ON THE MATHIEU GROUPS M_{22} AND M_{11}

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1. Introduction

In the list of known finite non-abelian simple groups there are infinitely many pairs of non-isomorphic simple groups which have the same order. The smallest known example of two such groups are the simple groups A_8 and $PSL(3, 4)$, of order 20,160.

It should be of interest to investigate the following general problem: "Are there any non-isomorphic non-abelian simple groups of the same order, other than the ones already known?"

This question has been answered for some particular orders; for example, see the papers by R. Brauer [4], T. M. Gagen [9] and R. Stanton [19].

Until recently, the five Mathieu groups were the only known finite non-abelian simple groups which did not fit into an infinite series. These groups have other interesting properties as well, and have therefore been the subject of a number of papers, especially in recent years. For example the group M_{11} has been characterized by R. Brauer [3] and W. Wong [20], the group M_{12} by Wong [21], the groups M_{22} and M_{23} by Z. Janko [15], and M_{24} will be characterized in a forthcoming paper of D. Held [14]. From the point of view of the general problem above, R. Stanton [19] has shown that the groups M_{24} and M_{12} are uniquely determined by their order. This problem for the group M_{23} is solved in a forthcoming paper by A. Bryce [7], while the present paper considers the problem for the two remaining groups M_{11} and M_{22}. In this paper we prove:

THEOREM A. *Let G be a finite non-abelian simple group of the order of* M_{22}; *i.e. of the order* 443,520. *Then G is isomorphic to* M_{22}.

THEOREM B. *Let G be a finite non-abelian simple group of the order of* M_{11}; *i.e. of order* 7920. *Then G is isomorphic to* M_{11}.

2. Notation and known results

Throughout this paper we use the notation of Janko [15]. Further G_p will denote a Sylow p-subgroup of a group G. By the word "character" we

will mean an "ordinary irreducible (complex) character", unless otherwise stated. If p^a is the maximal power of a prime p which divides the order $|G|$ of a group G, we write $p^a \top |G|$. Finally, p and q will denote distinct prime numbers throughout the paper.

The first part of the proof of both theorems relies heavily on the results of R. Brauer, H. F. Tuan and R. Stanton in the field of modular representations. We now state some of these results which will be used throughout this paper. (For a definition of the terms used below see [8], [1] or [2].)

RESULT 1 ([6], Lemma 3). *Let G be a finite group of order g, where $g = p^a q^b g^*$, $(pq, g^*) = 1$ and $a, b \geq 1$. Assume G has no elements of order pq. Then for every p-singular element x of G,*

$$\sum z_\mu \zeta_\mu(x) \equiv 0 \pmod{q^b}$$

where the sum extends over all characters ζ_μ belonging to a fixed p-block $B_\sigma(p)$ and to a fixed q-block $B_\tau(q)$. Here $z_\mu = \zeta_\mu(1)$.

Suppose now that for a finite group G we have $p \top |G|$. We write $C_G(G_p) = G_p \times V_p$. If V_p has l conjugate classes in the group $N_G(G_p)$, then G has l p-blocks of defect 1 (see [2]). Let t denote the number of conjugate classes of elements of order p in G.

To each of the l p-blocks $B_\lambda(p)$ of defect 1 there corresponds a certain multiple t_λ of t, where $t_\lambda | p - 1$, so that $B_\lambda(p)$ has $p - 1/t_\lambda$ characters ζ_μ which are p-conjugate only to themselves and one exceptional family of t_λ p-conjugate characters.

We let 1_G denote the principal character of G and $B_1(p)$ denote the p-block of G containing 1_G. This notation will be kept fixed throughout the paper.

RESULT 2 ([2], Theorem 11). *For the block $B_1(p)$, we have $t_1 = t$. The degrees z_μ of the characters ζ_μ of $B_1(p)$ satisfy:*

$$z_\mu \equiv \delta_\mu = \pm 1 \pmod{p}, \quad \text{if } \zeta_\mu \text{ is } p\text{-conjugate only to itself.}$$
$$z_\mu \equiv \delta_\mu/t = \pm 1/t \pmod{p}, \quad \text{otherwise.}$$

If $\zeta_1, \ldots, \zeta_j, \zeta_{j+1}$ are representatives from the different families of p-conjugate characters of $B_1(p)$ then

(1) $$1 + \delta_2 z_2 + \ldots + \delta_{j+1} z_{j+1} = 0, \qquad \left(j = \frac{p-1}{t} \right).$$

From this theorem and other results of [2] we get:

COROLLARY 1. *Suppose $p \top |G|$, $q^a \top |G|$, and G has elements of order pq. Then if χ is a character of G and $q^a | \chi(1)$, then $\chi \notin B_1(p)$.*

Let $p \top |G|$ and for each character ζ_μ of $B_1(p)$ let δ_μ be as defined in Result 2.

