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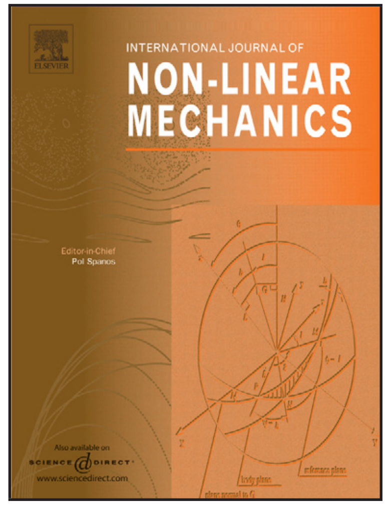




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# On the mathematical theory of vehicular traffic flow II Discrete velocity kinetic models

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**Abstract**

This paper deals with the modelling of vehicular traffic flow by methods of the discrete mathematical kinetic theory. The discretization is developed in the velocity space by a grid adapted to the local density. The discretization overcomes, at least in part, some technical difficulties related to the selection of the correct representation scale, while the adaptative grid allows an improved description of various phenomena related to vehicular traffic flow. Specific models are proposed and a qualitative and computational analysis is developed to show the properties of the model and their ability to describe real flow conditions. A critical analysis, proposed in the last part of the paper, outlines suitable research perspectives.

*Key words:* traffic flow; kinetic theory; nonlinear sciences; multiscale modelling.

## 1 Introduction

Methods of the mathematical kinetic theory have been developed, after the pioneer book by Prigogine and Hermann [29], to model vehicular traffic flow on roads and networks of roads. Prigogine's ideas have motivated research activity in the field by several authors, among others Paveri Fontana [28], Klar and Wegener [24], Nelson [27], Sopasakis [30], Lo Schiavo [26], Delitala [17], Coscia [12], Bellomo and Coscia [7], Darbha and Rajagopal [14], [15], [16]. The existing literature is reported in various review papers [20], [23], [25], [4], focused on different aspects: modelling, physics, development of computational schemes, analytic problems. A recent note by Delitala [17] suggests to use methods of the discrete mathematical kinetic theory [19] to model complex phenomena of vehicular traffic flow, taking also advantage of the methods of the generalized kinetic theory [1], [2]. As pointed out by Gatignol [19], the discrete Boltzmann equation is a model of the mathematical kinetic theory suitable to describe the behavior of a diluite gas of particles which can attain only a finite number of velocities. This general idea applied to traffic flow modelling rests on the assumption that vehicles move on the road with a finite number of velocities only. Moreover, methods of the generalized kinetic theory allow to include in the model, in addition to the usual variables related to the microscopic state of the interacting objects, also a number of variables related to a somehow self-organized behavior. Recent studies in the field are documented in the collection of surveys [5], as well as in some recent papers.

The idea of discretizing the velocity variable appears worth to be developed, not only because vehicles often move in clusters identified by a discrete set of velocities [22], but also considering that experiments developed to identify the parameters of the models can be effectively performed looking at groups of vehicles with the same velocity. The assessment of models is a necessary step to go through to get models which are effectively useful for applications, including flow of vehicles on roads with variable conditions [10], or networks of roads [3], [21].

The modelling of traffic flow phenomena can be developed, as detailed in [4], at different representation scales:

**Microscopic modelling** corresponds to model the dynamics of each single vehicle under the action of the surrounding vehicles;

**Statistical description**, in a framework close to the one of the kinetic theory of gases, consists in the derivation of an evolution equation for the distribution function on the position and velocity of the vehicle along the road;

**Macroscopic description**, analogous to the one of fluid dynamics, refers to the derivation, on the basis of conservation equations and material models, of an evolution equation for the mass density, linear momentum and energy.

It is also well understood that none of the above representation scales is consistent with physical reality. Indeed, continuity assumptions cannot be applied to vehicular flows considering that the inter-vehicular distances cannot be neglected, while methods of mathematical kinetic theory need a number of particles much greater than those involved in the road. Finally, the vehicle-driver system cannot be dealt with as a particle in classical mechanics, but, following the definitions adopted in [6], as **active particle** or **agent**. The above reasoning is supported by the sharp critical analysis proposed by Daganzo [13].

Extracting some specific ideas:

i) Experiments are generally developed in steady state conditions. On the other hand, modelling needs information in unsteady conditions.
ii) Measurements corresponding to repeated experiments provide different results with fluctuations around a certain mean value or most probable value. The output of experimental results can be regarded as a random variable with non negligible variance.
iii) Experiments related to gross quantities, such as density and mass velocity, are obtained, as it is explained in Section 2, by an averaging process, either in space or in time, of microscopic measurements. This procedure unavoidably generates errors.

Modelling by equations of the mathematical kinetic theory appears to be a rather natural way of approaching the problem as already suggested by Prigogine, [29]. However, it is worth going again back to some ideas posed by Daganzo:

iv) The width of a traffic shock only encompasses a few vehicles, and unlike molecules, vehicles have a personality (e.g. aggressive or timid) that remain unchanged by motion.
v) A fluid particle responds to stimuli from the front and from behind, but a car is an anisotropic particle that mostly responds to frontal stimuli.

Actually, the reasoning reported in (i)–(iii) is quite general and may possibly be applied to a large variety of equations of mathematical physics. On the other hand, items (iv) and (v) deal with specific features of the complex system we are dealing with. The analysis proposed in this paper will attempt to provide, at least in part, an answer to the above remarks. Indeed the discretization method with an adaptative grid contributes to weaken the above criticized continuity assumptions. Moreover, vehicles are considered as active, rather than classical, particles: interactions are modelled by a table of games[9] rather than by laws of points mechanics. This paper deals with the above mentioned topics with the objective of designing a new class of traffic flow models based on the idea of discretizing the velocity space in a way suitable to capture relevant aspects of the traffic flow complexity [22].

The paper is developed through six more sections which follow the above introduction. The first part of the paper refers to modelling, in details:

Section 2 deals with some essential features of the continuous and discrete distribution function. Furthermore the concept of adaptative grid on the velocity is introduced. The discretization is such that the number of collocation points is constant and the step grid of the velocity variable depends on the local density of vehicles.

Section 3 deals with the modelling of vehicular traffic flow in the presence of negligible density gradients. The evolution equation refers to a discrete distribution function which corresponds to the discrete velocities depending on time, but constant in space (space uniform). The model is derived by a mass balance equation implemented by suitable models of microscopic interactions. The model is stated in terms of a system of ordinary differential equations corresponding to the distribution function in the collocation points of the velocity space.

