

On the mathematical character of the relativistic transfer moment equations

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ABSTRACT

We consider general relativistic, frequency-dependent radiative transfer in spherical, differentially moving media. In particular, we investigate the character of the differential operator defined by the first two moment equations in the stationary case. We prove that the moment equations form a hyperbolic system when the logarithmic velocity gradient is positive, provided that a reasonable condition on the Eddington factors is met. The operator, however, may become elliptic in accretion flows and, in general, when gravity is taken into account. Finally, we show that, in an optically thick medium, one of the characteristics becomes infinite when the flow velocity equals $\pm c/\sqrt{3}$. Both high-speed, stationary inflows and outflows may therefore contain regions that are ‘causally’ disconnected.

Key words: radiative transfer – relativity.

1 INTRODUCTION

Radiative transfer in differentially moving media has been extensively investigated in the past, and a large body of literature is available on this subject (see Mihalas & Mihalas 1984, and references therein). Despite the great efforts, however, works dealing with the transfer of radiation through media moving at relativistic speeds are comparatively few. The special relativistic transfer equation was first derived by Thomas (1930) and, including Thomson scattering, by Simon (1963) and Castor (1972); a thorough derivation can be found in the monograph by Mihalas & Mihalas. Stationary solutions in spherical symmetry were discussed by Mihalas (1980), Mihalas, Winkler & Norman (1984) and Hauschildt & Wehrse (1991). Radiative transfer in curved space-times was investigated by Lindquist (1966), Anderson & Spiegel (1972), Thorne (1981), Schinder (1988), Schinder & Bludman (1989), Anile & Romano (1992) and Nobili, Turolla & Zampieri (1993). All the solutions to the relativistic transfer problem found so far have been obtained using essentially two different approaches: either the transfer equation is directly solved for the radiation intensity $I_\nu(r, \mu)$, or the angular dependence is removed by introducing the moments of $I_\nu(r, \mu)$ and the moment equations are then integrated. Each method has both advantages and disadvantages. The solution of the transfer equation gives $I_\nu(r, \mu)$ directly, but is a quite formidable numerical task and requires special techniques, like the discrete ordinate matrix exponential (DOME) method introduced by Hauschildt & Wehrse. On

the other hand, the moment equations, being of lower dimensionality, are easier to handle numerically but their solution alone does not specify $I_\nu(r, \mu)$ completely. Moreover, an exact solution for the moments themselves can be obtained only if the Eddington factors $f_E = K_\nu/J_\nu - 1/3$ and $g_E = N_\nu/H_\nu - 3/5$ are computed self-consistently (these are the definitions of f_E and g_E appropriate to the PSTF moment formalism; see below). An exact determination of f_E and g_E was obtained by means of the tangent ray method (TRM, originally due to Mihalas, Kunasz & Hummer 1975) by Mihalas (1980) in the special relativistic case and by Schinder & Bludman (1989) for a spherically symmetric, static atmosphere in a Schwarzschild space-time. While it is not entirely obvious that TRM can be fruitfully applied to more complex situations in which dynamics and gravity are both present, the moment method can still be used to obtain, at least, an approximate solution for $I_\nu(r, \mu)$ by introducing, ‘a priori’, reasonable expressions for the Eddington factors (see e.g. Minerbo 1978; Nobili et al. 1993).

In this paper we present an analysis of the stationary, spherically symmetric relativistic moment equations, placing particular emphasis on the character of the second-order differential operator implicitly defined by the zeroth- and first-order equations. This point has never received proper attention in the past, despite the fact that it appears to be of non-negligible relevance because the choice of the boundary conditions is crucially related to the character of the operator. Mihalas, Kunasz & Hummer (1976) discussed to some extent the problem of frequency conditions in the non-

relativistic case, concluding that for accelerated winds and decelerated inflows the transfer equation is of the Feautrier type, although they stressed that different velocity laws may produce anomalous behaviours. Here we prove that the special relativistic moment equations form a hyperbolic system for positive logarithmic velocity gradients at least if $f_E - g_E - 4/15 < 0$, so that they should be solved as a two-point boundary-value problem in space and an initial-value problem in frequency. In converging flows, however, advection and aberration effects may produce a region of ellipticity. Moreover, when gravity is taken into account, the operator may become elliptic even in the wind case. We point out that one of the two characteristics, when they exist, diverges when the flow velocity equals a ‘sound’ speed v_s , $1/3 \leq (v_s/c)^2 \leq 1$ depending on optical depth. The sphere of radius r_s is completely analogous to a sound horizon and behaves like a one-way membrane as far as the propagation of boundary data is concerned. Finally, we discuss the connection between the ‘pathologies’ in the moment equations and the vanishing of the coefficients of I_r -derivatives in the transfer equation.

