive wave equations") are clearly the correct model for electromagnetic phenomena in moving bodies. In this sense, the results of the present paper can also be considered as a refined derivation of the Maxwell–Hertz equations. But the primordial question is whether the progressive-wave equations (e.g., the version given in the present paper) can be construed to hold also *in vacuo*. The answer is obviously in the affirmative if one accepts the fact that what is currently called "physical vacuum" has to be a physical (material) continuum. This requirement means that one should not just consider a field in an empty geometric space, but understand that the space itself is a material continuum with interactions between the points following the rules for material continua.[37] This conjecture forms the conceptual basis of the present work which we name the *Material Invariance Principle* (MIP). A definition of MIP is as follows:

> There exist a three-dimensional absolute material continuum with specific rheology which obeys the principle of material invariance, including material frame indifference of the constitutive relations. The electromagnetic field is a manifestation in three dimensions of the dynamics of the absolute continuum, called the *metacontinuum*.

If EM is a manifestation of another reality that is on a deeper level, then the latter appears to be beyond (*meta*) EM, and the best way to convey this notion is to call it the *metacontinuum*. Although MIP involves the notion of an absolute medium, the *metacontinuum* of the present work is not the classic *aether* because it is not considered as a medium "surrounding" the particles and bodies, but as a materialization of the space itself. The particles and bodies are deformational patterns of the *metacontinuum*. This concept is an attempt to formulate on a quantitative level the proposal of Riemann (adopted by Einstein in General Relativity) that all physical forces are the manifestation of the geometry of space. If we are to implement this proposal we inevitably have to assume that the space is not a geometric volume of non-interacting points but is rather a continuum of interaction material points for which the material invariance principle should reign. The description of the *metacotinuum* from the point of view of theory of viscoelstic liquids is presented elsewhere.[9]

Note that there are no objections for this 3D continuum to be, in fact, a 3D hypersurface, such as a (hyper)membrane or (hyper)shell, immersed in a higher-order geometric (or material) space. The corollary of MIP is that the governing equations will have the same form in any coordinate system ("frame"), including coordinate systems that are in arbitrary deformational motion ("deformable frames"). The latter includes also the case of rotational motion of the frame. The simplest corollary of this principle is that the equations of electrodynamics will be Galilean invariant, since a rigid frame in rectilinear motion is the simplest limiting case of a deformable frame in general motion.

The drive to make the original Maxwell equations invariant took quite a bold turn more than a decade after Hertz's book was published. At that time it was discovered that some vestiges from the missing convective terms can be restored in the coordinate transformation, provided that time is no longer considered as an absolute parameter. Instead it was stipulated that time in the moving frame (parameterized by $x' = (x - vt)/\gamma$) transforms like $t' = (t - vx)/\gamma$, where γ is the Lorentz contraction factor. Such a transformation leaves the *form* of the linear wave equations for the potentials (Lorenz gauge) unchanged in a moving frame.

When translating the notion of Lorentz invariance in moving frames of the potentials to magnetic and electric fields, additional terms (forces) need to be added in Maxwell's equations.[22,31,32] In Faraday's law, it is the electromotive force that acts upon a moving electrical charge in a magnetic field. This is called the Lorentz force, because it was Lorentz who added it to Faraday's law as an integral part of the latter.[24] To make Faraday's law invariant in the moving frame, the electric field must transform according to the rule $\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$ which is the progenitor of the Lorentz-force term. In order that the Maxwell–Ampere equation be valid in a moving frame, then $\boldsymbol{B}' = \boldsymbol{B} - c^{-2}\boldsymbol{v} \times \boldsymbol{E}$ has to replace the magnetic field.[22,32] Thus, the Lorentz invariant formulation of the two dynamical Maxwell equations reads

$$-\nabla \times \boldsymbol{E} = \frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{B}, \tag{5}$$

$$c^2 \nabla \times \boldsymbol{B}' - \frac{\boldsymbol{j}'}{\epsilon_0} = \frac{\partial \boldsymbol{E}}{\partial t} + \nabla \times \boldsymbol{E} + \boldsymbol{v}(\nabla \cdot \boldsymbol{E}) = \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{E}. \tag{6}$$

When we presented the convective form of the equations (the last equalities in each of those), we made use of Eqs. (3) and (4). Note also, that the current has to transform according to $\boldsymbol{j} = \boldsymbol{j}' + \rho \boldsymbol{v} = \boldsymbol{j}' + \epsilon_0 (\nabla \cdot \boldsymbol{E})\boldsymbol{v}$. If this is not taken into account, the resulting system is not Galilean invariant.[23]