If ζ_μ is p-conjugate only to itself, we say that

or

$$\zeta_\mu \text{ is of type } 0 \text{ if } \delta_\mu = 1$$
$$\zeta_\mu \text{ is of type } 1 \text{ if } \delta_\mu = -1.$$

If ζ_μ belongs to the exceptional family of $B_1(p)$, then

$$\zeta_\mu \text{ is of type } 0 \text{ if } \delta_\mu = +1$$
$$\zeta_\mu \text{ is of type } 1 \text{ if } \delta_\mu = -1.$$

We can now state the "Block intersection theorem" of R. Stanton [19]:

RESULT 3. *Let G be a simple group such that $p \top |G|$, $q \top |G|$, and G has no elements of order pq. Let a_{ij} be the number of characters* in $B_1(p) \cap B_1(q)$ which are of type i for p and of type j for q $(i, j = 0, 1)$. Then*

$$a_{00} + a_{11} = a_{01} + a_{10}.$$

*If either the exceptional p-family or q-family (or both) occurs in $B_1(p) \cap B_1(q)$ then we count only one member of the family (or families), in the numbers a_{ij}, $(i, j = 0, 1)$. Note that no character can be "exceptional" for both p and q, and if a character is contained in $B_1(p)$ for any $p \mid |G|$, then so are all its algebraic conjugates ([19], Lemma 3).

3. Proof of theorem A

Throughout this section, G denotes a simple group of order $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$. By Sylow's theorem and Burnside's transfer theorem ([11], p. 203) we have the following possibilities for $N_G(G_{11})$:

$$|N_G(G_{11}) : C_G(G_{11})| = 2, 5, 10.$$
$$|G : N_G(G_{11})| = \text{(a) } 2^2 \cdot 3, \text{ (b) } 2^4 \cdot 3^2,$$
$$\text{(c) } 2^3 \cdot 7, \text{ (d) } 2^5 \cdot 7 \cdot 3,$$
$$\text{(e) } 2^7 \cdot 3^2 \cdot 7.$$

Result 2 is used to rule out the case $|N_G(G_{11}) : C_G(G_{11})| = 2$ immediately. In case (b) using Sylow's theorem G must have a subgroup of index 12. Case (a) and case (b) are now ruled out in the same way as A_{12} has no elements of order 77.

In cases (c), (d) G has elements of order 33. Using Sylow's theorem we see that if $C_G(G_{11}) = G_{11} \times V_{11}$ and $C_G(G_5) = G_5 \times V_5$ then $V_{11} \cap V_5 > \langle 1 \rangle$ in both cases. By R. Stanton ([19], Lemma 5) if $\zeta \in B_1(11)$, then

$$\zeta(1) = z \geqq 1 + 2 \cdot 5 \cdot 11 = 111.$$

Using this fact, Result 2, and the fact that $670^2 > |G|$ we get the following list of degrees of characters which could lie in $B_1(11) \cap B_1(5)$:

$$144 - (0, 1) \qquad\qquad 126 - (1, 0)$$
$$252 - (1, 1) \qquad\qquad 192 - (1, 1)$$
$$384 - (1, 1) \qquad\qquad 336 - (0, 0).$$

The numbers in brackets give the type of a character with that degree, where (i, j) stands for a character of type i for 11 and type j for 5.

By Result 3 $B_1(11) \cap B_1(5)$ has a character of degree 144 or 126. Both these degrees are divisible by $9 = 3^2$, in contradiction to Corollary 1, as G has elements of order 33.

LEMMA 1. *The Sylow* 11-*normalizer of* G *is a Frobenius group of order* 55.

It follows immediately that $B_1(11)$ has exactly two 11-conjugate characters of degree $z \equiv \pm 5 \pmod{11}$ and five characters ζ_μ $(\mu = 1, \ldots 5)$ with $\zeta_\mu(1) \equiv \pm 1 \pmod{11}$. All other characters of G have degrees divisible by 11. Also $11 \mid |G : N_G(G_p)|$ for $p = 2, 3, 5, 7$.