2 MOMENT EQUATIONS

General relativistic moment equations were derived by Thorne (1981), who introduced projected symmetric trace-free (PSTF) tensors to describe the moments of the radiation intensity. In spherical symmetry, the k th PSTF moment has just one independent component, the radial one denoted by w_k , and the formalism greatly simplifies. The first two radiation moments are the radiation energy density and flux measured by a comoving observer; the third PSTF moment is the radiative shear, $w_2 = 4\pi(K_v - J_v/3)$. An alternative form of the moment equations in a static, spherical space-time, making use of a Lagrangian comoving coordinate system, has been presented by Schinder (1988) and Schinder & Bludman (1989).

Using Thorne’s notation, with all moments in erg cm^{-3} , the first two stationary moment equations are (see also Nobili et al. 1993)

$$\frac{\partial w_1}{\partial \ln r} + 2w_1 + \frac{y'}{y} \left(w_1 - \frac{\partial w_1}{\partial \ln v} \right) + \frac{v}{c} \left[-\left(\frac{u'}{u} - 1 \right) \frac{\partial w_2}{\partial \ln v} \right. \quad (1a)$$

$$\left. + \frac{\partial w_0}{\partial \ln r} + \left(2 + \frac{u'}{u} \right) w_0 - \frac{1}{3} \left(2 + \frac{u'}{u} \right) \frac{\partial w_0}{\partial \ln v} \right] \frac{s_v^0 r}{y},$$

$$\frac{1}{3} \frac{\partial w_0}{\partial \ln r} + \frac{y'}{y} \left(w_0 - \frac{1}{3} \frac{\partial w_0}{\partial \ln v} \right) + \frac{\partial w_2}{\partial \ln r} + 3w_2 - \frac{y'}{y} \frac{\partial w_2}{\partial \ln v} \quad (1b)$$

$$+ \frac{v}{c} \left[\frac{\partial w_1}{\partial \ln r} + \frac{1}{5} \left(7 \frac{u'}{u} + 8 \right) w_1 - \frac{1}{5} \left(3 \frac{u'}{u} + 2 \right) \right. \\ \left. \times \frac{\partial w_1}{\partial \ln v} - \left(\frac{u'}{u} - 1 \right) \left(w_3 + \frac{\partial w_3}{\partial \ln v} \right) \right] = \frac{s_v^1 r}{y}.$$

Here, $y = \gamma \sqrt{1 - r_g/r}$ is the total energy per unit mass, $u = yv/c$ is the radial component of the fluid 4-velocity, and s_v^0 and s_v^1 are the source moments; a prime denotes the total derivative in the r -direction, $\gamma = (1 - v^2/c^2)^{-1/2}$ and r_g is the gravitational radius.

An analysis on the nature of the various dynamical terms appearing in the moment equations was presented by Buchler (1983); similar discussions for the transfer equation can be found in Castor (1972), Mihalas et al. (1976) and Hauschildt & Wehrse (1991). Terms of order v/c in equations (1) account both for the local Doppler shift of photons and for advection and aberration. Mihalas et al. (1976) have shown that advection and aberration produce a fractional variation on the solution which is $\sim 5v/c$ and can be safely neglected for small velocities. In some astrophysical situations, however, like photospheric supernova expansion, jets, accretion on to black holes and on to neutron stars, velocities of $\geq 0.1c$ are expected and such effects cannot be ignored. In the next sections we show that, apart from obvious quantitative effects, advection/aberration terms may change substantially the mathematical properties of the moment equations in spherical inflows.