Section 4 deals with the modelling in the case of spatial gradients. The model describes the evolution in terms of a system of partial differential equations corresponding to the distribution function, depending on time and space, in the collocation points of the velocity.

The second part of the paper refers to the qualitative and computational analysis of the mathematical problems generated by the application of the model to real flow conditions. Some research perspectives will also be brought to the attention of the reader. In details:

Section 5 develops a qualitative analysis (wellposedness and stability properties) of the model proposed in Section 3 in spatially homogeneous conditions. Some numerical simulations are reported which allow to define the asymptotic equilibrium distribution function and, moreover, to show how the model can describe the velocity distribution in steady space uniform flow conditions and its trend toward equilibrium.

Section 6 develops some simulations of the non homogeneous model proposed in Section 4, visualizing some pertinent physical phenomena described by the model.

Section 7, finally, proposes a critical analysis of the contents of the paper and indicates some research perspectives, with special attention to the development of models.



## 2 Discrete Statistical Distribution Function

The statistical representation particles system by methods of the mathematical kinetic theory is defined by a probability distribution over the microscopic state of particles. This paper is based, as already mentioned, on a representation by a discrete probability distribution over the velocity state of vehicles. This section is organized in two subsections. The first one briefly recalls the essentials of the continuous distribution functions; the second one, the essentials of the discrete velocity distribution function.

### *2.1 Continuous distribution function*

Consider the one dimensional and one directional flow of vehicles on a one lane road, and define the following quantities:

• $\ell$ is the length of the road;

• $n_M$ is the maximum number density of vehicles corresponding to bump-to-bump conditions;

• $v_M$ is the maximum mean velocity corresponding to free flow of vehicles;

• $T$ is the characteristic time chosen according to the condition $v_M T = \ell$, that means that $T$ is the time necessary to cover the whole road length at the maximum mean velocity in free flow $v_M$.

• $t = t_r/T$ is the dimensionless time variable referred to the characteristic time $T$, where $t_r$ is the real time.

The state, at the time $t$, of each vehicle, regarded as a geometrical point, is identified by the following variables:

• $x = x_r/\ell$ is the dimensionless position referred to the characteristic length of the road $\ell$, where $x_r$ is the real dimensional space;

• $V = V_r/v_M$ is the dimensionless velocity of vehicles referred to the maximum mean velocity $v_M$, where $V_r$ is the real velocity of the single vehicle;

Moreover gross mean quantities, useful in the analysis developed in what follows, are:

• $u = n/n_M$ is the dimensionless density referred to the maximum density $n_M$;

• $q$ is the dimensionless linear mean flux referred to the maximum admissible mean flux $q_M = n_M v_M$.

The above quantities have to be regarded as functions of time and space with values in the interval $[0, 1]$:

$$u = u(t, x)\,, \quad q = q(t, x)\,, \qquad x \in [0, 1]\,, \qquad u \in [0, 1]\,, \qquad q \in [0, 1[\,,$$

while $V$ can be larger than 1 considering that isolated vehicles can attain a velocity larger than $v_M$. This means that there exist a limit velocity such that

$$V \in [0, 1+\mu], \quad \mu > 0, \tag{1}$$

where $\mu$ is a parameter to be defined from experimental data.

As known, the overall distribution of vehicles in mathematical kinetic theory [29], is described by the distribution function:

$$f = f(t, x, V) : \quad \mathbb{R}_+ \times [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+ \,, \tag{2}$$

such that $f dx dV$ is the number of vehicles which at the time $t$ are in the element of the state space $[x, x+dx] \times [V, V+dV]$.

Macroscopic observable quantities can be obtained, under suitable integrability assumptions, as momenta of the distribution $f$, normalized with respect to the maximum density $n_M$ so that all the derived variables such as density, flux etc. will be given in a dimensionless form. Specifically, the dimensionless **local density** is given by

$$u(t, x) = \int_0^{1+\mu} f(t, x, V)\, dV \,, \tag{3}$$

and the **flux of vehicles** at the time $t$ is given by

$$q(t, x) = \int_0^{1+\mu} V\, f(t, x, V)\, dV \,, \tag{4}$$

while the dimensionless mean velocity is:

$$v(t, x) = \frac{q(t, x)}{u(t, x)} \,. \tag{5}$$

An experimental information which is useful for the modelling developed in what follows refers to the experimental identification of the dimensionless equilibrium velocity which depends, in steady space uniform flow conditions, on the local density, [8]. Specifically, the following expression for the equilibrium mean velocity $v_e(u)$ can be used:

$$v_e = v_e(u) = \exp\left\{ -\alpha\, \frac{u}{1-u} \right\} \,, \tag{6}$$

where $\alpha$ is a constant depending on the road and environmental conditions. The identification proposed in [8] shows that $\alpha$ takes values in the range $[1, 3]$, where lower values correspond to favorable flow conditions: drivers tend to keep high speeds even when the density increases. On the other hand, difficult road and weather conditions oblige to reduce the velocity on a relatively larger amount when the density increases.

An oversimplification of the above model is the following:

$$v_e = v_e(u) = 1 - u\,, \tag{7}$$

which is supposed to provide a rough description of dependence of $v_e$ on the density.

Of course, the maximum of the distribution function defines the most probable velocity $V_p$, and the distance between $V_p$ and the mean velocity $v_e$ is an interesting information, as critically analyzed later in this paper; nevertheless the difficulty of setting experiments does not allow to assess it precisely.

### *2.2 Discrete velocity distribution function*

As already mentioned in Section 1, this paper deals with the modelling in the framework of the discrete mathematical theory. This means that the velocity variable can attain a finite number of velocities. The discretization dealt with in what follows uses a fixed number of velocities with a size depending on the local density. In particular, the size of each velocity step decreases with increasing density in a way that when $u \to 1$, than all velocities tend to zero.

Bearing this in mind, consider the following discretization of the velocity variable:

$$I_V = \{V_1 = 0\,, \ldots\,, V_i\,, \ldots\,, V_n = v_e\,, \ldots\,, V_{2n-1} = (1+\mu)v_e\}\,, \tag{8}$$

where $v_e$ is the dimensionless mean velocity which may be delivered by Model (6), or even (7).