As no surprise, we found that the proper acknowledgment of all relevant terms stemming from the Lorentz transformation yields the standard convective derivatives. In other words, the artificial (from the point of view of a material continuum) device called the Lorentz transformation, yields equations in a form indistinguishable from that of Hertz provided that the velocity is constant. Yet this does not mean that the Lorentz transformation gives the Hertz form of the equations, because the former has very limited validity for $\boldsymbol{v} = \text{const}$, while the latter is valid for arbitrary velocities. For this reason, the attempts to generalize the Lorentz transformation to arbitrarily moving frames has not been successful.[31] In our opinion it cannot be generalized in principle, because it amounts to a relative time that is different from point to point. Even if it may be deemed acceptable, the fact that velocity depends on spatial coordinates and time, immediately destroys the invariant nature of the Lorentz transformations. It brings into consideration a host of additional terms involving spatial and temporal derivatives of the velocity components.

The Lorentz invariance can be viewed as a "poor man's material invariance" in the sense that the assumption of relativity of time (with mandatory time dilation) is a palliative solution to the problems of Maxwell's system in moving frames: a heuristic approach that can restore some terms of the convective derivatives. On the contrary, MIP introduced here is much more general than Lorentz invariance/covariance and fits better with the notion of "general covariance." In fact, it can be called "General Invariance" (GI) since the governing equations are in tensorial form. As mentioned in Ref. 31, the terminology involving "covariance" and/or "invariance" is quite often very loose. In a sense, the MIP proposed here makes the definition of GI non-ambiguous.

The fact that Eqs. (5) and (6) (especially Faraday's law) are also Galilean invariant has been spotted by many authors (see, e.g., the authoritative monographs, Ref. 18, pp. 212–213, Refs. 22 and 32). In order not to contradict the notion that the equations of electrodynamics cannot be Galilean invariant *in vacuo*, the authors of Refs. 18, 22 and 32, and many others, attempted to explain the apparent Galilean invariance as a limit of the Lorentz invariance for small velocities. Indeed, it is only natural to present this result as a limiting case in slowly moving frames, because the Lorentz force is observed in material bodies, and their velocities are not supposed to exceed the speed of light. The discussion on whether material bodies, or some other processes in the *metacontinuum*, can exceed the speed of light, goes beyond the scope of present work. The postulate that the speed of light cannot be exceeded is, in fact, an additional conjecture, regardless of the fact that many investigators assume that it stems from the postulate of the constancy of the speed of light. The speed of light being a limiting velocity for bodies in translatory motion does not contradict the absolutivity principle, MIP, of the present work. Drawing an analogy with acoustics, we can say that the small-amplitude (linear) waves cannot exceed the speed of light, but shock waves do.

And this effect is exclusively connected with the fact that convective time derivatives enter the model. For EM, this would mean that a body at supercritical (superluminal) speeds will behave as a shock wave of the metacontinuum (more precisely a tangential discontinuity or vortex sheet), which is equivalent to the statement that the body (in its original form and structure) cannot exceed the speed of light. Thus in MIP, the limiting property of the speed of light finds its natural explanation without imposing it as an additional postulate to the theory.

Yet, Eqs. (5) and (6) are still not the desired set of equations because one cannot derive from them an invariant equation for conservation of charge. This means that using merely the convective derivative does not make the model "material invariant," which is a more stringent requirement than Galilean invariance, and entails the latter. The way out of this situation is to exploit the above stated analogy between the Maxwell–Ampere law and Maxwell's constitutive relation for elastic liquids. The simplest constitutive laws, such as Hooke's law in elasticity and the Navier–Stokes law for Newtonian viscous liquids, establish pointwise ("local") connections between the stress tensor and the tensor of strains, or rate of strains. Such constitutive laws are local in the sense that they have no memory, and material invariance is trivially established by the transformation rule. It is a very different situation when a constitutive law involves also time derivatives (relaxation of stresses or retardation of strains). It is beyond doubt that adding a mere partial time derivative in a constitutive relation is not sufficient to ensure material invariance. It is interesting to note that taking the convective derivative, which is well suited to make the inertial terms in the momentum equations material invariant, is not enough to make a constitutive law fully material invariant (see Ref. 29). Nowadays, Oldroyd's notion of invariance of a constitutive law is incorporated in the so-called principle of material-frame indifference.[37]

The mathematically rigorous way to pursue the analogy between electrodynamics and the theory of elastic liquids is to consider the Maxwell displacement current as a constitutive relation; this is the object of the rest of the present work.