3 CHARACTERISTIC ANALYSIS

In the following we investigate the mathematical character of the second-order differential operator defined implicitly by equations (1). For the sake of simplicity, we shall assume that source moments contain no derivatives of the radiation moments; an extension of the present analysis will be needed to include radiative processes like non-conservative electron scattering treated in the Fokker–Planck approximation, which depends on both first and second v -derivatives of w_0 (see the discussion at the end of this section).

The characteristic analysis of a generic, linear system of first-order partial differential equations can be easily performed when the system is brought into the form (see e.g. Whitham 1974)

$$\frac{\partial u_i}{\partial t} + a_{ij} \frac{\partial u_j}{\partial x} + b_i(x, t; u_i) = 0, \quad i, j = 1, n. \quad (2)$$

In this case the characteristic velocities are the roots of the equation

$$|a_{ij} - \lambda \delta_{ij}| = 0 \quad (3)$$

and the system is hyperbolic if equation (3) has n different real roots. Rewriting the moment equations in the form (2) and introducing the Eddington factors $f_E = w_2/w_0 = K_v/J_v - 1/3$, $g_E = w_3/w_1 = N_v/H_v - 3/5$, we obtain, after some manipulations,

$$\frac{\partial w_0}{\partial \ln r} + \frac{1}{f_E + 1/3 - v^2/c^2} \left\{ \left[\frac{v^2}{c^2} F - \frac{y'}{y} \left(f_E + \frac{1}{3} - \frac{v^2}{c^2} \right) \right] \frac{\partial w_0}{\partial \ln v} \right. \\ \left. - \frac{v}{c} G \frac{\partial w_1}{\partial \ln v} \right\} + C_1 = 0, \quad (4a)$$

$$\frac{\partial w_1}{\partial \ln r} + \frac{1}{f_E + 1/3 - v^2/c^2} \left\{ -\frac{v}{c} \left(f_E + \frac{1}{3} \right) F \frac{\partial w_0}{\partial \ln v} \right. \\ \left. + \left[\frac{v^2}{c^2} G - \frac{y'}{y} \left(f_E + \frac{1}{3} - \frac{v^2}{c^2} \right) \right] \frac{\partial w_1}{\partial \ln v} \right\} + C_2 = 0. \quad (4b)$$

Here $F = (\beta - 1)f_E + (2 + \beta)/3 - y'/y$, $G = (\beta - 1)g_E + (2 + 3\beta)/5 - y'/y$, $\beta = u'/u$, $y'/y = \beta v^2/c^2 + r_g/2y^2 r$, and all terms

not containing derivatives of the moments are grouped together into C_1 and C_2 . Actually, C_1 and C_2 do contain derivatives of both f_E and g_E , but the Eddington factors are to be regarded as known functions, either coming from the solution of the transfer equation, as in the TRM, or being specified 'a priori' if a closure is assumed. The Eddington factors used here differ from the usual ones inasmuch as PSTF moments are originated by a Legendre polynomial expansion of the intensity; in particular, f_E and g_E are restricted in the ranges $0 \leq f_E \leq 2/3$ and $0 \leq g_E \leq 2/5$. We note that the matrix of the coefficients a_{ij} , defined in equation (2), is symmetric and that its eigenvalues are real, if $f_E = 2/3$ and $g_E = 2/5$. This means that the moment equations are always hyperbolic in the streaming limit, for any values of v and β .

The equation for the characteristic velocities is

$$\lambda^2 + \frac{1}{f_E + 1/3 - v^2/c^2} \left[2 \frac{y'}{y} \left(f_E + \frac{1}{3} - \frac{v^2}{c^2} \right) - \frac{v^2}{c^2} (F + G) \right] \lambda + \frac{1}{f_E + 1/3 - v^2/c^2} \left[\left(\frac{y'}{y} \right)^2 \left(f_E + \frac{1}{3} - \frac{v^2}{c^2} \right) - \frac{y'}{y} \frac{v^2}{c^2} (F + G) - \frac{v^2}{c^2} FG \right] = 0. \quad (5)$$

