On the matrix elements of the U(n) generators — Source link

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On the matrix elements of the U(n) generators

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A straightforward derivation of the matrix elements of the U(n) generators is presented using algebraic infinitesimal techniques. An expression for the general fundamental Wigner coefficients of the group is obtained as a polynomial in the group generators. This enables generalized matrix elements to be defined without explicit reference to basis states. Such considerations are important for treating groups such as Sp($2n$) whose basis states are not known.

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1. INTRODUCTION

It was shown in a previous publication[1] (herein referred to as I) that the polynomial identities satisfied by the infinitesimal generators of a semisimple Lie group may be applied to give a simple determination of the (multiplicity-free) Wigner coefficients of the group. In this paper we shall extend some of the techniques presented in I to give a simple self-contained derivation of the matrix elements of the U(n) generators.

An orthonormal basis for the finite dimensional irreducible representations of U(n) was first constructed by Gel'fand and Zetlin.[2] The matrix elements of the U(n) generators in this basis were first derived by Gel'fand and Zetlin[2] and rederived using boson-calculus techniques by Baird and Biedenharn.[3] In their discussion of the Gel'fand–Zetlin results Baird and Biedenharn made an important contribution by explicitly expressing the general matrix element as a product of a reduced matrix element and a Wigner coefficient. As a result the fundamental Wigner coefficients of U(n), for general n, were given for the first time. It is our principal aim to obtain these results using algebraic infinitesimal techniques in contrast to the integral techniques of Gel'fand and Zetlin and Baird and Biedenharn.

The relationship between our approach and that employed by Biedenharn *et al.*[3–5] has been discussed in I. Although the two approaches are closely related we feel that our approach offers some novel features. In the literature it is customary to obtain the matrix elements of generators of the form a^m_{m+1} and a^{m+1}_m $(m<n)$ first and to obtain the matrix elements of the remaining generators by repeated commutation. Making use of the concept of simultaneous shift operators which shift the representation labels of U(n) and each of its canonical subgroups in a certain prescribed way, we shall present an alternative derivation where the matrix elements of all generators are obtained just as easily as those of the form a^m_{m+1} and a^{m+1}_m. An expression for the general fundamental Wigner coefficients of U(n) is also given in terms of polynomials in the group generators constructed using the characteristic identities of U(n) and each of its canonical subgroups. The expressions obtained are clearly generalizable to more general groups.

We shall also obtain an expression for the U(n) : U($n-1$) reduced Wigner coefficients (or isoscalar factors) as a polynomial in the group generators. The simultaneous shift operators used in this paper are obviously related to the pattern calculus of Biedenharn *et al.*[5] and their concept of Wigner operator. The exact relationship between them will be discussed in a forthcoming publication.

The extension of this work to the discrete series of representations of the noncompact groups U(n,1) and the orthogonal groups O(n) and O(n,1) is evident.

2. WIGNER COEFFICIENTS AND REDUCED MATRIX ELEMENTS

The generators a^i_j $(i,j=1,\ldots,n)$ of the Lie group U(n) satisfy the commutation relations

$$[a^i_j,a^k_l]=\delta^k_j a^i_l-\delta^i_l a^k_j$$

and the Hermiticity property

$$(a^i_j)^\dagger=a^j_i.$$

These generators may be assembled into a square matrix a whose (i,j) entry is the generator a^i_j. Polynomials in a may be defined recursively by the formula

$$(a^{m+1})^i_j=(a^m)^i_k a^k_j=a^i_k(a^m)^k_j.$$

Associated with the matrix a is its adjoint $\bar a$ with entries $\bar a_j^{\ i}=-a^i_j$. Polynomials in $\bar a$ may be defined by

$$(\bar a^{m+1})^i_j=(\bar a^m)^k_j\bar a^i_k=\bar a_j^{\ k}(\bar a^m)^i_k.$$

It has been shown[5] on a finite dimensional irreducible representation of U(n) with highest weight $\lambda=(\lambda_1,\ldots,\lambda_n)$ that the matrices a and $\bar a$ satisfy the polynomial identities

$$\prod_{r=1}^n(a-\alpha_r)=0,\quad \prod_{r=1}^n(\bar a-\bar\alpha_r)=0, \tag{1}$$

where the roots α_r and $\bar\alpha_r$ are given by

$$\alpha_r=\lambda_r+n-r=n-1-\bar\alpha_r.$$

By virtue of the identities (1), projection operators $P[r]$ and $\bar P[r]$ may be constructed by setting

$$P[r]=\prod_{l\neq r}\left(\frac{a-\alpha_l}{\alpha_r-\alpha_l}\right),$$

$$\bar P[r]=\prod_{l\neq r}\left(\frac{\bar a-\bar\alpha_l}{\bar\alpha_r-\bar\alpha_l}\right).$$

The matrix elements of such projectors in unitary represen-

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tations of the group were shown in I to be bilinear combinations of Wigner coefficients. To be more explicit let $V(\lambda)$ be a finite dimensional irreducible representation with highest weight λ and let $|{}^{\lambda}_{(\nu)}\rangle$ and $|{}^{\lambda}_{(\nu')}\rangle$ be Gel'fand basis states in the space $V(\lambda)$. According to I we have

$$\left\langle {\lambda \atop (\nu)} \middle| P[r]^i_j \middle| {\lambda \atop (\nu')} \right\rangle = \sum_{(\mu)} \left\langle {\lambda \atop (\nu)} ; {\overline{\dot{10}} \atop i} \middle| {\lambda - \Delta_r \atop (\mu)} \right\rangle \left\langle {\lambda - \Delta_r \atop (\mu)} \middle| {\overline{\dot{10}} \atop j} ; {\lambda \atop (\nu')} \right\rangle, \tag{2}$$

where $|{}^{\overline{10}}_{i}\rangle$ constitutes an orthonormal basis for the contragredient vector representation and where $|{}^{\overline{10};\lambda}_{i;(\nu)}\rangle$ denotes the product state $|{}^{\overline{10}}_{i}\rangle \otimes |{}^{\lambda}_{(\nu)}\rangle$. Similarly, we have

$$\left\langle {\lambda \atop (\nu)} \middle| \bar{P}[r]^i_j \middle| {\lambda \atop (\nu')} \right\rangle = \sum_{(\mu)} \left\langle {\lambda \atop (\nu)} ; {\dot{10} \atop j} \middle| {\lambda + \Delta_r \atop (\mu)} \right\rangle \left\langle {\lambda + \Delta_r \atop (\mu)} \middle| {\dot{10} \atop i} ; {\lambda \atop (\nu')} \right\rangle, \tag{3}$$