3. INVARIANT TIME DERIVATIVE OF A VECTOR DENSITY

Directional and other invariant derivatives of tensors are investigated in numerous mathematical and physical works but in order to make the paper self-contained and to clarify the physical meaning, we present here the pertinent derivations. In addition, formulating the passage from geometric coordinates to embedded material coordinates highlights the main idea of present paper.

Consider the 3D space and a fixed system of coordinates $\{x^i\}$, in it. The fixed coordinate system can be assumed to be Cartesian without loosing the generality. Together with the fixed coordinate system, consider a generally curvilinear moving coordinate system $\{\bar{x}^i\}$, that is embedded in the material continuum occupying the geometric space in the sense that coordinate lines of the moving system consists always of the same material particles. Then the transformation $x^j = f^j(\bar{x}^i; t)$ presents the law of motion of a material particle, parameterized by the coordinate $\bar{x}^j$. Assume that at time t, the two coordinate systems coincide. Then at time $t + \Delta t$, the law of motion gives the infinitesimal transformation $x^j = \bar{x}^j + v^j \Delta t$, which can be resolved for the material coordinates:

$$\bar{x}^i = x^i - v^i(x^j)\Delta t, \tag{7}$$

where v^i is the contravariant velocity vector.

$$\frac{\partial \bar{x}^i}{\partial x^j} = \delta^i_j - \Delta t \frac{\partial v^i}{\partial x^j} + o(\Delta t), \qquad \frac{\partial x^j}{\partial \bar{x}^i} = \delta^j_i + \Delta t \frac{\partial v^j}{\partial x^i} + o(\Delta t). \tag{8}$$

Let $\boldsymbol{A}$ represent some mechanical quantity, e.g., stress vector, electric field, temperature flux, etc. For all these mechanical characteristics, the actual observable is the following integral (see, e.g., Ref. 34)

$$\int_D \boldsymbol{A}\, d^3\boldsymbol{x} = \int_{\bar{D}} \boldsymbol{A}\, d\bar{\boldsymbol{x}}. \tag{9}$$

The principle of material invariance requires that this integral be invariant under coordinate transformation, which means that the vector $\boldsymbol{A}$ is a tensor density (or what is called "relative tensor of unit weight"[25]). In component form, the integral in the left-hand side can be rewritten as

$$\int_D A^k\, dx^1\, dx^2\, dx^3 \equiv \int_{\bar{D}} \frac{\partial \bar{x}^k}{\partial x^j} J A^j\, d\bar{x}^1\, d\bar{x}^2 d\,\bar{x}^3, \tag{10}$$

where J is the Jacobian of the coordinate transformation,

$$J = \left| \frac{\partial x^i}{\partial \bar{x}^j} \right| = 1 + \Delta t \frac{\partial \bar{v}^i}{\partial x^i} + o\,(\Delta t) \tag{11}$$

and A^k are the contravariant components of $\boldsymbol{A}$. Being reminded that D is an arbitrary region, one finds the transformation rule for a vector density in contravariant components

$$\bar{A}^k = J \frac{\partial \bar{x}^k}{\partial x^l} A^l, \tag{12}$$

where a summation is understood if an index appears once as a superscript and once as a subscript. Material invariance (see Ref. 29) requires that in constitutive laws, the total time variance of a tensor density,

$$\frac{\mathfrak{d} A^j}{\mathfrak{d} t} = \lim_{\Delta t \to 0} \frac{\bar{A}^j(\bar{x}^k; t + \Delta t) - A^j(x^k; t)}{\Delta t} \tag{13}$$

is used. Taylor series with Eq. (7) acknowledged, yields

$$\begin{aligned} \bar{A}^j(\bar{x}^k; t + \Delta t) &= \bar{A}^j(x^k; t) + \Delta t \left[\frac{\partial \bar{A}^j}{\partial t} + v^l \frac{\partial \bar{A}^j}{\partial \bar{x}^l} \right] + o(\Delta t) \\ &= \bar{A}^j(x^k; t) + \Delta t \left[\frac{\partial A^j}{\partial t} + v^l \frac{\partial A^j}{\partial x^l} \right] + o(\Delta t), \end{aligned} \tag{14}$$