By introducing $U^2 = (v^2/c^2)/(f_E + 1/3)$, the discriminant of equation (5) can be written as

$$\Delta = \frac{U^2}{(U^2 - 1)^2} [(U^2 - 1)(F - G)^2 + (F + G)^2] = \frac{U^2}{(U^2 - 1)^2} [U^2(F - G)^2 + 4FG]. \quad (6)$$

From equation (6) it follows that the sign of Δ depends only on the sign of the term in square brackets, and it is $\Delta > 0$ if either $U^2 > 1$, that is to say $v^2/c^2 > f_E + 1/3$, or $FG > 0$, regardless of the values of the Eddington factors and of the velocity gradient. In order to make the analytical treatment feasible in the following we shall neglect gravity, so that $y'/y = \beta v^2/c^2$. In this case it is easy to see that F and G are opposite in sign and, consequently, Δ may become negative for flow velocities in the range $a < v^2/c^2 < b$, where

$$a = \min \left[f_E + \frac{1}{3} + \frac{1}{\beta} \left(\frac{2}{3} - f_E \right), g_E + \frac{3}{5} + \frac{1}{\beta} \left(\frac{2}{5} - g_E \right) \right], \quad (7)$$

$$b = \max \left[f_E + \frac{1}{3} + \frac{1}{\beta} \left(\frac{2}{3} - f_E \right), g_E + \frac{3}{5} + \frac{1}{\beta} \left(\frac{2}{5} - g_E \right) \right].$$

Let us consider the case $\beta > 0$ first. Since we have already shown that $\Delta > 0$ if $v^2/c^2 > f_E + 1/3$, it follows that only the velocity interval

$$g_E + \frac{3}{5} + \frac{1}{\beta} \left(\frac{2}{5} - g_E \right) < \frac{v^2}{c^2} < f_E + \frac{1}{3} \quad (8)$$

needs to be considered. It can be easily checked that the above conditions are never fulfilled if $0 < \beta \leq 1$. For $\beta > 1$, a sufficient condition for the positivity of Δ can be obtained

by demanding that

$$g_E + \frac{3}{5} + \frac{1}{\beta} \left(\frac{2}{5} - g_E \right) > f_E + \frac{1}{3},$$

which is equivalent to

$$(\beta - 1) \left(g_E - f_E + \frac{4}{15} \right) + \frac{2}{3} - f_E > 0.$$

In order for the left-hand side of the previous inequality to be positive, it is enough to ask that

$$f_E - g_E - \frac{4}{15} < 0 \quad (9)$$

which, we stress again, gives only a *sufficient* condition for the hyperbolicity of the moment equations for $\beta > 1$. On the other hand, we note that if condition (9) is violated there always exists a value of β ,

$$\beta = 1 + (2/3 - f_E)/(f_E - g_E - 4/15) > 1,$$

beyond which Δ may become negative.

Condition (9) cannot be proved to hold in full generality, and should be verified case by case. It should be taken into account, however, that the first two Eddington factors are not independent of each other, although we have avoided, until now, specifying any relation between them. In order to check whether condition (9) can be physically acceptable, we compare it with the results obtained by Fu (1987a,b). Using a statistical formalism to approximate the radiation intensity at all depths, he was able to derive constraints on the Eddington factors, showing that the values of K/J are bounded by two curves, the 'logarithmic' (upper) and 'hyperbolic' (lower) limits, in the H/J versus K/J plane. Expressed in terms of the more conventional Eddington factors K/J and N/H , condition (9) reduces to $K/J < N/H$. We have computed the logarithmic and hyperbolic limits of the second Eddington factor, and verified that it is $(K/J)_{\log} \leq (N/H)_{\log}$, $(K/J)_{\text{hyp}} \leq (N/H)_{\text{hyp}}$ for $H/J \leq 1$, although the inequality $(K/J)_{\log} \leq (N/H)_{\text{hyp}}$ is not satisfied for $H/J > 0.67$. On the other hand, since the hyperbolic (logarithmic) limit should be attained in the streaming (diffusion) regime, it seems more meaningful to compare values of K/J and N/H that describe statistical properties of the radiation field in the same physical conditions; so our request that $K/J < N/H$ at all depths seems indeed compatible with the results of Fu's analysis.