where $|{}^{10}_{i}\rangle$ forms the usual basis for the fundamental vector representation. Substituting $i = j = n$ into Eqs. (2) and (3) we obtain

$$\left\langle {\lambda \atop (\nu)} \middle| P[r]^n_n \middle| {\lambda \atop (\nu')} \right\rangle = \delta_{(\nu),(\nu')} \left| \left\langle {\lambda \atop (\nu)} ; {\overline{\dot{10}} \atop n} \middle| {\lambda - \Delta_r \atop (\nu)} \right\rangle \right|^2,$$

$$\left\langle {\lambda \atop (\nu)} \middle| \bar{P}[r]^n_n \middle| {\lambda \atop (\nu')} \right\rangle = \delta_{(\nu),(\nu')} \left| \left\langle {\lambda \atop (\nu)} ; {\dot{10} \atop n} \middle| {\lambda + \Delta_r \atop (\nu)} \right\rangle \right|^2. \tag{4}$$

It is our aim now to apply the $U(n)$ characteristic identities (1) to evaluate the operators $P[r]^n_n$ and $\bar{P}[r]^n_n$ which, by Eqs. (4), are essentially squares of Wigner coefficients.

We now turn our attention to the group $U(n+1)$ whose generators a^i_j $(i,j = 1,...,n+1)$ may be assembled into a matrix b whose (i,j) entry is the generator a^i_j. The matarix b satisfies an $n+1$ degree polynomial identity analogous to the $U(n)$ matrix a:

$$\prod_{k=1}^{n+1} (b - \beta_k) = 0,$$

where the β_k take constant values on an irreducible representation with highest weight $\lambda = (\lambda_1,...,\lambda_{n+1})$ given by $\beta_k = \lambda_k + n + 1 - k$. In a similar way we define the adjoint matrix $\bar{b}$ whose roots $\bar{\beta}_k$ are given by $\bar{\beta}_k = n - \beta_k$.

As for $U(n)$ we may construct the $U(n+1)$ projection operators

$$Q[k] = \prod_{\substack{l=1 \\ \neq k}}^{n+1} \left(\frac{b - \beta_l}{\beta_k - \beta_l} \right),$$

$$\bar{Q}[k] = \prod_{\substack{l=1 \\ \neq k}}^{n+1} \left(\frac{\bar{b} - \bar{\beta}_l}{\bar{\beta}_k - \bar{\beta}_l} \right).$$

Also, according to I, if $p(x)$ is any polynomial we may write

$$P(b) = \sum_{k=1}^{n+1} P(\beta_k) Q[k]. \tag{5}$$

From the $U(n+1)$ identity we have

$$bQ[k] = \beta_k Q[k].$$

Taking the $(i,n+1)$ entry of this matrix equation we may write

$$\sum_{l=1}^{n+1} a^i_l Q[k]^l_{n+1} = \beta_k Q[k]^i_{n+1}, \quad i = 1,...,n.$$

Rearranging this expression we obtain

$$a^i_{n+1} Q[k]^{n+1}_{n+1} = (\beta_k - a)^i_l Q[k]^l_{n+1}. \tag{6}$$

Similarly, we may write

$$Q[k]^{n+1}_{n+1} a^{n+1}_i = Q[k]^{n+1}_l (\beta_k - a)^l_i. \tag{7}$$

For simplicity let us for the moment denote the $U(n)$ invariant $Q[k]^{n+1}_{n+1}$ by C_k. Clearly, the C_k are $U(n+1)$ analogs of the operators $P[r]^n_n$ whose matrix elements are squares of Wigner coefficients. It is our aim to express C_k as a function of the β_k and α_r. Note that Eq. (5) implies

$$P(b)^{n+1}_{n+1} = \sum_{k=1}^{n+1} P(\beta_k) C_k,$$

which enables a systematic evaluation of $U(n)$ invariants of the form $P(b)^{n+1}_{n+1}$ once the C_k have been determined.

We may invert Eqs. (6) and (7) by writing

$$Q[k]^i_{n+1} = [(\beta_k - a)^{-1}]^i_j a^j_{n+1} C_k,$$
$$Q[k]^{n+1}_i = C_k a^{n+1}_j [(\beta_k - a)^{-1}]^j_i, \tag{8}$$

where $(\beta_k - a)^{-1}$ denotes the matrix

$$(\beta_k - a)^{-1} = \sum_{r=1}^{n} (\beta_k - \alpha_r)^{-1} P[r].$$

Throughout the remainder of this section let ψ denote the $U(n)$ vector operator with components $\psi^i = a^i_{n+1}$, $i = 1,...,n$, with adjoint $\psi^\dagger$ whose components are given by $\psi^\dagger_j = a^{n+1}_j$. Following Green and Bracken,[6] the vector operator ψ and its contragredient $\psi^\dagger$ may be resolved into a sum of shift vectors

$$\psi = \sum_{r=1}^{n} \psi[r], \quad \psi^\dagger = \sum_{r=1}^{n} \psi^\dagger[r]$$

which alter the $U(n)$ representation labels according to

$$\lambda_k \psi[r] = \psi[r](\lambda_k + \delta_{kr}),$$

$$\lambda_k \psi^\dagger[r] = \psi^\dagger[r](\lambda_k - \delta_{kr}).$$

Such shift vectors may be constructed by application of the $U(n)$ projectors $P[r]$ and $\bar{P}[r]$ as follows:

$$\psi[r] = P[r]\psi = \psi\bar{P}[r],$$
$$\psi^\dagger[r] = \bar{P}[r]\psi^\dagger = \psi^\dagger P[r].$$

Decomposing the $U(n)$ vector ψ into its shift components allows us to write Eqs. (8) in the form

$$Q[k]^i_{n+1} = \sum_{r=1}^{n} \psi[r]^i (\beta_k - \alpha_r - 1)^{-1} C_k,$$

$$Q[k]^{n+1}_i = \sum_{r=1}^{n} C_k (\beta_k - \alpha_r - 1)^{-1} \psi^\dagger[r]_i. \tag{9}$$

However, from Eq. (5), we have

$$\sum_{k=1}^{n+1} Q[k]^i_{n+1} = \delta^i_{n+1} = 0, \quad \text{for } i = 1,...,n.$$

Hence, summing Eqs. (9) over k from 1 to $n+1$, we obtain

$$\sum_{r=1}^{n} \psi[r] \left(\sum_{k=1}^{n+1} (\beta_k - \alpha_r - 1)^{-1} C_k \right) = 0.$$