Assuming, for the sake of simplicity, $\mu = 1$, we get a symmetric grid which will be used in the sequel:

$$I_V = \{V_1 = 0, \ldots, V_i, \ldots, V_n = v_e, \ldots, V_{2n-1} = 2v_e\},\; V_i = \frac{i-1}{n-1}v_e(u)\,. \tag{9}$$

The corresponding discrete representation is obtained by linking to each $V_i$

the discrete distribution functions:

$$f_i = f_i(t,x) : \quad \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+ \qquad i = 1, \ldots, 2n-1 . \tag{10}$$

**Remark 1** *The above discretization naturally implies that the number of vehicles with a velocity larger than $(1+\mu)v_e$ can be disregarded. In other words, it is technically assumed that the presence of vehicles with velocity much larger than the maximum mean velocity corresponding to the given density is negligible.*

**Remark 2** *In view of the dependence of $v_e = V_n$ on the density $u$, the continuous velocity interval is discretized in steps of different size depending on the density. The number of nodes is fixed, and the grid width stretches at low density while, for large density, it shrinks as $v_e$ goes to zero.*

According to the above discretization, the following gross quantities are obtained by weighted sums:

$$u(t,x) = \sum_{i=1}^{2n-1} f_i(t,x) , \tag{11}$$

and

$$q(t,x) = \sum_{i=1}^{2n-1} V_i f_i(t,x) , \qquad v(t,x) = \frac{q(t,x)}{u(t,x)} . \tag{12}$$

We also define, following the gas kinetic theory, two quantities that can give important information about the vehicles flux: the temperature (i.e. the variance of velocities) and the H function.

$$\Theta(t,x) = \sigma(t,x) = \frac{1}{u(t,x)} \sum_{i=1}^{2n-1} (V_i - v(t,x))^2 f_i(t,x). \tag{13}$$

$$H(t,x) = \sum_{i=1}^{2n-1} f_i(t,x) \log f_i(t,x). \tag{14}$$

The above discretization, as it will be discussed later in the paper, can be technically modified assuming that $v_M$ depends not only on the density, but also on the local space gradients as suggested in [18].

## 3 Models for Steady Uniform Flow Conditions

This section deals with the modelling of traffic flow phenomena for flow conditions such that the space gradients of the number density $u = u(t, x)$ are negligible.

In general, the model consists in a set of evolution equations for the densities $f_i$ linked to the velocities $V_i$. The mathematical structure of the evolution equations in the spatially homogeneous case, is, according to [9], the following:

$$\frac{df_i}{dt} = \sum_{h=1}^{2n-1} \sum_{k=1}^{2n-1} \eta_{hk} A^i_{hk} f_h f_k - f_i \sum_{k=1}^{2n-1} \eta_{ik} f_k \,, \tag{15}$$

for $i = 1, \ldots, 2n-1$, where:

• $\eta_{hk}$ is the encounter rate (number of interactions per unit time) of vehicles with velocities $V_h$ and $V_k$;

• $A^i_{hk}$ is the probability density that a vehicle with velocity $V_h$, **the test vehicle**, reaches the velocity $V_i$ after the interaction with the vehicle with velocity $V_k$, the **field vehicle**.

The above terms have the property of a discrete probability density:

$$\sum_{i=1}^{2n-1} A^i_{hk} = 1 \,, \qquad \forall\, h, k = 1, \ldots, 2n-1 \,. \tag{16}$$

Reversibility, which is a typical feature of classical particles, is not here claimed. Indeed the output of the interaction depends on the ability of drivers to organize the dynamics of the vehicle.

The derivation of the model, consistently with the above framework, means *modelling the encounter rates and the transition probability densities according with specific phenomenological behavior of the system* we are dealing with.

Modelling the encounter rate is a matter of mechanical calculations. This quantity depends on the relative velocity though a dimensional parameter $\gamma$:

$$\eta_{hk} = \gamma |V_h - V_k| \,, \tag{17}$$

and we choose for technical calculations $\gamma = 1$.

On the other hand, the modelling of the terms $A^i_{hk}$ which, according to the terminology proposed in [9], define the **table of games**, needs a mathematical interpretation of the microscopical phenomenology of the system. The modelling proposed in this section is based on the idea that the terms $A^i_{hk}$ may depend on the distribution functions $f_i$ through the local number density. In fact, drivers adjust the dynamics of the vehicle according to the local density conditions. The specific assumptions which generate the model are the following:

**Assumption 3.1** *Interactions modify the velocity of the test and field vehicles only if $V_h$ and $V_k$ are sufficiently close:*

$$A^i_{hk} = 0 \qquad if \quad |h-k| > 1\,; \qquad i = 1, .., 2n-1\,. \tag{18}$$

**Assumption 3.2** *The test vehicle can modify its velocity only by jumping to a neighboring velocity value.*

$$A^i_{hk} = 0 \quad if \quad |i-h| > 1 \quad for \quad k = h-1\,,\, h\,,\, h+1\,. \tag{19}$$

According to the above stated assumptions, we are lead to consider only the non null terms of matrices $A^i_{hk}$,

$$A^{i=h-1}_{h\,h-1}\,,\; A^{i=h}_{h\,h-1},\; A^{i=h+1}_{h\,h-1}\,,\; A^{i=h+1}_{h\,h+1}\,,\; A^{i=h}_{h\,h+1}\,,\; A^{i=h-1}_{h\,h+1}\,. \tag{20}$$

**Assumption 3.3** *We distinguish the case $h < n$ and $h > n$. In fact, when $h < n$, the test vehicle is slower than the equilibrium velocity and the driver has tendency to increase his speed when he/she interacts with a "fast" vehicle, while no changes occurs if he/she interacts with a slower vehicle. The opposite trend (acceleration) is expected when the test vehicle, faster than the equilibrium velocity, $h > n$, interacts with a slower vehicle.*

According to the above notations, the following **table of games** provides a mathematical interpretation of the phenomenology of the system:

$$
\begin{array}{llr}
\text{if } h \leq n \text{ and } k = h+1 & A_{hh+1}^{h+1} = \varepsilon_a \quad A_{hh+1}^{h} = 1 - \varepsilon_a\,, & \\[2ex]
\text{if } h \geq n \text{ and } k = h-1 & A_{hh-1}^{h-1} = \varepsilon_d \quad A_{hh-1}^{h} = 1 - \varepsilon_d\,, & \\
 & & (21) \\
\text{if} \quad |h-k| > 1 \quad \text{or} \quad k = h & A_{hk}^{h} = 1, & \\[2ex]
\text{otherwise} & A_{hk}^{i} = 0, &
\end{array}
$$

for $(i, h = 1, \ldots, 2n-1\,;\ k = h-1\,,\ h\,,\ h+1)$.