Unfortunately, the study of the limits given by equation (7) is not so straightforward if $\beta < 0$ and when gravity is taken into account. If the gravitational field is strong enough and/or the gas flow is almost in free-fall, however, the existence of velocity ranges where Δ changes sign, and the operator becomes elliptic, is certainly possible, even if v/c is small.

Until now we have discussed the conditions for the existence of real characteristics without considering the actual behaviour of the characteristics themselves. Assuming $\beta > 0$, $f_E - g_E - 4/15 < 0$ and neglecting gravity, equation (5) can be used to analyse how the characteristics change on varying v/c . In the limit of vanishing velocity, there is just the double root $\lambda = 0$ which indicates that the two moment equations decouple (no 'frequency mixing' between the

moments). As v/c increases, the characteristics become distinct. It is possible to prove that the solutions of equation (5) are opposite in sign, but not equal in magnitude, if $(v/c)^2 < f_E + 1/3$. From equation (5), in fact, it follows that the product of the roots, $\lambda_1 \lambda_2$, is

$$\lambda_1 \lambda_2 = \frac{v^2/c^2}{f_E + 1/3 - v^2/c^2} \left[\beta^2 \left(f_E + \frac{1}{3} - \frac{v^2}{c^2} \right) \frac{v^2}{c^2} - \beta(F + G) \frac{v^2}{c^2} - FG \right]. \quad (10)$$

The term in square brackets can be written as

$$\begin{aligned} & \beta \left[\beta \left(f_E + \frac{1}{3} - \frac{v^2}{c^2} \right) - F \right] \frac{v^2}{c^2} - G \left(\beta \frac{v^2}{c^2} + F \right) \\ &= - \left(\beta \frac{v^2}{c^2} + G \right) \left(\frac{2}{3} - f_E \right) - \beta \left(f_E + \frac{1}{3} \right) G. \end{aligned}$$

Since $G > 0$ for $(v/c)^2 \leq f_E + 1/3$ (see condition 9), the previous expression is always negative and we can conclude that $\lambda_1 \lambda_2 < 0$ for $(v/c)^2 < f_E + 1/3$.

As equation (5) shows, one of the characteristics switches from positive to negative through a pole at $(v/c)^2 = f_E + 1/3$. The existence of a diverging characteristic implies that the two spatial regions separated by the line $r = r_s$, where $(v/c)^2 = f_E + 1/3$, are causally disconnected in the sense that the behaviour of the solution for $(v/c)^2 < f_E + 1/3$ is not influenced by what happens for $(v/c)^2 > f_E + 1/3$. The surface of radius r_s behaves like a one-way membrane in the same way as the sound horizon does in transonic flows. As a consequence, if the flow velocity equals the ‘sound’ speed $v_s = (f_E + 1/3)^{1/2} c$, the moment equations cannot be solved as a two-point boundary-value problem in space and an initial-value problem in frequency, contrary to the case in which $v < v_s$ everywhere. Now the solution depends only on the data assigned on the spatial boundary of the ‘subsonic’ region plus the two initial frequency conditions.

The presence of a ‘sound’ horizon is not an artefact introduced by the moment expansion, as can be seen by examining the special relativistic form of the transfer equation [see e.g. Hauschildt & Wehrse (1991), equation (1)]. The r -derivative of the radiation intensity is, in fact, multiplied by the factor $e = \gamma(\mu + v/c)$, which is zero at $v/c = -\mu$. This means that, if a non-vanishing velocity field is present, the transfer equation becomes necessarily singular on the surface $v(r)/c = -\mu$ in the r - μ - v space, where the dimensionality of the equation lowers from 3 to 2. Moment equations contain the same kind of degeneracy but, being obtained by angle-averaging the transfer equation, the coefficients of the space derivatives vanish at a fixed value of μ which is just

$$\sqrt{\langle \mu^2 \rangle} = 1/\sqrt{3}$$

in the Eddington approximation. We note that the transfer equation does not exhibit any singularity when written in its characteristic form in the r - μ plane, as in the tangent ray method, because this amounts to use of a coordinate system that establishes a one-to-one, regular map between the integral surface and the integration domain.