However, the shift vectors $\psi[r]$ form a linearly independent set since they effect different shifts. This implies that

$$\sum_{k=1}^{n+1} (\beta_k - \alpha_r - 1)^{-1} C_k = 0, \quad r = 1,...,n. \tag{10}$$

This set of equations together with the condition

$$\sum_{k=1}^{n+1} C_k = 1\Big(\sum_{k=1}^{n+1} Q\,[k\,]^{n+1}_{n+1} = \delta^{n+1}_{n+1}\Big)$$

uniquely determine the C_k. These equations are easily solved (using Cramer's rule for example) and yield the solution

$$C_k = \prod_{\substack{p=1\\ \neq k}}^{n+1} (\beta_k - \beta_p)^{-1} \prod_{l=1}^{n} (\beta_k - \alpha_l - 1). \tag{11}$$

Similarly, using the adjoint projectors $\bar{Q}\,[k\,]$, one may deduce the equations

$$\bar{Q}\,[k\,]^i_{n+1} = \sum_{r=1}^{n} \bar{C}_k(\beta_k - \alpha_r)^{-1}\psi[r]^i,$$

$$\bar{Q}\,[k\,]^{n+1}_i = \sum_{r=1}^{n} \psi^\dagger[r]_i(\beta_k - \alpha_r)^{-1}\bar{C}_k, \tag{12}$$

where $\bar{C}_k$ is shorthand notation for $\bar{Q}\,[k\,]^{n+1}_{n+1}$ which may be expressed in terms of the β's and α's according to

$$\bar{C}_k = \prod_{\substack{p=1\\ \neq k}}^{n+1} (\beta_k - \beta_p)^{-1} \prod_{l=1}^{n} (\beta_k - \alpha_l). \tag{13}$$

The U(n) invariants C_k and $\bar{C}_k$ are the $U(n+1)$ analogs of the operators $P\,[r\,]^n_n$ and $\bar{P}\,[r\,]^n_n$ which may likewise be expressed in terms of the roots in the U(n) and U($n-1$) identities. This then enables us to evaluate the fundamental Wigner coefficients (4) as required. However, in order to determine the matrix elements of the group generators we must also determine the reduced matrix elements of ψ and $\psi^\dagger$.

Since the matrix elements of the projectors $P\,[r]$ and $\bar{P}\,[r]$ are bilinear combinations of Wigner coefficients, the Wigner–Eckart theorem allows us to write

$$\psi[r]\psi^\dagger[r] = \bar{M}_r\, P\,[r],$$

$$\psi^\dagger[r]\psi[r] = M_r\, \bar{P}\,[r], \tag{14}$$

where the M_r ($\bar{M}_r$) are U(n) invariants whose eigenvalues determine the squares of the reduced matrix elements of ψ ($\psi^\dagger$). Equation (14) is clearly an operator generalization of the Wigner–Eckart theorem and may be derived using purely algebraic techniques as demonstrated in Ref. 7. By taking the traces of Eqs. (14) we obtain the result

$$\bar{M}_r = \frac{\psi[r]^i\psi^\dagger[r]_i}{t_r(P\,[r])},$$

$$M_r = \frac{\psi^\dagger[r]_i\psi[r]^i}{t_r(\bar{P}\,[r])}, \tag{15}$$

which enables the invariants $\bar{M}_r$ and M_r to be expressed as a function of the β_k and α_r using techniques similar to those used in the derivation of the C_k and $\bar{C}_k$ (see Ref. 7 for further details). We obtain

$$\bar{M}_r = (-1)^n \prod_{p=1}^{n+1} (\beta_p - \alpha_r)\prod_{l\neq r} (\alpha_r - \alpha_l - 1)^{-1}, \tag{16}$$

$$M_r = (-1)^n \prod_{p=1}^{n+1} (\beta_p - \alpha_r - 1) \prod_{l\neq r} (\alpha_r - \alpha_l + 1)^{-1}. \tag{17}$$

One may check directly from Eq. (15) and the Wigner–Eckart theorem that the $\bar{M}_r$ and M_r in fact determine the reduced matrix elements as required.

Taking the (n,n) entries of Eqs. (14), we obtain

$$\psi[r]^n\psi^\dagger[r]_n = \bar{M}_r P\,[r]^n_n,$$

$$\psi^\dagger[r]_n\psi[r]^n = M_r \bar{P}\,[r]^n_n, \tag{18}$$

which, using formulas (11), (13), (16), and (17), enables us immediately to write down the matrix elements of the generators a^n_{n+1} and a^{n+1}_n. However, in order to obtain the matrix elements of the remaining generators we need more information. To this end we obtain a relationship between the U($n+1$) and U(n) projection operators which, as we shall later see, reflects the properties of U($n+1$) : U(n) reduced Wigner operators.

First of all it is easily seen, as a trivial property of Wigner coefficients and Eqs. (2) and (3), that the following relations holds:

$$Q\,[k\,]^i_{n+1} (C_k)^{-1} Q\,[k\,]^{n+1}_j = Q\,[k\,]^i_j,$$

$$\bar{Q}\,[k\,]^{n+1}_i (\bar{C}_k)^{-1} \bar{Q}\,[k\,]^j_{n+1} = \bar{Q}\,[k\,]^j_i.$$

A proof of this result which exploits only the Lie algebra commutation relations is presented in Ref. 7 (see also Green[8]). By applying the U(n) projectors $P\,[r]$ ($\bar{P}\,[r]$) to both sides of the above equations, we obtain, by virtue of Eqs. (9) and (12), the result

$$\sum_{l,m=1}^{n} P\,[r]^i_l Q\,[k\,]^l_m P\,[r]^m_j$$

$$= \psi[r]^i C_k(\beta_k - \alpha_r - 1)^{-2}\, \psi^\dagger[r]_j.$$

We now note, from the form of C_k given by Eq. (11), that $C_k(\beta_k - \alpha_r - 1)^{-1}$ is independent of α_r and hence commutes with $\psi[r]$. We therefore obtain

$$P\,[r]^i_l Q\,[k\,]^l_m P\,[r]^m_j$$

$$= C_k(\beta_k - \alpha_r - 1)^{-1}(\beta_k - \alpha_r)^{-1}\psi[r]^i\psi^\dagger[r]_j.$$

Using Eqs. (14) this in turn may be written

$$P\,[r]^i_l Q\,[k\,]^l_m P\,[r]^m_j$$

$$= C_k\bar{M}_r(\beta_k - \alpha_r - 1)^{-1}(\beta_k - \alpha_r)^{-1}P\,[r]^i_j. \tag{19}$$

Similarly, we obtain

$$\bar{P}\,[r]^l_i \bar{Q}\,[k\,]^m_l \bar{P}\,[r]^j_m$$

$$= \bar{C}_k M_r(\beta_k - \alpha_r - 1)^{-1}(\beta_k - \alpha_r)^{-1}\bar{P}\,[r]^j_i. \tag{20}$$

As we shall see Eqs. (19) and (20) are essentially all we need to determine the matrix elements of the U(n) generators.