We distinguish between the accelerating probability $\varepsilon_a$ and the slowing down probability $\varepsilon_d$. These coefficients represent the reactivity of vehicles to different events and in principle may assume different values, if one of the events is more effective, e.g. braking usually has a shorter reaction time with respect to acceleration. It is useful to rewrite them as

$$
\varepsilon_a = \varepsilon \quad \text{and} \quad \varepsilon_b = \nu\,\varepsilon, \tag{22}
$$

to identify a basic reaction probability $\varepsilon$ and a measure of its asymmetry $\nu$.

The above assumption models the fact that slow vehicles have a trend to reach the mean velocity when "motivated" by fast vehicles, while the opposite behavior is observed for fast vehicles which are "motivated" to decelerate by the presence of slow vehicles. The interaction is supposed to be effective only when the velocity distance is not too large.

It is worth stressing that, due to the adaptive grid, interactions are related to the local density considering that the velocity distribution is compressed for increasing density.

The above model is identified only by five parameters: $\mu$, $\gamma$, $n$, $\varepsilon$ and $\nu$.

• $\mu$ as defined in (1) is a parameter that identifies the maximum velocity of the single vehicle, an can be directely identified from experimental data. In the sequel, we will choose $\mu = 1$.

• $\gamma$ as in (17) is a dimensional parameter which will be included in the time variable and set equal to one.

• $n$ defines the number of nodes.

• $\varepsilon$ corresponds to the tendency of the driver to modify his/her velocity when interacting with a vehicle with a velocity in the above defined interacting velocity range.

• $\nu$ takes into account the different tendency in accelerating or braking of driver. We will see in the sequel that the choice of $\nu$ heavily affects the shape of the equilibrium distribution. This is the reason we can suggest some requirements (inequalities) for $\nu$ to be satisfied.

Following the remarks of Section 5, we claim that suitable experimental information on the most probable velocity provides estimates useful for the identification of parameters $n$, $\varepsilon$ and $\nu$.

Assumptions 3.1 and 3.2 and Eq. (17) allow us to rewrite Eq. (15) as follows:

$$\frac{df_i}{dt} = \sum_{h=i-1}^{i+1} \sum_{k=h-1}^{h+1} \eta_{hk} A^i_{hk} f_h f_k - f_i \sum_{k=h-1}^{h+1} \eta_{ik} f_k \,, \tag{23}$$

for $i = 1, \ldots, 2n-1$

**Remark 3** *It worth pointing out that the parameter $n$ defining the number of nodes is not a mere discretization parameter since, as stated in the introduction, the discretization not only reduces the computational complexity but it is justified from the modelling viewpoint.*
*In particular, $n$ defines a velocity range of sensitivity, in which interactions are effective. Indeed, bearing in mind the adaptative grid (9), Assumptions 3.1 and 3.2, $n$ defines both the interacting velocity ranges to which the drivers are sensitive to changing their velocity, and the width of the velocity jumps of vehicles when braking or accelerating. Therefore the above model is consistent only if $n$ is finite and not too large.*

## 4 Models for Unsteady Flow Conditions

The modelling method proposed in Section 3 can be technically generalized to the analysis of models in the presence of density gradients. In this case the reference framework, still according to [9], is the following:

$$\frac{\partial f_i}{\partial t} + \frac{\partial}{\partial x}(V_i\, f_i) = \sum_{h=1}^{2n-1} \sum_{k=1}^{2n-1} \eta_{hk} A^i_{hk} f_h f_k - f_i \sum_{k=1}^{2n-1} \eta_{ik} f_k \,, \tag{24}$$

for $i = 1, \ldots, 2n-1$.

This is a nonhomogeneous system of hyperbolic first order equations with a quadratic right-hand-side term in the unknowns. Moreover, since the velocity $V_i$ depends on $u$ following (9), i.e. on the $f_i$, the left term is non linear too. Interactions between vehicles are modelled both explicitly by the table of games and implicitly by the empirical equation $v_e(u)$, which instantaneously adapts the velocity grid to the density.

Choosing, as a particular case, to deal with the relatively simpler model

$$v_e(u) = 1 - u, \tag{25}$$

the system particularizes as follows:

$$\begin{gathered}\frac{\partial f_i}{\partial t} + \frac{\partial}{\partial x}\left(\frac{i-1}{n-1}\left(1 - \sum_{i=1}^{2n-1} f_i\right) f_i\right) = \\ \sum_{h=1}^{2n-1}\sum_{k=1}^{2n-1} \eta_{hk} A_{hk}^i f_h f_k - f_i \sum_{k=1}^{2n-1} \eta_{ik} f_k,\end{gathered} \tag{26}$$

for $i = 1 \dots 2n-1$.

Equation (26) can be rewritten in vector nonconservative as follows

$$\frac{\partial \mathbf{f}}{\partial t} + \mathbf{A}(u)\frac{\partial \mathbf{f}}{\partial x} = \mathbf{R}[\mathbf{f}], \tag{27}$$

where

$$\mathbf{f} = (f_i)\,,$$

$$\mathbf{A}(u) = \left(\frac{\partial q_i}{\partial f_l}\right) = \frac{d\,V_i(u)}{d\,u} f_i + V_i \delta_{il},$$

and

$$\mathbf{R}[\mathbf{f}] = \sum_{h=1}^{2n-1}\sum_{k=1}^{2n-1} \eta_{hk} A_{hk}^i f_h f_k - f_i \sum_{k=1}^{2n-1} \eta_{ik} f_k.$$

Equation (27) has positive characteristic velocities, the eigenvalues of $\mathbf{A}(u)$, which control the velocity propagation of the perturbations. The total mass, i.e. the number of vehicles $n$, is conserved.

## 5 Qualitative and Computational Analysis of the Model for Steady Uniform Flow Conditions

This section deals with a qualitative analysis of the initial value problem for the discrete velocity traffic flow model proposed in Section 3. The analysis is first finalized to show the wellposedness of the initial value problem. Then, in some simple case, equilibrium solutions are found and their stability properties partially analyzed. Finally, the numerical approach highlights the trend of solutions to equilibrium and their shape for different densities.

### *5.1 Qualitative Analysis*

The mathematical statement of the problem is as follows:

$$\begin{cases} \dfrac{df_i}{dt} = J_i[f] \\ \\ \displaystyle\sum_{i=1}^{2n-1} f_{i0} = u_0 \quad \text{with} \quad f_{i0} = f_i(t=0)\,, \end{cases} \tag{28}$$

where $J_i$ is defined by the right-hand-side term of Eq. (23), and the transition densities $A^i_{hk}$ are defined by the table of games reported in (21).