For $N_G(G_7)$ we get the following possibilities:

$$|N_G(G_7) : C_G(G_7)| = 2, 3, 6.$$
$$|G : N_G(G_7)| = (1)\ 2 \cdot 11, \qquad (2)\ 2^4 \cdot 11, \qquad (3)\ 2^7 \cdot 11,$$
$$(4)\ 2 \cdot 3 \cdot 5 \cdot 11,\ (5)\ 2^4 \cdot 3 \cdot 5 \cdot 11,$$
$$(6)\ 2^3 \cdot 3^2 \cdot 11,\ (7)\ 2^6 \cdot 3^2 \cdot 11,\qquad (8)\ 3^2 \cdot 11,$$
$$(9)\ 2^7 \cdot 3 \cdot 5 \cdot 11.$$

The case $|N_G(G_7) : C_G(G_7)| = 2$ is ruled out by Result 2 and by ([6], Lemma 1), and so cases (6), (7), (8) cannot occur.

If $|N_G(G_7) : C_G(G_7)| = 6$, $B_1(7)$ has seven characters ζ_μ $(\mu = 1, \ldots 7)$, all of which satisfy $\zeta_\mu(1) \equiv \pm 1 \pmod 7$. We give only an example of the methods used to rule out this case. For simplification, if ζ is a character of degree z, and if ζ belongs to $B_\tau(p)$, then we write $B_\tau(p) = \{z, \ldots\}$ i.e. we let the degree of a character stand for the character. If ζ has a number of p-conjugates $\zeta', \zeta'', \ldots$ in $B_\tau(p)$ we write

$$B_\tau(p) = \{z, z', z'', \ldots\}.$$

We now give a list of all possible degrees of characters which could occur in $B_1(11)$ and all degrees (of characters) which are congruent to $\pm 1 \pmod 5$.

TABLE I

12,	45,	56,	144,	210,	320	$\equiv +1 \pmod{11}$
10,	21,	32,	120,	252,	384, 560	$\equiv -1 \pmod{11}$
5,	16,	60,	126,	192,	280	$\equiv +5 \pmod{11}$
6,	28,	72,	105,	160,	336	$\equiv -5 \pmod{11}$

<div align="center">TABLE II</div>

11,	16,	21,	56,	66,	126,	176,	231,	336,	396,	616	$\equiv +1 \pmod 5$
44,	99,	144,	154,	264,	384						$\equiv -1 \pmod 5$

From table 1 and by Result 3 we get that $B_1(7) \cap B_1(11) = \{1, 120, \ldots\}$ or $\{1, 160, 160', \ldots\}$. By Corollary 1, G has no elements of order 35, and so we are in cases (4) or (5). We now consider the following cases:

$$B_1(5) \cap B_1(7) = \{1, 99\}, \quad B_1(5) \cap B_1(11) = \{1, 21\},$$
$$B_1(7) \cap B_1(11) = \{1, 120\} \text{ or } \{1, 160, 160'\}.$$

Under these assumptions if $\zeta \in B_1(5)$, then $\zeta(1) \equiv \pm 1 \pmod 5$ and by using table 2 and Result 2 we get:

(α)	$B_1(5) = \{1, 21, 99, \ 21, \ \ 56\},$	$1 + 21 - 99 + \ 21 + \ 56 = 0$
(β)	$B_1(5) = \{1, 21, 99, \ 66, \ \ 11\},$	$1 + 21 - 99 + \ 66 + \ 11 = 0$
(γ)	$B_1(5) = \{1, 21, 99, \ 99, \ 176\},$	$1 + 21 - 99 - \ 99 + 176 = 0$
(η)	$B_1(5) = \{1, 21, 99, 154, 231,\}$	$1 + 21 - 99 - 154 + 231 = 0.$

The equations on the right is the equation (1) of Result 2 for each of the four cases. By Corollary 1, as $B_1(5) \supseteq \{99\}$ for all four cases, G has no elements of order 15, so we apply Result 1 to $B_1(5)$ and $B_2(3)$, the second being the 3-block of defect 1 containing the character of degree 21. The cases (α), (β), (γ) are ruled out by Result 1, as

$$21 \not\equiv 0 \pmod 9, \quad 21 + 21 = 42 \not\equiv 0(9), \quad 21 + 66 = 87 \not\equiv 0(9),$$

and because all characters in 3-blocks of defect 1 have degrees divisible by 3, but not by 9 (see [1]). (Note that in case (η), $21 + 231 = 252 \equiv 0 \pmod 9$.)