3. SIMULTANEOUS SHIFTS

The Lie group U(n) admits the canonical[9] chain of subgroups

$$U(n)\supset U(n-1)\supset\cdots\supset U(1), \tag{21}$$

where each group U(m) occurring in this chain has infinitesimal generators consisting of the U(n) generators $a^i_{\ j}$ for values of i and j in the range $1,...,m$. Before proceeding we establish some notation. We denote the U(m) matrix whose (i,j) entry is the U(m) generator $a^i_{\ j}$ $(i,j = 1,...,m)$ simply by a_m. We denote the characteristic roots of a_m by $\alpha_{r,m}$ $(r = 1,...,m)$. They take constant values on a finite dimensional irreducible representation of U(m) with highest weight $(\lambda_{1m},...,\lambda_{mm})$ given by $\alpha_{r,m} = \lambda_{r,m} + m - r$. We denote the corresponding U(m) projectors simply by $P\binom{m}{r}$ and $\bar{P}\binom{m}{r}$:

$$P\binom{m}{r} = \prod_{l \neq r} \left(\frac{a_m - \alpha_{l,m}}{\alpha_{r,m} - \alpha_{l,m}}\right),$$

$$\bar{P}\binom{m}{r} = \prod_{l \neq r} \left(\frac{\bar{a}_m - \bar{\alpha}_{l,m}}{\bar{\alpha}_{r,m} - \bar{\alpha}_{l,m}}\right),$$

where $\bar{a}_m$ is the U(m) adjoint matrix whose roots $\bar{\alpha}_{l,m}$ are given by $\bar{\alpha}_{l,m} = m - 1 - \bar{\alpha}_{l,m}$. We denote the (m,m) entries of these projectors by $C_{r,m}$ and $\bar{C}_{r,m}$, respectively. From the previous section we know that these operators are essentially squares of Wigner coefficients whose eigenvalues are given by [cf. Eqs. (11) and (13)]

$$\begin{aligned} C_{r,m} &= \prod_{\substack{k=1 \\ \neq r}}^{m} (\alpha_{r,m} - \alpha_{k,m})^{-1} \prod_{l=1}^{m-1} (\alpha_{r,m} - \alpha_{l,m-1} - 1), \\ \bar{C}_{r,m} &= \prod_{\substack{k=1 \\ \neq r}}^{m} (\alpha_{r,m} - \alpha_{k,m})^{-1} \prod_{l=1}^{m-1} (\alpha_{r,m} - \alpha_{l,m-1}). \end{aligned} \tag{22}$$

Finally we denote the U(m) vector operator $\{a^i_{\ m+1}\}$ $(i = 1,...,m)$ simply by $\psi(m)$. Its Hermitian conjugate $\psi^\dagger(m)$ constitutes a contragredient vector operator with components $\psi^\dagger(m)_i = a_i^{\ m+1}$. We denote the shift components of these operators by $\psi\binom{m}{r}$ and $\psi^\dagger\binom{m}{r}$, respectively. According to Eqs. (14) we may write

$$\begin{aligned} \psi\binom{m}{r}\psi^\dagger\binom{m}{r} &= \bar{M}_{r,m} P\binom{m}{r}, \\ \psi^\dagger\binom{m}{r}\psi\binom{m}{r} &= M_{r,m} \bar{P}\binom{m}{r}, \end{aligned} \tag{23}$$

where the U(m) invariants $\bar{M}_{r,m}$ and $M_{r,m}$ (the squared reduced matrix elements) are given by

$$\begin{aligned} M_{r,m} &= (-1)^m \prod_{k=1}^{m+1} (\alpha_{k,m+1} - \alpha_{r,m} - 1) \\ &\quad \times \prod_{l \neq r} (\alpha_{r,m} - \alpha_{l,m} + 1)^{-1}, \\ \bar{M}_{r,m} &= (-1)^m \prod_{k=1}^{m+1} (\alpha_{k,m+1} - \alpha_{r,m}) \\ &\quad \times \prod_{l \neq r} (\alpha_{r,m} - \alpha_{l,m} - 1)^{-1}. \end{aligned} \tag{24}$$

The (m,m) entries of Eqs. (23) yield the relations

$$\begin{aligned} \psi\binom{m}{r}^m \psi^\dagger\binom{m}{r}_m &= \bar{M}_{r,m} C_{r,m}, \\ \psi^\dagger\binom{m}{r}_m \psi\binom{m}{r}^m &= M_{r,m} \bar{C}_{r,m}, \end{aligned} \tag{25}$$

which determines the matrix elements of the generators $a^m_{\ m+1}$ and $a_m^{\ m+1}$.

If U($m+1$) and U(m) are two canonical subgroups of U(n), we have already remarked that the operator $\psi(m)$ with components $\psi(m)^l = a^l_{\ m+1}$ constitutes a U(m) vector operator. Hence, each operator $a^l_{\ m+1}$ may be written as a sum of shift components $\psi\binom{m}{r}$ which alter the representation labels of the group U(m). However, if k is a positive integer less than m, then the components $\psi(m)^i$ $(i = 1,...,k)$ also constitute a vector operator with respect to the subgroup U(k). Hence, any given operator of the form $a^l_{\ m+1}$ $(l < m+1)$ transforms as a component of a vector operator with respect to the subgroups U(m), U($m-1$),...,U(l).