It is useful to introduce the following scaled variables:

$$g_i(t) = \frac{1}{u_0} f_i(t)\,, \qquad g_{i0} = \frac{1}{u_0} f_{i0}\,. \tag{29}$$

In this way, the system (28) writes:

$$\begin{cases} \dfrac{dg_i}{dt} + g_i L_i[g] = Q_i[g] \\ \\ \displaystyle\sum_{i=1}^{2n-1} g_{i0} = 1 \quad \text{with} \quad g_{i0} = g_i(t=0)\,, \end{cases} \tag{30}$$

where

$$Q_i[g] = \sum_{h=i-1}^{i+1} \sum_{k=h-1}^{n+1} \eta_{hk} A^i_{hk} g_h(t) g_k(t)\,, \tag{31}$$

and

$$L_i[g](t) = \sum_{k=1}^{2n-1} \eta_{ik} g_k(t)\,. \tag{32}$$

The well posedness is assured by the following Theorem:

**Theorem 4** *The solution $g(t) = (g_1(t), \ldots, g_{2n-1}(t))$ of the initial value problem* (30) *(satisfying (16)) exists, is unique, and is positive defined, for all $t \in [0, +\infty)$, for any given set $\{g_{i0}\}_{i=1,\ldots,2n-1}$ with $g_{i0} \geq 0$. Moreover, solutions are continuously dependent on the initial conditions.*

*In particular, one has*

$$\forall t \geq 0 : \qquad g_i(t) \geq 0 \quad \textit{for any } i = 1, \ldots, n \quad \textit{and} \quad \sum_{i=1}^{2n-1} g_i(t) = 1\,. \tag{33}$$

**Proof:** Using the same reasoning of [9], the right-hand-side in System (30) is of class $C^1$ (actually, $C^\infty$) on $\mathbb{R}^n$, and hence locally Lipschitz. This guarantees existence and uniqueness of solutions, $\{g_i(t)\}_{i=1,..,2n-1} \in C^\infty(\mathbb{R}^n, [0, t^*))$ of the Cauchy problem, as well as their continuous dependence on the initial data, for some $t^* > 0$. Moreover, the *a priori* estimates (33), that result from direct computation using the hypotheses on the initial conditions, imply that $|g_i(t)|$ is bounded for any $t > 0$ ($i = 1, .., 2n-1$). This is sufficient in order the problem (30) to be globally well posed. ∎

Theorem 4 makes us confident in using numerical schemes on our model.

On the other hand, a crucial point in the qualitative analysis of the model is to look at its equilibrium points, if any, and their stability properties. In steady uniform flow conditions, the evolution of the "vector" distribution function $\mathbf{g} = (g_i, .., g_{2n-1})$ corresponding to a prescribed initial data $\mathbf{g}_0 = (g_{0_1}, .., g_{0_{2n-1}})$ can be regarded as the flow $\mathbf{g}(\mathbf{g}_0, t)$ of the autonomous dynamical system generated by problem (30), where the normalization condition in $(30)_2$ reduces the number of unknowns to $2n-2$. The equilibrium points are solutions to:

$$J_i[g] = Q_i[g] - g_i L_i[g] = 0 \qquad i = 1, .., 2n-1 \tag{34}$$

In view of the difficulty to treat the general case with $n$ arbitrary we consider some particular value of the discretization index, starting with $n = 2$. In this case equation (15) reads:

$$
\begin{cases}
\dfrac{dg_1}{dt} = V_M(\varepsilon_d - \varepsilon_a) g_1 g_2 \\[2ex]
\dfrac{dg_2}{dt} = V_M(\varepsilon_a - \varepsilon_d) g_2 (g_1 - g_3) \\[2ex]
\dfrac{dg_3}{dt} = -V_M(\varepsilon_d - \varepsilon_a) g_2 g_3.
\end{cases}
\tag{35}
$$

If $\varepsilon_d = \varepsilon_a$, then any "position" $P = (g_1^*, g_2^*, g_3^*)$ satisfying condition $(30)_2$ is an equilibrium point that, consequently, is trivially stable.

If $\varepsilon_d \neq \varepsilon_a$ we have two equilibrium solutions, $P_0 = (0,1,0)$ and $P_1 = (a, 0, 1 - a)$, $0 < a < 1$. We limit ourselves to investigate the stability of such solutions with respect to "small" perturbations. Looking at the eigenvalues of the jacobian matrix:

$$
\mathbf{J} = V_M(\varepsilon_d - \varepsilon_a) \begin{pmatrix} g_2 & g_1 & 0 \\ -g_2 & g_3 - g_1 & g_2 \\ 0 & -g_3 & -g_2 \end{pmatrix},
$$

we deduce that $P_0$ is unstable, while $P_1$ is stable provided:

$$
(2a-1)(\varepsilon_a - \varepsilon_d) < 0,
$$

that is, provided the velocity distribution is asymmetric, with a prevalence of slow vehicles if $\varepsilon_a < \varepsilon_d$, and a prevalence of fast vehicles in the opposite case $\varepsilon_a > \varepsilon_d$. It is worth to remark that since $P_1$ is not an isolated, but a line of, equilibrium points ($\mathbf{J}$ has in $P_1$ a multiplicity-2 zero eigenvalue), its stability is, strictly speaking, marginal.

The picture is a bit more rich when $n = 3$. In this case (15) becomes:

$$\begin{cases} \dfrac{dg_1}{dt} = -\dfrac{1}{2} V_M\, \varepsilon_a\, g_1\, g_2 \\[2ex] \dfrac{dg_2}{dt} = \dfrac{1}{2} V_M\, g_2\, (\varepsilon_a\, (g_1 - g_3) + \varepsilon_d\, g_3) \\[2ex] \dfrac{dg_3}{dt} = \dfrac{1}{2} V_M\, (\varepsilon_a - \varepsilon_d)\, g_3\, (g_2 - g_4), \\[2ex] \dfrac{dg_4}{dt} = \dfrac{1}{2} V_M\, g_4\, \left(\varepsilon_a\, g_3 + \varepsilon_d\, (-g_3 + g_5)\right), \\[2ex] \dfrac{dg_5}{dt} = -\dfrac{1}{2} V_M\, \varepsilon_d\, g_4\, g_5. \end{cases} \tag{36}$$

The results, obtained in a way similar as outlined above, are summarized in the following way:

*Let $n = 3$. If $\varepsilon_a \neq \varepsilon_d$, the equation (34) has solutions:*

- $P_0 = (a, 0, 1 - a - b, 0, b)$
- $P_1 = (0, c, 0, 1 - c, 0)$
- $P_2 = (0, d, 0, 0, 1 - d)$
- $P_3 = (e, 0, 0, 1 - e, 0)$.