CASE (η). Using Result 1, table 1, and the theory of blocks of defect 1 developed by R. Brauer (see [1]), we get three possibilities for $B_1(11)$:

$$B_1(11) = \{1, 21, \ \ 21, 21, \ldots\} \qquad (\eta_1)$$
$$B_1(11) = \{1, 21, \ \ 12, \ \ \ \ldots\} \qquad (\eta_2)$$
$$B_1(11) = \{1, 21, 210, \ \ \ \ldots\} \qquad (\eta_3)$$

By Result 1 and since $3 \mathop{\top} 120$, $B_1(11) \cap B_1(7) = \{1, 160, 160'\}$ in case (η_1). Equation (1) of Result 2 becomes:

$$1 - 21 - 21 - 21 + 160 + \delta_1 z_1 = \ \ 0,$$

hence
$$z_1 = 98,$$

where z_1 denote the degree of the remaining character of $B_1(11)$. Since $98 \mathop{\dagger} |G|$, case (η_1) cannot occur.

In case (η_2) using Result 2, the only possibility for equation (1) is: $1+12-21-120-32+160 = 0$. However this leads to a contradiction by Result 1, as $3 \top 120$.

In case (η_3) equation (1) becomes:

$$1-21+210+160-560+210 = 0, \text{ if } B_1(7) \cap B_1(11) = \{1, 160, 160'\}.$$

Again we have a contradiction to Result 1. Hence $B_1(7) \cap B_1(11) = \{1, 120\}$ and using Result 1 we see that another degree z occurs in $B_1(11)$ such that $3 \top z$. We finally get

$$B_1(11) = \{1, 21, 210, 120, 210, 280, 280'\}.$$

From this we see G as two 3-blocks of defect 1 having non-trivial intersection with $B_1(11)$. We can write these 3-blocks as

$$B_2(3) = \{21, 210, 231\}, \quad B_3(3) = \{120, 210, 330\}.$$

However if we sum the squares of the degrees determined so far we get that their sum is greater than the order of G.

Suppose now that $|N_G(G_7) : C_G(G_7)| = 3$. $B_1(7) \cap B_1(11)$ must contain a character of one of the following degrees: 45, 120, 60, 160, 32 (by Result 3). Using Corollary 1, in the first four cases G can have no elements of order 35. In the last case by the methods outlined above, $B_1(7) = \{1, 32, 32', 99, \ldots\}$ or $B_1(7) = \{1, 32, 32', 55, \ldots\}$. It follows that either G has no elements of order 35 or no elements of order 21 and so cases (1), (2), (3) cannot occur.

In case (4) using Sylow's theorem and P. Hall's solubility theorem ([11], p. 141) we get a contradiction by Lemma 1.

If we are in case (5) then $|N_G(G_7)| = 2^3 \cdot 3 \cdot 7$, and $C_G(G_7) = G_7 \times T$, where T is a 2-group of order 8. By Sylow's theorem $|N_G(T)| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ or $2^6 \cdot 3 \cdot 7$. If $1 < K \leq T$ with $2^7||N_G(K)|$ we get a contradiction by Lemma 1. Let S denote the Sylow 2-subgroup (of order $64 = 2^6$ in both cases) of $N_G(T)$, and G_2 be a Sylow 2-subgroup of G containing S.

If $\alpha \in G_2 \setminus S$ then $S = T \times T^\alpha$ by the above remarks.

Suppose that $|N_G(T)| = 2^6 \cdot 3 \cdot 7$. If T is abelian, then $S/T \simeq T$ is elementary abelian, and $N_G(T)$ is soluble. By Gaschütz ([11], p. 246), $N_G(T) = T \times K$, where K is a soluble group of order $2^3 \cdot 3 \cdot 7$. It follows that the Sylow 2-subgroup of K is normal in K, and hence $S \triangleleft N_G(T)$, which means that $11||N_G(S)|$, in contradiction to Lemma 1.

If T is non-abelian, $N_G(T)$ is non-soluble, and $T \simeq D_8$, the dihedral group of order 8. We now have $N_G(T)/T \simeq PSL(2, 7)$, and by Gaschütz ([11], p. 246), $C_G(T) = Z(T) \times X$ where $X \simeq PSL(2, 7)$. Let $A \leq X$ such that $A \simeq A_4$, and E^α is the Sylow 2-subgroup of A, where E is some four group of T and $\alpha \in G_2 \setminus S$. By Sylow's theorem, $|N_G(E)| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$, and by the result of P. Hall, $N_G(E)$ is non-soluble. It follows that $O_2 = O_2(N_G(E))$

is of order 8, $O_2 \leqq N_G(G_7)$ and $N_G(E)/O_2 \simeq A_7$. Hence $T = O_2$ and so we have that $|N_G(T)| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$.