where the fact is also acknowledged that at the moment t, vectors $\boldsymbol{A}$ and $\bar{\boldsymbol{A}}$, and their gradients coincide. Now, the contravariant components A^k transform according to the rule from Eq. (12), which gives

$$\begin{aligned} \bar{A}^j(x^k; t) &= \left(1 + \frac{\partial v^i}{\partial x^i} \Delta t\right) \left(A^j(x^k) - \frac{\partial v^k}{\partial x^m} A^m(x^k) \Delta t\right) \\ &= A^j(x^k) + \frac{\partial v^i}{\partial x^i} A^j(x^k) \Delta t - \frac{\partial v^k}{\partial x^m} A^m(x^k) \Delta t + o(\Delta t). \end{aligned} \tag{15}$$

After making use of Eq. (11) and neglecting the higher order terms in (Δt), Eq. (15) yields

$$\begin{aligned} \frac{\mathfrak{d} A^j}{\mathfrak{d} t} \overset{\text{def}}{=} \lim_{\Delta t \to \infty} \frac{\bar{A}^j(x^k) - A^j(x^k)}{\Delta t} &= \frac{\partial A^j}{\partial t} + \mathcal{L}_v A^j \\ = \frac{\partial A^j}{\partial t} + v^k \frac{\partial A^j}{\partial x^k} - \frac{\partial v^j}{\partial x^m} A^m &+ \frac{\partial v^i}{\partial x^i} A^j, \end{aligned} \tag{16}$$

where $\mathcal{L}_v$ is the Lie derivative along the vector field v^i (see Ref. 25 for a mixed tensor density of arbitrary rank). The first term in the invariant derivative (the partial time derivative) accounts for the changes of the components as functions of time, and the second term (the Lie derivative) represents the changes due to the fact that the coordinate system and the associated basis are also changing with time (being "convected" with the velocity field). In abstract vector notations valid in any coordinate system, one gets

$$\frac{\eth \boldsymbol{A}}{\eth t} = \frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{A} - \boldsymbol{A} \cdot \nabla \boldsymbol{v} + (\nabla \cdot \boldsymbol{v})\boldsymbol{A}. \tag{17}$$

What Oldroyd did, actually amounted to taking the directional derivative of a contravariant tensor density along the contravariant velocity vector of the material point at which the constitutive relation was written, which is a generalization of the advective part of the usual material derivative. For a pointed exposition of different issues connected with invariant derivatives, we refer the reader to the recent article[36] and the literature cited therein.

Following the established terminology,[3] we can call Eq. (17) "the upper convected" material derivative of vector $\boldsymbol{A}$. Note that if $\boldsymbol{A}$ was not a tensor density, but an absolute tensor, then the last term in Eq. (17) would be absent (see also Ref. 36). As shown in Ref. 29, there is a difference in the frame indifferent material derivatives of a contravariant and a covariant tensor, and the choice was left open to additional mechanical considerations in the mechanics of viscoelastic liquids.[3]

For the purpose of present work, it suffices to adopt the argument from Ref. 34, namely, that the electric field behaves as a contravariant tensor density. If so, one has to use Oldroyd's upper convected derivative. An interesting point is whether a covariant formulation can be derived. Naturally, in such a derivation the lower convected derivative would appear. However, as shown later in this work, the upper convected formulation fits precisely within the model, explaining the continuity equation for the charge in a moving frame while it can be demonstrated that the lower convected derivative cannot accomplish this result. Hence, in electrodynamics there is no alternative to the upper convected derivative.

4. MATERIAL INVARIANT MAXWELL-HERTZ ELECTRODYNAMICS

Guided by the analogy with elastic liquids, we find that the natural way to formulate electrodynamics *in vacuo* in a manner invariant under the changes of material frames, is to replace the partial time derivative of Maxwell's displacement current with the full material invariant time derivative (Oldroyd's derivative in this case) which secures that it is material frame indifferent. We propose that the Maxwell–Ampere law be re-formulated using the Oldroyd upper convected derivative, $\frac{\eth \boldsymbol{E}}{\eth t}$, *in lieu* of the partial time derivative. This means that in the Hertz System, we replace the Maxwell–Ampere law, Eq. (2), by

$$\frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{E} - \boldsymbol{E} \cdot \nabla \boldsymbol{v} + (\nabla \cdot \boldsymbol{v})\boldsymbol{E} = c^2 \nabla \times \boldsymbol{B} - \frac{\boldsymbol{j}}{\epsilon_0}. \tag{18}$$