*with $a, b > 0$, $0 < a + b < 1$, $0 < c < 1$, $0 < d < 1$ and $0 < e < 1$. These equilibrium solutions are unstable except $P_1$, which is linearly (marginally) stable provided $(2c - 1)(\varepsilon_a - \varepsilon_d) < 0$, (cf. the case $n = 2$). Otherwise, if $\varepsilon_a = \varepsilon_d$, the equilibrium solutions are:*

- $P_0 = (a, 0, g_3^0, 0, 1 - a)$
- $P_1 = (0, c, g_3^0, 1 - c, 0)$
- $P_2 = (0, d, g_3^0, 0, 1 - d)$
- $P_3 = (e, 0, g_3^0, 1 - e, 0)$.

*with $f_3^0, a, c, d, e > 0$, $0 < a + g_3^0 < 1$, $0 < c + g_3^0 < 1$, $0 < d + g_3^0 < 1$ and $0 < e + g_3^0 < 1$. As before, all these solutions are unstable except $P_1$, which is linearly unconditionally marginally stable.*

As $n$ increases, the problem becomes formally more and more involved. However, the general behavior outlined in the simple cases $n = 2$ and $n = 3$ is preserved. In particular, we still observe concentration toward central values, alternation of vanishing and non vanishing values, asymmetry (in the case

$\varepsilon_a \neq \varepsilon_d$) as well as, when $\varepsilon_a = \varepsilon_d$, the conservation in time on the "central" component $g_n$. This aspect, as well as the general behavior depicted in this section, will be actually observed in the numerical simulations.

### *5.2 Computational Analysis*

Based upon the above analytical background, some simulations can be developed to visualize the model's trend toward equilibrium distribution.

Simulations are developed with the aim of showing that the stationary picture expected from the model is similar to the experimentally observed one. We notice that the system actually tends to reach an equilibrium distribution. As we have seen, asymptotic states do not depend on $\varepsilon$, but on $\nu$. Moreover, these equilibrium solutions are not isolated points in the space of solutions, but regions of points of some hyperplanes. Indeed, there are some "degrees of freedom" in the system, where the dependence of the solution on the initial condition remains: the system keeps a memory of its initial conditions.

The main features of these equilibria are concentration toward the central values and alternation between zero and nonzero values. This feature, which comes directly from Assumptions 3.1, is physically expected and seems to mimic the physical phenomenon of packing of vehicles with similar velocity.

Specifically, we consider initial conditions which are uniform with respect to the discrete velocity variable:

$$g_{i0} = \frac{1}{2n-1}\,. \tag{37}$$

The result of simulations is shown in Figures 1 and 2, visualizing the asymptotic (in time) distribution and the initial uniform distribution. We notice that if $\nu \neq 1$ i.e. the reactivity is asymmetric, there is a trend to asymmetry, which eventually superposes to the asymmetry of the initial data.

It is worth to remark that, in the simulations, the equilibrium solution corresponds to a minimum of the functions $H$ and $\Theta$ defined in (13) and (14).

We should also remark that the most probable velocity $V_P$ is assumed in $V_{n-1}$ or $V_{n+1}$, depending on the value of $\nu$. This fact, joined with the meaning of $n$ (see Remark 3), could be used to identify the parameter by experimental data showing the above feature.

It is interesting pointing out that the discretization is such that the numerical equilibrium distribution has the same shape for all densities $u_0$, while it is

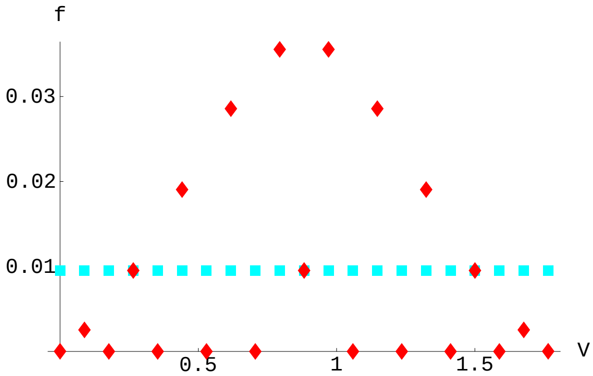


Fig. 1. Initial uniform (▮) and equilibrium (◇) discrete distributions for $\nu = 1$.

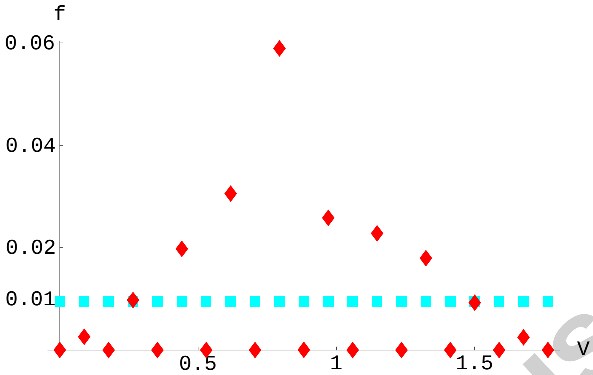


Fig. 2. Initial uniform (▮) and equilibrium (◇) discrete distributions for $\nu = 1.1$.

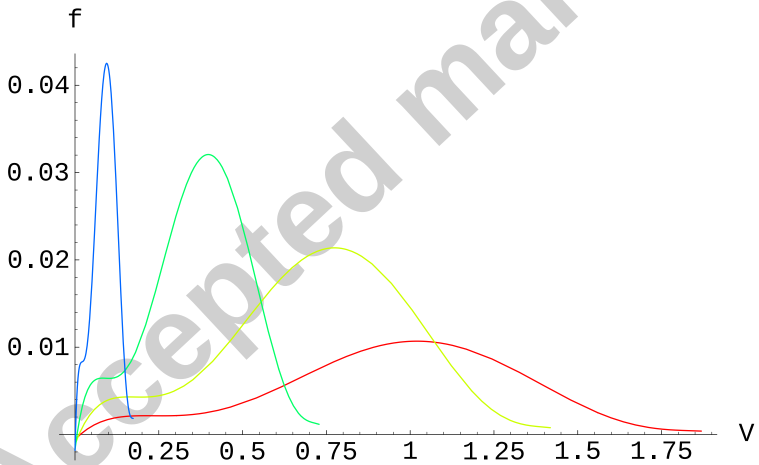


Fig. 3. Polynomial fit of the equilibrium discrete distributions for different densities: $u_0 = .05$, $u_0 = .1$, $u_0 = .3$, $u_0 = .5$.

compressed toward small velocities when the density increases. This is visualized in Figure 3, which shows a polynomial fit of the discrete equilibrium distribution.