As before $N_G(T)$ is non-soluble, $T \simeq D_8$, $N_G(T)/T \simeq A_7$ and $C_G(T) \simeq Z(T) \times A_7$. In the same way as above we see that T possesses an elementary four group E which is conjugate in G to the Sylow 2-subgroup of a subgroup A of $C_G(T)$ where $A \simeq A_4$. Hence $3^3 | |N_G(E)|$, clearly a contradiction. Case (9) is the only remaining case, so we have proved:

LEMMA 2. *The Sylow 7-normalizer of G is a Frobenius group of order* 21. We now suppose that

(I) G has no elements of order 15,
(II) $|N_G(G_5) : C_G(G_5)| = 4$.

We get only two possibilities:

(a) $|G : N_G(G_5)| = 2 \cdot 3^2 \cdot 7 \cdot 11$
(b) $|G : N_G(G_5)| = 2^5 \cdot 3^2 \cdot 7 \cdot 11$.

CASE (a). We have $C_G(G_5) = G_5 \times T$ where T is a 2-group of order 16. Let S be a Sylow 2-subgroup of $N_G(G_5)$, and G_2 a Sylow 2-subgroup of G containing S. (Note that $|G_2 : S| = 2$). As $T \lhd S$, there is an involution $t \in T$ such that $C_G(t) \geqq G_2$. Using the two lemmas and Sylow's theorem, we have $|C_G(t)| = 2^7 \cdot 3 \cdot 5$. Hence $C = C_G(t)$ is non-soluble. Further, we must have $O_2(C) = T$, $C/T \simeq S_5$, and $C = N_G(T)$. As $G_5 \leqq C_G(T)$, $C_G(T)$ cannot be soluble and so $C(T)/Z(T)$ is isomorphic to either A_5 or S_5.

If u is an involution in $T \setminus \langle t \rangle$ we claim that $3^2 \nmid |C_G(u)|$. This is shown by way of contradiction. If $3^2 | |C_G(u)|$, then $u \notin Z(T)$, $C_G(u)$ is non-soluble and $O_2(C_G(u)) \leqq T \cap C_G(u)$. (The non-solubility follows from assumption (I).) As $C_G(u)/O_2(C_G(u))$ is non-soluble, from our assumptions we get that

$$C_G(u)/O_2(C_G(u)) \simeq N \quad \text{where} \quad \text{Aut}(A_6) \geqq N \geqq A_6.$$

From the order of C, and because A_6 is simple and $G_5 \leqq C_G(T)$, then $t \notin O_2(C_G(u))$. Since $O_2(C_G(u)) < C$, $O_2(C_G(u))$ is normal in $C(u) \cap C$, and as $C_C(u)$ is non-soluble, $C_C(u)/O_2(C_G(u))$ is non-soluble. It follows that $C_C(u)/O_2(C_G(u)) \simeq L \times B$, where L is a non-trivial 2-group (since $\langle t \rangle \lhd C$) and $A_5 \leqq B \leqq S_5$. As $C_C(u) < C_G(u)$ we have a contradiction to the structure of $\text{Aut}(A_6)$.

Using the structure of C, if K is a Sylow 3-subgroup of C, then $|C_C(K)| = 2^5 \cdot 3$. Denote by Y the Sylow 2-subgroup of $C_C(K)$ and R the Sylow 2-subgroup of $C_G(K)$. If $Y < R$ then $|R : Y| = 2$, and $|R : T| = 4$. Thus there is an involution $j \in T$ with $j \in Z(R)$. As $T \leqq C_G(j)$, $T = O_2 (C_G(j))$ and hence $T \lhd R$. This contradicts $C = N_G(T)$ and hence $R = Y$.

We have shown that $D = C_G(K)$ is of order $2^5 \cdot 3^2$. Since for any element $x \in T$, $3 \nmid |C_G(x)|$, either $C_D(G_3) = G_3$ or $C_D(G_3) = N_D(G_3) > G_3$, where G_3 is a Sylow 3-subgroup of G containing K.