Note, that in Eq. (1), material invariance is ensured by the usual material derivative, while Eq. (18) involves the Oldroyd upper convected derivative. A similar situation is observed in viscoelastic fluids, where the momentum equations involve the usual material derivative, while the rheology is based on the upper convected derivative. The reason for this asymmetry is that the magnetic field $\boldsymbol{B}$ can be explained as the *curl* of the velocity field of the *metacontinuum* (aka the vector potential in the Lorenz gauge), and hence Faraday's law is a straightforward corollary of the momentum equations (Cauchy balance).[6–8]

The formulation proposed here is also instrumental in deriving a material invariant continuity equation for the charge. To see this we take the divergence of Eq. (18), and after the cancellation of similar terms (not possible in the case of lower convected derivative), we get

$$\begin{aligned}
&\nabla \cdot [\boldsymbol{E}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{E} - \boldsymbol{E} \cdot \nabla \boldsymbol{v} + (\nabla \cdot \boldsymbol{v})\boldsymbol{E}] \\
&\quad = (\nabla \cdot \boldsymbol{E})_t + \boldsymbol{v} \cdot \nabla(\nabla \cdot \boldsymbol{E}) + \nabla \boldsymbol{v} \colon \nabla \boldsymbol{E} - \nabla \boldsymbol{E} \colon \nabla \boldsymbol{v} - \boldsymbol{E} \cdot (\nabla \cdot \boldsymbol{v}) \\
&\qquad + \boldsymbol{E} \cdot (\nabla \cdot \boldsymbol{v}) + (\nabla \cdot \boldsymbol{v})(\nabla \cdot \boldsymbol{E}) \\
&\quad = (\nabla \cdot \boldsymbol{E})_t + \boldsymbol{v} \cdot \nabla(\nabla \cdot \boldsymbol{E}) + (\nabla \cdot \boldsymbol{v})(\nabla \cdot \boldsymbol{E}) \\
&\quad = (\nabla \cdot \boldsymbol{E})_t + \nabla \cdot [(\nabla \cdot \boldsymbol{E})(\nabla \cdot \boldsymbol{v})] = {\epsilon_0}^{-1} [\rho_t + \nabla \cdot (\rho \boldsymbol{v})],
\end{aligned} \tag{19}$$

where the last equality is obtained after the expression for charge density, Eq. (4), is substituted. Consequently, this gives the following equation for the charge density

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\boldsymbol{j} + \rho \boldsymbol{v}) = 0, \tag{20}$$

which is the accepted form of the continuity equation in a moving (laboratory) frame.[15,18] While a naive approach to material invariance would have been to take just the usual material derivative, in doing so, one would not obtain the proper equation of conservation of charge. Taking the full fledged invariant derivative is the *only* way to make the full system of electrodynamics material invariant.

It is clear that without the last term in Eq. (20), one cannot explain any electromagnetic phenomena in a moving frame and it is already a part of the model of electrodynamics. The main difference here is that we do not arbitrarily add the convective term. Rather, it appears as an integral part of the model, just in the same way as the Lorentz force does.

In closing of this section, we mention that the metacontinuum model which is valid in any coordinate frame (moving with acceleration, rotation, and even with deformation), is valid also in rigid frames moving translatory with constant velocity. The material invariance principle entails as a small corollary the Galilean invariance in inertially moving non-deformable frames. However, it has always been of interest whether a model is Galilean invariant or not. Indeed, in a moving frame one can introduce the new variables $\hat{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{V}t, \hat{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{V}, \hat{\boldsymbol{j}} = \boldsymbol{j} + \rho\boldsymbol{V},$) and if $\widehat{\nabla}$ is the nabla vector, and $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{B}}$ are the electric and magnetic fields in the new frame, then the governing system has exactly the same form as Eqs. (1), (18), and (20), namely

$$\begin{gathered}\frac{\partial \hat{\boldsymbol{B}}}{\partial t} + \hat{\boldsymbol{v}} \cdot \widehat{\nabla} \hat{\boldsymbol{B}} = \widehat{\nabla} \times \hat{\boldsymbol{E}}, \\ \frac{\partial \hat{\boldsymbol{E}}}{\partial t} + \hat{\boldsymbol{v}} \cdot \widehat{\nabla} \hat{\boldsymbol{E}} - \hat{\boldsymbol{E}} \cdot \widehat{\nabla} \hat{\boldsymbol{v}} + (\widehat{\nabla} \cdot \hat{\boldsymbol{v}}) \hat{\boldsymbol{E}} = c^2 \widehat{\nabla} \times \hat{\boldsymbol{B}} - \frac{\hat{\boldsymbol{j}}}{\epsilon_0}, \\ \frac{\partial \rho}{\partial t} + \widehat{\nabla} \cdot (\hat{\boldsymbol{j}} + \rho \hat{\boldsymbol{v}}) = 0. \end{gathered} \tag{21}$$