As visualized in Figures 4 and 5, we also get a good agreement with the qualitative picture of the velocity and fundamental diagrams, [4], describing the mean velocity and the flux as functions of density.

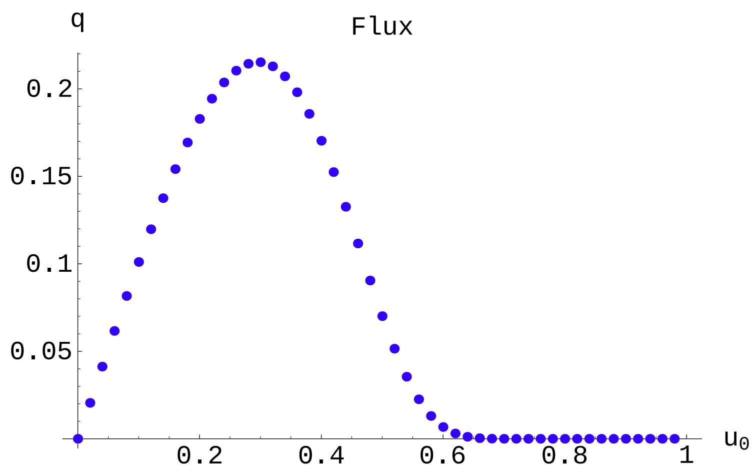


Fig. 4. Equilibrium flux versus density. $n = 11$, $\nu = .9$.

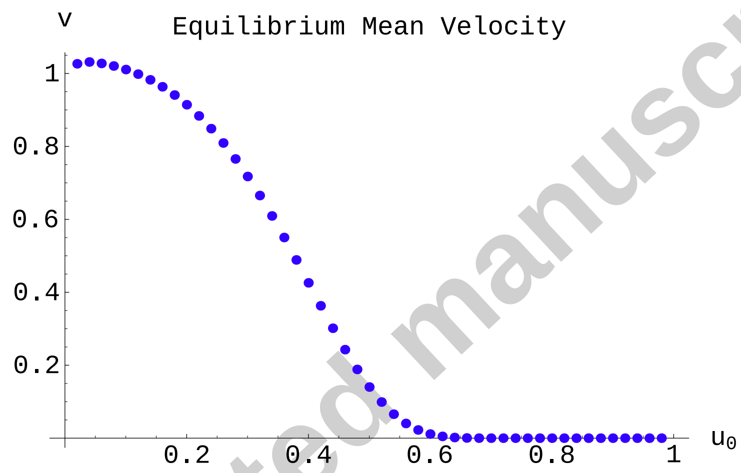


Fig. 5. Equilibrium velocity versus density. $n = 11$, $\nu = .9$.



## 6 Simulations for Unsteady Flow Conditions

We have already mentioned in Section 4 that solutions, in the general case, are characterized by a nonlinear propagation, leading to generation of shocks. This implies that the corresponding hyperbolic equation has to be solved using a method designed for conservation laws. As an example, in this Section we present some numerical simulations related to the evolution of the system from different specific initial conditions obtained by a first-order up–wind method.

We consider the initial value problem:

$$\begin{cases} \dfrac{\partial f_i}{\partial t} + \dfrac{\partial}{\partial x}(V_i f_i) = J[f] \\ \\ f_i(t=0,x) = f_{i0}(x) \\ \\ f_i(t,x=0) = f_i(t,x=1) \end{cases} \tag{38}$$

corresponding to vehicles on a ring road. According to three case studies we consider different initial conditions $f_{i0}(x)$ and thus different initial density and mean velocity profiles.

• *Evolution of a jam*

The jam is originated by an initial concentration of slow vehicles (initial condition). As time goes on, the jam damps and the density and mean velocity profiles become more uniform. This evolution of the density is showed in the 3D Figure 6.

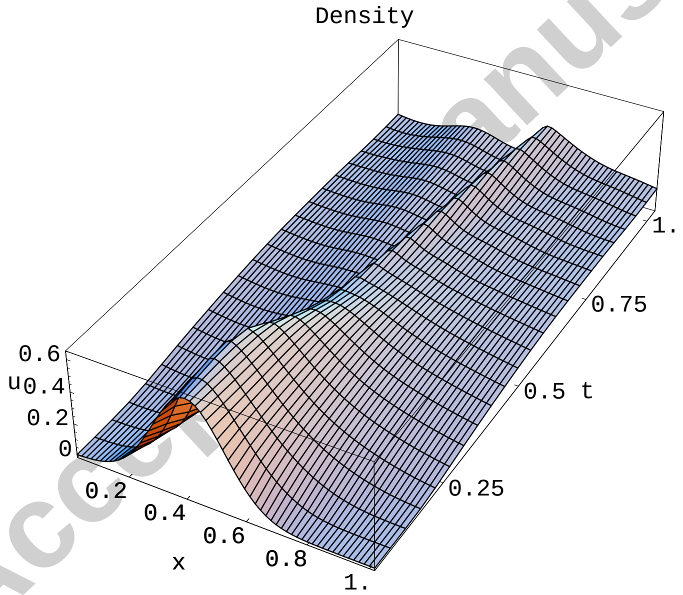


Fig. 6. Space and time evolution of a density perturbation: total density $u(x,t)$.

• *Braking of fast vehicles* A jam arises when quicker vehicles reach a cluster of slower ones and interact with them, as showed in Figure 7.

• *Vacuum formation* Vacuum arises between fast vehicles running ahead and slower ones left behind. The density graphs are showed in Figure 8.