In the first case $O = O_2(C_G(K))$ is of order 16 and $|T \cap O| = 8$. In the second case, $Y \lhd D$, and by a transfer lemma of John G. Thompson [17], D has a subgroup M of index 2. Again, $\bar{O} = Y \cap M$ is normal in D, of order 16, and $|\bar{O} \cap T| = 8$. We may put $O = \bar{O}$ and note that there is an (inner) automorphism σ of D, $\sigma^3 = 1$, such that σ acts fixed-point-free on O. Hence there exists a subgroup E of T such that $E^\sigma = E$, and E is elementary of order four. Since $G_5 < C_G(E)$, we get $C_G(E)/O_2(C_G(E)) = B$ where $A_5 \leq B \leq S_5$. However as $|N_G(E) : C_G(E)| = 6$ and

$$\mathrm{Aut}(S_5) = \mathrm{Aut}(A_5) = S_5,$$

G has elements of order 15, contradicting (I). Case (a) is ruled out.

We return to the block theoretic argument and show that assumptions (I) and (II) actually hold in G.

Using the lemmas proved above and the methods outlined previously we get that

$$B_1(7) = \{1, 22, 22, 45, 45'\}$$

or

$$B_1(7) = \{1, 99, 55, 45, 45'\}.$$

In the first case the characters of degrees 1, 22, 22 are modular 7-irreducible and by Theorem 13 of [1], the two characters of degree 22 must be complex for some 7-regular element of G (and hence they are complex conjugates), as the two characters of degree 45 are real for 7-regular elements.

Applying Result 3 it follows immediately that the two characters of degree 22 are the two 5-conjugate characters of the special family in $B_1(5)$. It follows that $|N_G(G_5) : C_G(G_5)| = 2$ and the two characters of degree 22 take real values on 5-singular elements and hence take real values on all elements of G, a contradiction.

Using Results 1, 2, 3 and the results of [1] (on blocks of defect 1) we quickly get the following block decomposition for G:

$$B_1(11) = \{1, 45, 45, 21, 210, 280, 280'\}$$
$$B_1(7) \ = \{1, 99, 55, 45, 45'\}$$
$$B_1(5) \ = \{1, 21, 99, 154, 231\}$$
$$B_2(3) \ = \{21, 210, 231\},$$

where $B_2(3)$ is a 3-block of defect 1. Since $B_1(5)$ consists of 5 characters and since the degree 99 occurs in $B_1(5)$, assumptions (I) and (II) must hold, and we have:

LEMMA 3. *The Sylow 5-normalizer of G is a Frobenius group of order* 20.

By Lemmas 1, 2, 3 we see that the degree of any character of G not listed above must be divisible by $5 \cdot 7 \cdot 11$. Summing the squares of the degrees of the characters determined so far we get that there is only one

remaining character of degree $385 = 5 \cdot 7 \cdot 11$. We have therefore proved that G has twelve conjugate classes.

Using Result 1 we determine $B_1(3)$ and see that it is the only 3-block of defect 2 (i.e. of lowest type). Applying [5], Theorem 2, we have proved that $C_G(G_3) = G_3$ i.e. a Sylow 3-subgroup of G is self-centralizing.

So far we have determined the order of a representative of nine of the twelve conjugate classes of G. Further, G must have at least one conjugate class of elements of order 6 by M. Suzuki, [16].

Using the results so far determined and Sylow's theorem,

$$|N_G(G_3) : C_G(G_3)| = 8.$$

The structure of $GL(2, 3)$ shows that $N_G(G_3)/G_3$ is isomorphic to either the quaternion, cyclic or dihedral group of order 8.

The values of the twelve characters of G can be determined immediately for elements of order 5, 7, or 11 by the results of Brauer, [2]. The values of all (irreducible, complex) characters on elements of order 3 or 6 can be determined by using the orthogonality relations, Brauer's relations for characters in blocks of defect 1 (see [1]), and the fact that all characters in $B_2(3)$ take integral values. At the same time we determine that if c, d are elements of orders 3, 6 respectively, then $|C_G(c)| = 2^2 \cdot 3^2 = 36$, $|C_G(d)| = 2^2 \cdot 3 = 12$. Using these values and the orthogonality relations for ζ_{12}, where $\zeta_{12}(1) = 385$, we prove that the Sylow 2-subgroup of $N_G(G_3)$ is not dihedral. Hence all elements of order 3 are conjugate in G, and it then follows that G has only one class of elements of order 6. Using these results and summing the orders of conjugate classes of G we get that the orders of the centralizers of representatives of the remaining four conjugate classes are $2^7 \cdot 3$, 2^5, 2^4, 2^3. We have proved:

LEMMA 4. *The group G has one class of elements of order three and one class of elements of order 6. The centralizer of an element of order three is of order 36 and has a normal elementary Sylow 2-subgroup of order four. The centre of a Sylow 2-subgroup is elementary, all central involutions are conjugate and have centralizers of order $2^7 \cdot 3$.*

From Burnside's lemma ([11], p. 203) for any Sylow 2-subgroup of G, $|Z(G_2)| = 4$ or 2.