In the limiting case when $l = m$ we see that $a^m_{\ m+1}$ can only be a component of a vector operator with respect to the subgroup U(m). In this case $a^m_{\ m+1}$ can only alter the representation labels of the subgroup U(m) and we may resolve $a^m_{\ m+1}$ into its U(m) shift components according to

$$a^m_{\ m+1} = \sum_{r=1}^{m} \psi\binom{m}{r}^m.$$

Suppose now we consider a generator of the form $a^{m-1}_{\ m+1}$ which transforms as a component of a vector with respect to the subgroups U($m-1$) and U(m). Firstly, $a^{m-1}_{\ m+1}$ must alter the representation labels of the subgroup U(m) and we obtain a primary decomposition into U(m) shift components

$$a^{m-1}_{\ m+1} = \sum_{r=1}^{m} \psi\binom{m}{r}^{m-1},$$

where

$$\psi\binom{m}{r}^i = P\binom{m}{r}^i_{\ j} a^j_{\ m+1} = a^j_{\ m+1} \bar{P}\binom{m}{r}^i_{\ j}.$$

Now each $\psi\binom{m}{r}^{m-1}$ is also a component of a vector operator with respect to U($m-1$). Hence, we may further decompose $\psi\binom{m}{r}^{m-1}$ into its U($m-1$) shift components according to

$$\psi\binom{m}{r}^{m-1} = \sum_{l=1}^{m-1} \psi\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}^{m-1},$$

where

$$\begin{aligned} \psi\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}^{m-1} &= P\binom{m-1}{l}^{m-1}_{\ \ i} \psi\binom{m}{r}^i \\ &= \psi\binom{m}{r}^i \bar{P}\binom{m-1}{l}^{m-1}_{\ \ i}. \end{aligned}$$

Hence, we obtain the resolution

$$a^{m-1}_{\ m+1} = \sum_{r=1}^{m} \sum_{l=1}^{m-1} \psi\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}^{m-1},$$

where each component $\psi\left(\begin{smallmatrix} m & m-1 \\ r & l \end{smallmatrix}\right)$ simultaneously alters the representation labels of $U(m)$ and its subgroup $U(m-1)$ according to

$$\lambda_{k,m} \psi\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix} = \psi\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}(\lambda_{k,m} + \delta_{kr}),$$

$$\lambda_{k,m-1} \psi\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix} = \psi\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}(\lambda_{k,m-1} + \delta_{kl}).$$

By our construction the shift components $\psi\left(\begin{smallmatrix} m & m-1 \\ r & l \end{smallmatrix}\right)$ are given by

$$\psi\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix} = P\begin{pmatrix} m-1 & m \\ l & r \end{pmatrix}\psi(m)$$
$$= \psi(m)\bar{P}\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix},$$

where $P\binom{m\ \ m-1}{r\ \ \ l}$ may be interpreted as an $(m-1)\times m$ matrix of operators with entries

$$P\begin{pmatrix} m-1 & m \\ l & r \end{pmatrix}^i_j = \sum_{k=1}^{m-1} P\begin{pmatrix} m-1 \\ l \end{pmatrix}^i_k P\begin{pmatrix} m \\ r \end{pmatrix}^k_j,$$
$$i = 1,\ldots,m-1, \quad j = 1,\ldots,m.$$

Similarly, we define the operators $\bar{P}\binom{m\ \ m-1}{r\ \ \ l}$.

More generally, an operator a^l_{m+1} $(l < m+1)$ may be decomposed into a sum of shift components which simultaneously alter the representation labels of the subgroups $U(m)$, $U(m-1)$,...,$U(l)$. We write this decomposition as

$$a^l_{m+1} = \sum_{i(\mathbf{k})} \psi\begin{pmatrix} m & m-1 & & l \\ i(m) & i(m-1) & \cdots & i(l) \end{pmatrix}^l, \tag{26}$$

where the summation symbol is shorthand for

$$\sum_{i(m)=1}^{m} \sum_{i(m-1)=1}^{m-1} \cdots \sum_{i(l)=1}^{l}.$$

Each shift component simultaneously alters the representation labels of the subgroups $U(m)$,...,$U(l)$ according to

$$\lambda_{k,p}\psi\begin{pmatrix} m & m-1 & & l \\ i(m) & i(m-1) & \cdots & i(l) \end{pmatrix}$$
$$= \psi\begin{pmatrix} m & m-1 & & l \\ i(m) & i(m-1) & \cdots & i(l) \end{pmatrix}(\lambda_{k,p} + \delta_{k,i(p)}) \tag{27}$$

for $p = l,\ldots,m$ and $k = 1,\ldots,p$.

These shift components may be constructed by repeated application of the subgroup projectors as in the a^{m-1}_{m+1} case. Let us denote the $l\times m$ matrix of operators with entries

$$\sum_{r=1}^{l} \cdots \sum_{q=1}^{m-2} \sum_{p=1}^{m-1} P\begin{pmatrix} l \\ i(l) \end{pmatrix}^i_r P\begin{pmatrix} l+1 \\ i(l+1) \end{pmatrix}^r \cdots P\begin{pmatrix} m-1 \\ i(m-1) \end{pmatrix}^q_p$$
$$\times P\begin{pmatrix} m \\ i(m) \end{pmatrix}^p_j$$

simply by

$$P\begin{pmatrix} l & & m-1 & m \\ i(l) & \cdots & i(m-1) & i(m) \end{pmatrix}.$$

It is clear that these operators project out the simultaneous shift components of the generator $a^l_{m+1} = \psi(m)^l$ from the left:

$$\psi\begin{pmatrix} m & & l \\ i(m) & \cdots & i(l) \end{pmatrix} = P\begin{pmatrix} l & & m \\ i(l) & \cdots & i(m) \end{pmatrix}\psi(m). \tag{28}$$

Similarly, we define the projectors

$$\bar{P}\begin{pmatrix} m & m-1 & & l \\ i(m) & i(m-1) & \cdots & i(l) \end{pmatrix},$$

whose (i,j) entry is given by

$$\sum_{p=1}^{m-1} \sum_{q=1}^{m-2} \cdots \sum_{r=1}^{l} \bar{P}\begin{pmatrix} m \\ i(m) \end{pmatrix}^p_i \bar{P}\begin{pmatrix} m-1 \\ i(m-1) \end{pmatrix}^q_p \cdots \bar{P}\begin{pmatrix} l \\ i(l) \end{pmatrix}^j_r$$

for $i = 1,\ldots,m$ and $j = 1,\ldots,l$. Clearly, these operators project out the simultaneous shift components of the generator $\psi(m)^l = a^l_{m+1}$ from the right:

$$\psi\begin{pmatrix} m & & l \\ i(m) & \cdots & i(l) \end{pmatrix} = \psi(m)\bar{P}\begin{pmatrix} m & & l \\ i(m) & \cdots & i(l) \end{pmatrix}. \tag{29}$$

In a similar way we define the operators

$$\bar{P}\begin{pmatrix} l & & m-1 & m \\ i(l) & \cdots & i(m-1) & i(m) \end{pmatrix}$$

and (30)

$$P\begin{pmatrix} m & m-1 & & l \\ i(m) & i(m-1) & \cdots & i(l) \end{pmatrix},$$

defined in the same way but with the order reversed.