5. DISCUSSION

One of the consequences of Galilean invariance is that the speed of propagation of small disturbances (speed of light) will be the same in any inertial frame. Hence, there is no need to *postulate* the absolutivity of the speed of light as an additional postulate. Consequently, the material invariant electrodynamics formulated here resembles the modern formulation of Maxwell's theory of elastic liquids.

The difference between material invariant and Lorentz invariant electrodynamics must show up in the formula for the Doppler effect. There is no place in the new formulation for the concept of the relativistic Doppler effect (whatever this might mean), because the absolutivity of the speed of light is an inherent feature of the model. Hence, no need for "relativistic addition" of velocities when considering the propagation of waves. The scale factors for the relativistic, R_{d}, and classic, C_{d}, Doppler effects can be written as[14]

$$R_{\mathrm{d}} = \sqrt{\frac{1 + v/c}{1 - v/c}}, \quad C_{\mathrm{d}} = \sqrt{1 - v^2/c^2}, \quad \rightarrow \quad R_{\mathrm{d}} = C_{\mathrm{d}} + O(v^2/c^2). \tag{22}$$

What kind of Doppler effect is present in Nature (either classic or relativistic) can be easily found experimentally using interplanetary probes. This is because their velocity can be inferred from the rate of change of their spatial position, while, on the other hand, the Doppler shift is independently measured. If the wrong formula is used for the Doppler effect, then discrepancies between the two kind of measurements of the speed of a craft must arise that are of order of v^2/c^2 with v being the relative velocity of the craft with respect to Earth.

The superb measurements performed by Pioneer 10 and 11 space probes can be used to assess which Doppler effect takes place. As has been well documented (see Ref. 1 and the works cited therein), there is an apparent blue shift in the Doppler data when compared with the velocity as computed from the trajectory. The magnitude of the blue shift is of order of $10^{-8} = O(v^2/c^2)$. This discrepancy is believed now to have been caused by some kind of unexplained acceleration toward the sun. This has been called the "Pioneer anomaly" because no clear cause has yet been officially accepted explaining this phenomenon. In our opinion, it is premature to implicate an acceleration, or any other physical effect, for the discrepancy because in the mentioned works, the relativistic Doppler formula was used. That was also the point of Renshaw[(33)], who did a comparison between the relativistic and classic Doppler effect for this case. He showed that the expected discrepancy when using the relativistic Doppler effect *in lieu* of the classic one is quite close to the reported difference of approximately $-8 \times 10^{-8}\,\mathrm{cm/s^2}$. However, it is not possible to take this analysis to a more quantitative level without access to the raw data, and for this reason it is not done in the present paper. Yet, considering the fact that the order of magnitude of the anomaly is so close to the difference between the two Doppler formulas one should not disregard the possibility that the relativistic Doppler formula is the culprit, thus warranting re-examination of the data using the classic Doppler formula.

Another strong point in favor of the existence of an absolute material continuum is the discovery that there is anisotropy in the Doppler shift of the cosmic microwave blackbody radiation (CMBR), which was reported as early as in 1976 in Ref. 11 and confirmed in 1977.[(35)] Since then, the anisotropy has been verified many times over (see Ref. 2) and can be regarded as one of the most important scientific discoveries of the last quarter of the 20th century. The anisotropy of the Doppler shift was clearly observed to follow the cosine rule with the axis of symmetry pointed approximately toward constellation Leo. The velocity corresponding to this anisotropy was estimated to be 270 ± 60 km/s in Ref. 11, and at 390 ± 60 km/s in Ref. 35. From the viewpoint of the present paper, this must be the velocity of Earth with respect to the *metacontinuum*, where the background radiation propagates. In other words, the Earth moves relative to the *metacontinuum* which creates the observer's Doppler effect, manifested in the above-mentioned anisotropy.