Let z be a central involution of G_2 and let Q be a Sylow 3-subgroup of $C = C_G(z)$. Since $C_C(Q) = Q \times E$ where E is an elementary four group, $O_2(C)$ must be of order $2^6 = 64$. Clearly $E < O_2(C)$ and $Z(G_2) < O_2(C)$.

LEMMA 5. *E is not normal in $O_2(C)$.*

PROOF. Suppose E is normal in $O_2(C)$. Then $E \triangleleft C$ and we have $|N_G(E) : C| = 3$ by Lemma 4. If $O_2(C) \cap C(E) < O_2(C)$, then

$$C_G(E) \cap O_2(C) = S$$

is of order 32. S is "Q-invariant", (regarding Q as an automorphism group of $O_2(C)$), and hence $E < C_G(Q) \cap S$, a contradiction. Therefore $O_2(C)$ is a subgroup of $C_G(E)$ and so $O_2(C) \lhd N_G(E)$ as $O_2(C) = O_2(C_G(E))$. The Frattini subgroup $D = D(O_2(C))$ of $O_2(C)$ is normal in $N_G(E)$ and hence of order 4 or 16.

Let $Q < G_3 < N_G(E)$ for some Sylow 3-subgroup G_3 of $N_G(E)$, and put $G_3 = Q \times K$. The subgroup D is certainly invariant under G_3.

Let $|D| = 16$. If K acts fixed-point-free on D, D is elementary abelian. Otherwise, $C(K) \cap D$ is of order four, and D is again elementary. Applying Maschke's Theorem, $C_D(K) \times E = D$. If $|D| = 4$, then $D = E$ and $O_2(C)/D$ is elementary abelian. G_3D/D cannot act fixed-point-free on $O_2(C)/D$ and we get a group W of order 16, such that $W \lhd O_2(C)$, W is G_3-invariant and $E < W$. We can then put $W = D$, when D is of order 16. We may now suppose that QD/D centralizes $O_2(C)/D$ and hence Q normalizes a subgroup L of $O_2(C)$ of order 32 with $D \leq L$, which is a contradiction, since $E \leq L$. The lemma is proved.

LEMMA 6. *The centre of a Sylow 2-subgroup of G is (cyclic) of order 2.*

PROOF. Suppose $Z = Z(G_2)$ is elementary of order four. By Lemma 4, the involutions of Z are conjugate in G, hence in $N_G(G_2)$, which must then be of order $2^7 \cdot 3$. From Lemma 5, $\bar{Z} = Z(O_2(C))$ is of order eight, and $\langle z \rangle = \bar{Z} \cap E$. $\bar{Z}$ is elementary abelian, as is the group $J = \bar{Z} \cdot E$, of order 16. Because Q normalizes $N(J) \cap O_2(C)$, it follows that $J \lhd C$, $C_C(J) = J$ and $C/J \cong S_4$.

Using Lemma 4 and Burnside's transfer theorem, the Sylow 2-subgroup U of $N_C(Q)$ (and hence of $N_G(Q)$) is dihedral of order eight. Hence $|N_G(E)| = 2^5 \cdot 3^2$, and $J \lhd N_G(E)$ as $C_G(E) = C_C(E) = J \cdot Q$. As we have shown that $N_G(J) > C$, from the structure of A_8, we must have $N_G(J)/J \cong A_6$.

Certainly J is not normal in $N_G(G_2)$, and hence J has two other conjugate subgroups J_2, J_3 in $N_G(G_2)$. If $J \cap J_2 \cap J_3 = Y$ is of order eight, then $Y \cap \bar{Z} = Z$, and $Y \cdot \bar{Z} = J$. As Y, $\bar{Z}$ are both normal in G_2 and both contain Z, $J \leq Z_2(G_2)$ (the second centre of G_2), and hence $|Z_2(G_2)| = 32$ as $G_2/J \cong D_8$. Hence $Z_2(G) = J \cdot J_2 \cdot J_3$ and $Z_2(G)$ is elementary, contradicting $C_G(J) = J$.