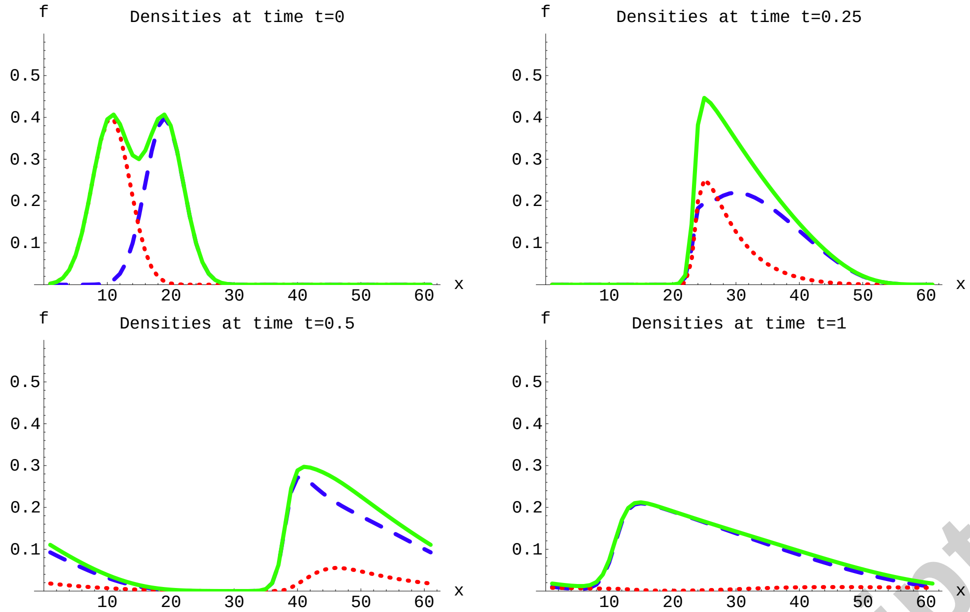


Fig. 7. Space evolution of total density (continuous line), quicker vehicles (dotted line) and slower ones (dashed line), at different times.

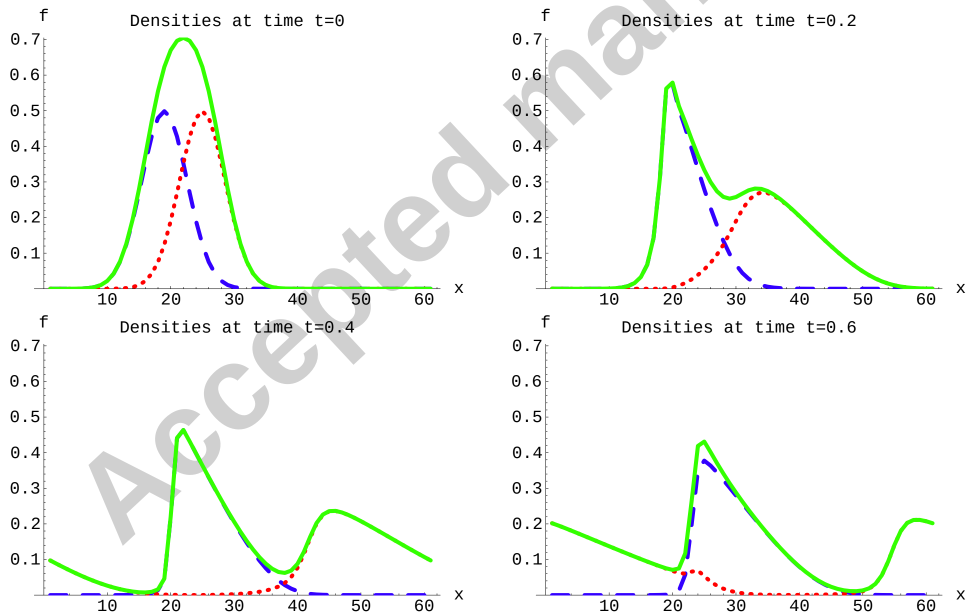


Fig. 8. Space evolution of total density (continuous line) of slower vehicles behind (dashed line) and quicker ones (dotted line) in front at different times. Vacuum occurs.

## 7 Conclusion and Perspectives

This paper developed a discrete kinetic theory approach to the modelling of vehicular traffic flow.

Before proposing a critical analysis of the contents of this paper it is worth stressing that the discrete velocity approach has not been developed with the aim of reducing computational complexity, but rather to improve the consistency of the kinetic theory approach to the modelling of the granular flow of vehicles. Indeed, as already observed by Daganzo [13], methods of the mathematical kinetic theory are designed to apply to systems with a large number of particles, while flow of vehicles involves a small number of particles: in this case the continuity of the distribution function cannot be claimed. Therefore, discretization of the velocity space can be regarded as a way to represent the system without needing artificial continuity assumptions, as well as to involve a relatively larger number of vehicles in the interactions scheme.

Several interesting problems are left open and can be regarded as challenging research perspectives.

From the qualitative viewpoint, it might be interesting to develop more powerful tools of analysis, which could be used in general in the class of discrete statistical models. We mostly think to equilibrium stability methods from classical dynamics and to a deeper understanding, guided by the mechanical kinetic theory [11], of the role of $\Theta$ and $H$ functions, as suggested from the physics of traffic [22]. Moreover we have to remember, in order to get a true insight into the physics, the importance of parameter identification.

¿From the modelling viewpoint, the car-driver system cannot be regarded as a classical particle. Adopting a commonly used terminology [6], it should rather be regarded as an *active particle*, where the skill of the drivers modifies the laws of classical mechanics. This feature has been somehow taken into account considering that the table of games describes a behavior which does not derive from mechanical interactions. On the other hand all microscopic entities have been assumed to behave in the same manner, that does not correspond to physical reality where different types of vehicles circulate (from trucks to sport cars), while the driver may be aggressive or shy [13]. This would lead to allow in our model more than one population, each with its own properties.

Active particles may be more properly characterized by a microscopic state which includes, in addition to geometrical and mechanical variables as position and velocity, also an additional variable, called *activity*, related to the ability of particles to organize their dynamics. This approach has been already developed in various fields of applied sciences, say social dynamics [9], immunology [6], living fluid dynamics [31], while various other applications are briefly reported in [6].

An interesting research perspective consists in developing this mathematical approach also in the case of traffic flow modelling. It means introducing an additional microscopic variable related to the specific characteristic of the driver-vehicle system. Then, the design of the tables of games must also refer to the above variable. Hopefully, the analysis already developed in this paper should be regarded as a useful basis to deal with the above outlined perspective.

An additional improvement can be proposed by introducing, according to [18], the concept of apparent density related to the actual density $u$ and its gradients by the phenomenological model proposed in [18]. This model simulates the behavior of a driver who feels an effective density larger than the real one in the presence of positive gradients, while the behavior corresponds to a relatively smaller density in the presence of negative gradients.

In conclusion, the model proposed in this paper can be read as a first effective implementation of the discrete kinetic framework proposed in [17]. We have showed that it is able to catch several features of the traffic phenomena and this observation encourages us to develop this approach following the above guiding lines.

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