By taking the Hermitian conjugate of Eqs. (26)–(29) we see that the generator a^{m+1}_l $(l < m+1)$ may also be resolved into its simultaneous shift components according to

$$a^{m+1}_l = \sum_{i(\mathbf{k})} \psi^\dagger\begin{pmatrix} m & m-1 & & l \\ i(m) & i(m-1) & \cdots & i(l) \end{pmatrix}_l,$$

where each component

$$\psi^\dagger\begin{pmatrix} m & & l \\ i(m) & \cdots & i(l) \end{pmatrix}$$

may be constructed by aplying the projectors (30):

$$\psi^\dagger\begin{pmatrix} m & m-1 & & l \\ i(m) & i(m-1) & \cdots & i(l) \end{pmatrix}$$
$$= \bar{P}\begin{pmatrix} l & & m-1 & m \\ i(l) & \cdots & i(m-1) & i(m) \end{pmatrix}\psi^\dagger(m)$$
$$= \psi^\dagger(m)P\begin{pmatrix} m & m-1 & & l \\ i(m) & i(m-1) & \cdots & i(l) \end{pmatrix}.$$

We conclude this section by obtaining a generalization of Eqs. (23) for the multiple shift vectors

$$\psi\begin{pmatrix} m & m-1 & & l \\ i(m) & i(m-1) & \cdots & i(l) \end{pmatrix}.$$

We have

$$\psi^\dagger\begin{pmatrix} m & & l \\ i(m) & \cdots & i(l) \end{pmatrix}_l \psi\begin{pmatrix} m & & l \\ i(m) & \cdots & i(l) \end{pmatrix}^l$$
$$= \sum_{i,j=1}^{m} \bar{P}\begin{pmatrix} l & & m \\ i(l) & \cdots & i(m) \end{pmatrix}^i_l \psi^\dagger(m)_i\psi(m)^j \bar{P}\begin{pmatrix} m & & l \\ i(m) & \cdots & i(l) \end{pmatrix}^l_j$$
$$= \sum_{i,j=1}^{m-1} \bar{P}\begin{pmatrix} l & & m-1 \\ i(l) & \cdots & i(m-1) \end{pmatrix}^i_l \psi^\dagger\begin{pmatrix} m \\ i(m) \end{pmatrix}_i \psi\begin{pmatrix} m \\ i(m) \end{pmatrix}^j$$
$$\times \bar{P}\begin{pmatrix} m-1 & & l \\ i(m-1) & \cdots & i(l) \end{pmatrix}^l_j. \tag{31}$$

However, from Eqs. (23), we know that

$$\psi^\dagger\begin{pmatrix} m \\ i(m) \end{pmatrix}_i \psi\begin{pmatrix} m \\ i(m) \end{pmatrix}^j = M_{i(m),m}\bar{P}\begin{pmatrix} m \\ i(m) \end{pmatrix}^j_i$$

and it follows that Eq. (31) may be written

$$M_{i(m),m}\bar{P}\begin{pmatrix} l & & m-1 \\ i(l) & \cdots & i(m-1) \end{pmatrix}\bar{P}\begin{pmatrix} m \\ i(m) \end{pmatrix}\bar{P}\begin{pmatrix} m-1 & & l \\ i(m-1) & \cdots & i(l) \end{pmatrix}^l_l. \tag{32}$$

By repeated application of Eq. (20) this in turn may be written

$$\prod_{p=l+1}^{m} (\alpha_{i(p),p} - \alpha_{i(p-1),p-1} - 1)^{-1}(\alpha_{i(p),p} - \alpha_{i(p-1),p-1})^{-1}$$
$$\times \prod_{r=l}^{m} M_{i(r),r}\bar{C}_{i(r),r}. \tag{33}$$

Similarly we have

$$\psi\begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l)\end{pmatrix}^{l}\psi^{\dagger}\begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l)\end{pmatrix}_{l}$$
$$= \prod_{p=l+1}^{m} (\alpha_{i(p),p} - \alpha_{i(p-1),p-1} - 1)^{-1}$$
$$\times(\alpha_{i(p),p} - \alpha_{i(p-1),p-1})^{-1}\prod_{r=l}^{m} \overline{M}_{i(r),r} C_{i(r),r}. \qquad (34)$$

These are the required generalizations of Eqs. (25). They in fact determine the squares of the matrix elements of a^{l}_{m+1} and a^{m+1}_{l}, respectively.

4. MATRIX ELEMENTS OF THE GROUP GENERATORS

Throughout this section we assume that we are working in a finite dimensional irreducible representation of U(n) and we shall adopt the usual Gel'fand basis notation. Our aim is to evaluate the matrix elements of the generators a^{l}_{m+1} and a^{m+1}_{l} ($l \leqslant m$). The matrix of a^{m+1}_{m+1} is of course diagonal with entries

$$\sum_{i=1}^{m+1} \lambda_{i,m+1} - \sum_{i=1}^{m} \lambda_{i,m}.$$

Suppressing the labels of U($m+2$), we may write an arbitrary Gel'fand pattern in the form

$$\left|\begin{matrix} \lambda_{i,m+1} \\ \lambda_{i,m} \\ \vdots \\ \lambda_{i,l-1} \\ (\nu) \end{matrix}\right\rangle,$$

where (ν) denotes a Gel'fand pattern for the subgroup U($l-2$). Let us fix this Gel'fand pattern and write it in the form $|\lambda_{j,k}\rangle$ for ease of notation. We begin by obtaining the matrix elements of the generators a^{m}_{m+1} and a^{m+1}_{m}.