Hence $Z = J \cap J_2 \cap J_3$, and as z has 15 conjugates in $N_G(J)$, it follows that $C \leq N_G(J) \cap N_G(J_2) \cap N_G(J_3)$. We have therefore that J_2, J_3 are normal in C, and hence $Z \lhd C$, clearly a contradiction.

LEMMA 7. *If z is the involution in the (cyclic) centre of a Sylow 2-subgroup of G then $C_G(z)$ is an extension of an elementary abelian subgroup of order 16 by S_4.*

PROOF. Denote by F a maximal normal elementary abelian subgroup of C. If $F \cap E = \langle z \rangle$ then F has order 2, 8, or 32. Using [12], p. 18, $|F| = 32$ gives that $Z(G_2) > \langle z \rangle$. If $|F| = 8$, then $C_C(F) > F$ and as $C/C_C(F)$ is isomorphic to a subgroup of $PSL(2, 7)$, either $C_C(F) = F \cdot E$ or $C_C(F) = O_2(C)$. In the first case $C_C(F)$ is elementary abelian, and in the second, using [12], p. 18 again, $Z(G_2) > \langle z \rangle$.

If $F = \langle z \rangle$, as E is not normal in C, $\langle z \rangle = Z(O_2(C))$ also. Let D denote the Frattini subgroup of $O_2(C)$. If $|D| = 32$, D is cyclic; if $|D| = 4$, $D = E$; if $|D| = 2$, $D = Z(O_2(C)) = O_2(C)' = \langle z \rangle$ and $O_2(C)$ is an extra special 2-group; if $|D| = 8$, since $D \triangleleft C$, D must be isomorphic to Q_8, the quaternion group of order 8. All these cases are impossible, the last by a result of Gaschütz, [10]. Finally, if $|D| = 16$, and if it is abelian it must be elementary abelian. Hence D is non-abelian, and as Q acts faithfully on D, $\langle z \rangle = Z(D) = D' = D(D)$; i.e. D is an extra special 2-group of order 16, which is impossible.

We have shown that $E \leqq F$, and as E is not normal in C by Lemma 5, F is of order 16. Because $C_C(F) \triangleleft C$, then $C_G(F) = C_C(F) = F$ and hence $C/F \cong S_4$. The lemma is proved.

By a theorem of D. Held [13], Lemma 7 and the fact that $|G| = 443,520$, we get finally that G is isomorphic to M_{22}.

4. Proof of Theorem B

In this section G denotes a simple group of order $7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$. If n is the degree of a (complex, irreducible) character of G, then $6 < n < 89$, since $89^2 > 7920$, and $n > 6$ because of a result of Tuan, [18].

Using the methods outlined at the beginning of § 3 we can quickly show:

The Sylow 11-normalizer of G is a Frobenius group of order 55.

We then get the following block decomposition for G:

$$B_1(11) = \{1, 10, 10, 10, 45, 16, 16'\}$$
$$B_1(5) \ = \{1, 16, 16, 11, 44\}.$$

Hence G has no elements of order ten, by Corollary 1, and so we have:

The Sylow 5-normalizer of G is a Frobenius group of order 20.

Any character not in $B_1(11)$ or $B_1(5)$ has degree divisible by $11.5 = 55$. Summing the squares of the degrees of the known characters, G has only one further character of degree 55.

The determination of $B_1(3)$ yields that this block is the only 3-block of defect 2, and hence $C_G(G_3) = G_3$ by [5], Theorem 2. By Sylow's theorem and the transfer theorem, $|N_G(G_3) : G_3| = 4$ or 16, and in both cases G_3 is elementary abelian. Further, $N_G(G_3)/G_3$ cannot be elementary of order four,

otherwise we have too many conjugate classes in G.

If $N_G(G_3)/G_3$ is cyclic of order four, then if c is an element of order 3, $T \lhd C_G(c)$ where T is the Sylow 2-subgroup of $C_G(c)$. If $T = G_2$, G has at most one class of elements of order 3. Hence T is elementary of order four and $|C_G(c)| = 2^2 \cdot 3^2$. G has only one class of involutions, by considering the number of classes so far determined. It now follows that the centralizer of any involution of G is 2-closed, contradicting Suzuki's result [16].

From the structure of $GL(2, 3)$ we have:

The Sylow 2-subgroup of $N_G(G_3)$, and hence the Sylow 2-subgroup of G is isomorphic to the semi-dihedral group of order 16.

Using the results proved so far, and the assumption on the order of G, a result of Wong ([20], Theorem 3) gives that $G \cong M_{11}$.

The theorem is proved.

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