The same kinds of considerations can be used to relate the possible ellipticity of the moment equations to the vanishing

of the coefficient of the v -derivative in the transfer equation,

$$g = \frac{\gamma}{r} \frac{v}{c} [1 - \mu^2 + \mu(\mu + v/c)\beta] \quad (11)$$

[see equation (3c) of Hauschildt & Wehrse, where our definition of β was used]. It can be shown that g is always non-negative for $-1 \leq \mu \leq 1$ only if $v > 0$ and $0 \leq \beta \leq 2$; for all other values of the velocity and of the velocity gradient there exists a value of μ at which g vanishes. The transfer equation may, therefore, become singular even in the outflow case, and this is consistent with the fact that the moment equations can be proved to be hyperbolic without any additional constraint only for $0 \leq \beta \leq 1$. In general, the degeneracy occurs along certain lines in the r - μ plane. Actually, in the moment equations (4), the coefficients of the v -derivatives, which are obviously related to g , contain some averaged value of μ , and they can change sign at a certain value of r in the integration domain. In particular, since these coefficients depend on the two Eddington factors f_E and g_E , they can change sign at two different radii, say r_1 and r_2 . As a consequence, an interval (r_1, r_2) , in which the system of differential equations becomes elliptic appears.

As stressed by Mihalas et al. (1976) and Hauschildt & Wehrse (1991), both e and g depend on the flow velocity only if advection and aberration are taken into account, even to first order in v/c . In this respect, it is interesting to note that the moment equations reduce to a parabolic system in the diffusion limit for any given β if only the local Doppler shift of photons is retained (Nobili et al. 1993; see also Blandford & Payne 1981a,b; Payne & Blandford 1981). It is, therefore, the inclusion of advection and aberration terms, which act as singular perturbations, that introduces degeneracies either into the transfer equation or into the system of the moment equations. A similar conclusion, although in a different context, was reached recently by Gombosi et al. (1993) who studied energetic particle transport by means of a moment expansion of the distribution function which is very similar to the one used here for the radiation intensity. A situation like this arises also when non-conservative scattering is included in the source term. Assuming that it can be treated in the Fokker-Planck approximation, the presence of v -derivatives of the radiation intensity produces an effect analogous to advection/aberration. In this case (see Colpi 1988), it can be shown that the transfer equation, in the diffusion limit and retaining only local Doppler shift, is always of the elliptic type and it must be integrated, prescribing suitable conditions on all of the boundary of the integration domain. Actually, we want to stress that a general analysis of the mathematical character of the transfer equation is not possible ‘a priori’, depending on the input physics included in the source term. In the present study, we have dealt with the more complete form of the moment equations in dynamical flows, but assuming that only conservative scattering and isotropic true emission-absorption processes are present. Even under these assumptions, we have shown that in accretion flows the operator defined by equations (1) may become of the mixed type, switching from hyperbolic to elliptic. The presence of a spatially limited elliptic region implies that, there, conditions must be specified at both the frequency boundaries, although the problem remains two-point boundary-valued in space.

As a final point, let us briefly discuss the effects induced by the presence of a gravitational field on the existence of real characteristics and, consequently, on the character of the operator defined by the moment equations. Both the expressions for F and G contain, now, an extra term $-r_g/2y^2r$ with respect to the special relativistic case and, even if $0 \leq \beta \leq 1$, Δ could be either positive or negative, depending on the sign of $F = F_{\text{SR}} - r_g/2y^2r$ and $G = G_{\text{SR}} - r_g/2y^2r$, where F_{SR} and G_{SR} are the special relativistic expressions derived earlier. This leads to the conclusion that, irrespective of the sign of β , the presence of a gravitational field can induce a change in the character of the moment equations: in particular, if $0 \leq \beta \leq 1$ the possible appearance of regions of ellipticity is completely due to gravity.

4 CONCLUSIONS

We have analysed the mathematical character of the system formed by the first two relativistic transfer moment equations. It has been shown that, similarly to the non-relativistic case, the differential operator is of the hyperbolic type when the flow velocity is a monotonically increasing function of the radial coordinate. In contrast, in converging flows and when gravity is taken into account, the character of the operator is much more complex and the system of equations may become of the mixed type. This result is of interest in connection with models of spherical accretion on to compact objects, and seems to be caused by advection and aberration effects.

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