Resolving a^{m}_{m+1} into its U(m) shift components, we have

$$a^{m}_{m+1}|\lambda_{j,k}\rangle = \sum_{r=1}^{m} \psi\begin{pmatrix} m \\ r\end{pmatrix}^{m}|\lambda_{j,k}\rangle$$
$$= \sum_{r=1}^{m} N^{m}_{r}[\lambda_{j,m+1};\lambda_{j,m};\lambda_{j,m-1}]$$
$$\times|\lambda_{j,k} + \Delta_{r,m}\rangle,$$

where $|\lambda_{j,k} + \Delta_{r,m}\rangle$ is shorthand notation for the state obtained from $|\lambda_{j,k}\rangle$ by increasing the label $\lambda_{r,m}$ of the group U(m) by one unit and leaving the remaining labels unchanged. The matrix elements N^{m}_{r}, in view of the Hermiticity property

$$\psi^{\dagger}\begin{pmatrix} m \\ r\end{pmatrix}_{m} = \left[\psi\begin{pmatrix} m \\ r\end{pmatrix}^{m}\right]^{\dagger}$$

and Eq. (25), are given by

$$N^{m}_{r}(\lambda_{j,m+1};\lambda_{j,m};\lambda_{j,m-1}) = \langle\lambda_{j,k}|M_{r,m}\bar{C}_{r,m}|\lambda_{j,k}\rangle^{1/2}. \qquad (35)$$

(Strictly speaking, this matrix element is to be multiplied by a phase factor. However, it is customary to choose the phases of the matrix elements of a^{m}_{m+1} to be real and positive. The question of phases shall be discussed more fully in the next section.) Substituting for $M_{r,m}$ and $\bar{C}_{r,m}$ using Eqs. (22) and (24) gives the result

$$N^{m}_{r} = \left(\frac{(-1)^{m}\Pi^{m+1}_{p=1}(\lambda_{p,m+1} - \lambda_{r,m} + r - p)\,\Pi^{m-1}_{l=1}(\lambda_{r,m} - \lambda_{l,m-1} + l - r + 1)}{\Pi^{m}_{l=1, \neq r}(\lambda_{r,m} - \lambda_{l,m} + l - r)(\lambda_{r,m} - \lambda_{l,m} + l - r + 1)}\right)^{1/2}. \qquad (36)$$

Similarly, the matrix elements of a^{m+1}_{m} are

$$\bar{N}^{m}_{r}(\lambda_{j,m+1};\lambda_{j,m};\lambda_{j,m-1}) = \langle\lambda_{j,k}|\overline{M}_{r,m}C_{r,m}|\lambda_{j,k}\rangle^{1/2}$$
$$= \left(\frac{(-1)^{m}\Pi^{m+1}_{p=1}(\lambda_{p,m+1} - \lambda_{r,m} + r - p + 1)\,\Pi^{m-1}_{=1}(\lambda_{r,m} - \lambda_{l,m-1} + l - r)}{\Pi^{m}_{l=1, \neq r}(\lambda_{r,m} - \lambda_{l,m} + l - r)(\lambda_{r,m} - \lambda_{l,m} + l - r - 1)}\right)^{1/2}. \qquad (37)$$

The method for calculating the matrix elements of a^{l}_{m+1} and a^{m+1}_{l} is similar and, in view of Eqs. (33) and (34), no more difficult. Resolving a^{l}_{m+1} ($l \leqslant m$) into its simultaneous shift components, we have

$$a^{l}_{m+1}|\lambda_{j,k}\rangle = \sum_{i(\mathbf{k})} \psi\begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l)\end{pmatrix}^{l}|\lambda_{j,k}\rangle$$
$$= \sum_{i(\mathbf{k})} N\begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l)\end{pmatrix}$$
$$\times|\lambda_{j,k} + \Delta_{i(m),m} + \cdots + \Delta_{i(l),l}\rangle,$$

where $|\lambda_{j,k} + \Delta_{i(m),m} + \cdots + \Delta_{i(l),l}\rangle$ denotes the state obtained from $|\lambda_{j,k}\rangle$ by increasing the representation label $\lambda_{i(r),r}$ of the subgroup $U(r)$, $r = l,\ldots,m$, by one unit and leaving the other labels unchanged. In this case the matrix elements

$$N\begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l)\end{pmatrix}$$

are given by

$$\pm\langle\lambda_{j,k}|\psi^{\dagger}\begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l)\end{pmatrix}_{l}\psi\begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l)\end{pmatrix}^{l}|\lambda_{j,k}\rangle^{1/2},$$

which, by virtue of Eqs. (33) and (35), equals

$$\pm\prod_{r=l}^{m} N^{r}_{i(r)}\prod_{r=l+1}^{m}[(\lambda_{i(r),r} - \lambda_{i(r-1),r-1} + i(r-1) - i(r))^{-1}$$
$$\times(\lambda_{i(r),r} - \lambda_{i(r-1),r-1} + i(r-1) - i(r) + 1)^{-1}]^{1/2}, \qquad (38)$$

where $N^{r}_{i(r)}$ are the matrix elements of the generator a^{r}_{r+1} which are given by Eq. (36). The undetermined phase ($\pm$) will be obtained in the next section.

Clearly,

$$N\begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l) \end{pmatrix}$$

corresponds to the matrix element

$$\left\langle \begin{matrix} \lambda_{j,m+1} \\ (\lambda') \end{matrix} \middle| a^l_{m+1} \middle| \begin{matrix} \lambda_{j,m+1} \\ (\lambda) \end{matrix} \right\rangle,$$

where $(\lambda') = (\lambda)$ except for $\lambda'_{i(r),r} = \lambda_{i(r),r} + 1$, $r = l,...,m$.

Similarly, the matrix elements

$$\bar{N}\begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l) \end{pmatrix}$$

of the generator a^{m+1}_l $(l < m+1)$ are given by

$$\pm \left\langle \lambda_{j,k} \middle| \psi\begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l) \end{pmatrix}^l \psi^\dagger \begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l) \end{pmatrix}_l \middle| \lambda_{j,k} \right\rangle^{1/2},$$

which, in view of Eq. (34), equals

$$\pm \prod_{r=l}^{m} \bar{N}^r_{i(r)} \prod_{r=l+1}^{m} [(\lambda_{i(r),r} - \lambda_{i(r-1),r-1} + i(r-1) - i(r) + 1)^{-1} (\lambda_{i(r),r} - \lambda_{i(r-1),r-1} + i(r-1) - i(r))^{-1}]^{1/2},$$

where $\bar{N}^r_{i(r)}$ are the matrix elements of the generator a^{r+1}_r which are given by Eq. (37).

5. CHOICE OF PHASES

In obtaining the matrix elements of the U(n) generators there is a degree of freedom in that the phases of the generators a^m_{m+1} may be chosen arbitrarily. Following Baird and Biedenharn,[3] we have chosen these phases to be positive [which agrees with the Condon–Shortley convention for SU(2)]. By Hermiticity it follows that the phases of the generators a^{m+1}_m are also positive. The phases of the remaining generators are then dictated by the Lie algebra commutation relations. It follows from these considerations that the general matrix element

$$N\begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l) \end{pmatrix}$$

has phase[3]

$$S(i(m-1) - i(m))S(i(m-2) - i(m-1))\cdots S(i(l) - i(l+1)),$$

where $S(x)$ is the sign of x and $S(0) = 1$.