It is easy to understand why the principle of relativity is correct for point particles (Galileo) and may be incorrect for continua (Poincare–Einstein), such as the electromagnetic field. A particle (at least in Galilean–Newtonian physics) does not have structure, while a wave has spatial structure and the changes in this spatial structure, e.g., Doppler shift, provide the necessary information about the underlying metacontinuum, including information about the state of motion of the laboratory frame. And this is exactly what happens in CMBR anisotropy.

Additional support for the notion of material invariance can be found in Ref. 12, where the planetary magnetic fields are explained as a result of the presence of convective derivatives of the electric field $\boldsymbol{E}$. This amounts to the assumption that the Hertzian version of Maxwell's equations reflects better the nature in this case than the original (non-invariant) Maxwell's system.

The last issue to be addressed is how it is possible that the particles can move *through* an absolute continuum without disturbing the latter? This was the key question whose unsatisfactory answer led to the downfall of the 19th century aether theories. The way out of this conceptual quagmire is to realize the fact that particles *are not moving through* the metacontinuum: they are propagating *over* it. In 1884–1885, Hinton (see, e.g., Ref. 17) was the first to suggest that a separate material space adjacent to our physical space can be assumed as the medium for the propagation of light. The intended meaning in Ref. 17 was that a point moves on one 3D hypersurface and the wave propagates on another 3D hypersurface. These two are embedded in a 4D space and have minute thicknesses which cannot readily be detected. The two hypersurfaces exchange momentum and energy because they touch each other tangentially. In a sense, the particles glide over the surface of the absolute medium without entraining it. At the same time, they can create wave-like disturbances which propagate though the adjacent medium and interact with different particles at different positions. The concept of minute thickness alongside the fourth dimension formed the basis of the 5D theories proposed by Kalutza[20] in 1921 and elaborated by Klein,[21] over four decades after the original work of Hinton.

The proposal of Ref. 17 is not the only way to avoid the "aether drift" fallacy. There is another possibility that looks much more plausible in the light of the recent development in soliton theory (see Refs. 4,28); solitons being shown to behave as particles upon collisions and are, in fact, called "quasi-particles." Following this idea, one can consider a particle as a localized wave of metacontinuum. It goes beyond the scope of present paper to present the details of such a concept and this will be done elsewhere. A preliminary account can be found in Ref. 8. The main point here is that a soliton/quasi-particle will *propagate* over the metacontinuum in much the same way as a water wave propagates on the surface without carrying along the molecules of water with it. This concept is also fully compatible with the concept of wave-particle duality. One can easily see that a propagating soliton in a material continuum is a phase pattern that undergoes Lorentz contraction. In a sense, the Lorentz contraction of quasi-particles is another manifestation of the Doppler effect.[(8)] Assuming that the particles are the coarse-grain description of localized waves of very short length, one arrives at a unified model in which the Lorentz contraction is mandatory and is a manifestation of the presence of the metacontinuum, the latter being the progenitor of waves and particles alike. This concept can also contribute to the resolution of a long standing paradox pointed out by Einstein,[(13)] (p. 21), that one cannot rationally reconcile the absolute speed of light with the apparent relativity of rectilinear motion. In the light of the proposed here MIP, the speed of light is a characteristics of the absolute medium, while the speed of rectilinear propagation of a pattern is in fact a phase speed and is by definition relative. Moreover, the phase speed of a pattern cannot exceed the characteristic speed of the continuum without the pattern radically metamorphosing into a discontinuity (a shock, vortex sheet, etc.)

6. CONCLUSIONS

In the present paper, it is argued that the partial time derivative of the electric field representing Maxwell's displacement current can be replaced by a material invariant time derivative in the same vein as in Ref. 29 reformulation of the constitutive relations for viscoelastic liquids. It is shown that together with the Faraday's law in its form augmented by the Lorentz force, the new formulation is material invariant *in vacuo*, including the continuity equation for the conservation of electric charge. As a result, the electromagnetic field *in vacuo* can be considered as a material continuum, even without the presence of gross matter. This led us to propose the existence of an *metacontinuum* and of an material invariance principle (absolutivity principle) for the GI of the latter.

On the basis of this material reformulation, we reassessed several significant outstanding issues in contemporary physics which currently invoke controversy. We showed that some of the perceived discrepancies encountered in the explanations can be resolved if the electromagnetic field is endowed with the ubiquitous properties of a material continuum.

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