It is interesting to note that the choice of phases may be obtained algebraically using the U(n) characteristic identities as demonstrated in Baird and Biedenharn.[4]

6. ANALYSIS OF RESULTS

We have shown that the only nonvanishing matrix elements of the generator a^l_{m+1} are of the form [suppressing the labels of the subgroup U($m+1$)]

$$\left\langle \begin{matrix} \lambda' \\ (\mu') \end{matrix} \middle| a^l_{m+1} \middle| \begin{matrix} \lambda \\ (\mu) \end{matrix} \right\rangle, \tag{39}$$

where λ' is of the form $\lambda' = \lambda + \Delta_{i(m)}$, where $\Delta_{i(m)}$ is the U(m) weight with 1 in position $i(m)$ and zero elsewhere. Also, since a^l_{m+1} is a vector with respect to the subgroups U(l),...,U($m-1$), we see that the only allowed patterns (μ') are of the form $(\mu') = (\mu)$ except $\mu'_{i(r),r} = \mu_{i(r),r} + 1$ for $r = l,...,m-1$ and some $i(r)$ in the range 1,...,r. The matrix element in this case is

$$N\begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l) \end{pmatrix}$$

and is given by Eq. (38). On the other hand, using the Wigner–Eckart theorem, this matrix element may also be written

$$\langle \lambda + \Delta_{i(m)} \| \psi(m) \| \lambda \rangle \left\langle \begin{matrix} \lambda \\ (\mu) \end{matrix}; \begin{matrix} \dot{1}0 \\ l \end{matrix} \middle| \begin{matrix} \lambda + \Delta_{i(m)} \\ (\mu') \end{matrix} \right\rangle, \tag{40}$$

where the first term is the U(m) reduced matrix element $(M_{i(m),m})^{1/2}$.

In the notation of Baird and Biedenharn[3] let us denote the U(l) Wigner coefficients $(\bar{C}_{i(l),l})^{1/2}$ by $\binom{i(l):l}{l-1}$, the reduced matrix element $(M_{i(r),r})^{1/2}$ by $\binom{r+1}{i(r):r}$, and the corresponding "reduced Wigner coefficients" by $\binom{i(r):r}{i(r-1):r-1}$. Then the matrix element

$$N\begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l) \end{pmatrix}$$

may be written in terms of reduced matrix elements, Wigner coefficients, and reduced Wigner coefficients according to[3]

$$N\begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l) \end{pmatrix} = \begin{pmatrix} m+1 \\ i(m):m \end{pmatrix} \prod_{r=l+1}^{m} \begin{pmatrix} i(r):r \\ i(r-1):r-1 \end{pmatrix} \begin{pmatrix} i(l):l \\ l-1 \end{pmatrix}.$$

It is interesting to note that by taking the trace of Eq. (20) we obtain the result

$$t_r\left(\bar{P}\begin{pmatrix} m & m+1 & m \\ r & k & r \end{pmatrix}\right) = \bar{C}_{k,m+1} M_{r,m} \bar{C}_{r,m} (\alpha_{k,m+1} - \alpha_{r,m} - 1)^{-1} \times (\alpha_{k,m+1} - \alpha_{r,m})^{-1}.$$

In terms of reduced Wigner coefficients this relation may be written in the form

$$t_r\left(\bar{P}\begin{pmatrix} m & m+1 & m \\ r & k & r \end{pmatrix}\right) = \begin{pmatrix} k:m+1 \\ r:m \end{pmatrix}^2 t_r\left(\bar{P}\begin{pmatrix} m \\ r \end{pmatrix}\right),$$

which shows that the reduced Wigner coefficients are determined solely by the subgroup projectors.

Finally, from Eq. (32) we may write the matrix element (39) in the form

$$\left\langle \begin{matrix} \lambda \\ (\mu) \end{matrix} \middle| M_{i(m),m} \bar{P}\begin{pmatrix} l & \cdots & m \\ i(l) & \cdots & i(m) \end{pmatrix} \bar{P}\begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l) \end{pmatrix}^l_l \middle| \begin{matrix} \lambda \\ (\mu) \end{matrix} \right\rangle^{1/2}.$$

Comparing this with the Wigner–Eckart factorization (40), we see that the general Wigner coefficient is given by

$$\left\langle \begin{matrix} \lambda \\ (\mu) \end{matrix} \middle| \bar{P}\begin{pmatrix} l & \cdots & m \\ i(l) & \cdots & i(m) \end{pmatrix} \bar{P}\begin{pmatrix} m & \cdots & l \\ i(m) & \cdots & i(l) \end{pmatrix}^l_l \middle| \begin{matrix} \lambda \\ (\mu) \end{matrix} \right\rangle = \left| \left\langle \begin{matrix} \lambda \\ (\mu) \end{matrix}; \begin{matrix} \dot{1}0 \\ l \end{matrix} \middle| \begin{matrix} \lambda + \Delta_{i(m)} \\ (\mu') \end{matrix} \right\rangle \right|^2. \tag{41}$$

This is clearly a generalization of Eq. (4) in Sec. 2.

7. CONCLUSION

Equation (41) shows that the general fundamental Wigner coefficients may be obtained solely from a knowledge of the subgroup projection operators. This form for the Wigner coefficients is useful and clearly may be generalized to arbitrary (multiplicity free) Wigner coefficients corresponding to the reduction of $V(\lambda) \otimes V(\mu)$, where $V(\lambda)$ is one of the tensor representations. One simply applies the U(n) projectors corresponding to the U(n) tensor identity with reference representation $V(\lambda)$ (see Ref. 1 for further details) and the canonical subgroup tensor projectors with reference representation given by the decomposition of $V(\lambda)$ into irreducible representations of its subgroups. By this means we may give a general expression for the U(n) Wigner operators of Biedenharn *et al.* as a polynomial in the group generators. This procedure is probably best described in the context of the pattern calculus and will be discussed more fully in a later publication.

Finally, we note that we have given an expression for the general matrix element (and the corresponding Wigner coefficients) as a polynomial in the group generators. This enables us to discuss "generalized matrix elements" without explicit reference to a basis state. It is therefore suggestive that this approach may be useful for obtaining generalized matrix elements for groups whose basis states are not known. In particular, it is hoped that useful information concerning the symplectic groups may be obtained by this method.

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