Open Journal of Discrete Mathematics, 2016, 6, 70-73
Published Online April 2016 in SciRes. http://www.scirp.org/journal/ojdm
http://dx.doi.org/10.4236/ojdm.2016.62006

On the Maximum Number of Dominating Classes in Graph Coloring

Bing Zhou

Department of Mathematics, Trent University, Peterborough, Canada
Email: bzhou@trentu.ca

Received 9 September 2015; accepted 3 April 2016; published 6 April 2016

Abstract

We investigate the dominating-χ-color number, $d_\chi(G)$, of a graph G. That is the maximum number of color classes that are also dominating when G is colored using $\chi(G)$ colors. We show that $d_\chi(G \vee H) = d_\chi(G) + d_\chi(H)$ where $G \vee H$ is the join of G and $\dot{H}$. This result allows us to construct classes of graphs such that $d_\chi(G) > 1$ and $d_\chi(G) = \chi(G)$ thus provide some information regarding two questions raised in [1] and [2].

Keywords

Graph Coloring, Dominating Sets, Dominating Coloring Classes, Chromatic Number, Dominating Color Number

1. Introduction

Let G be a graph with vertex set V and edge set E. A subset I of V is independent if no two vertices in I are adjacent. A subset S of V is a dominating set if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. We define a coloring C of G with k colors to be a partition of V into k independent sets:

$$C = \{C_1, C_2, \cdots, C_k\}$$

such that

$$C_1 \cup C_2 \cup \cdots \cup C_k = V$$

and C_i is independent for $i = 1, 2, \cdots, k$. The minimum of k for which such a partition is possible is the chromatic number of G, denoted $\chi(G)$. The dominating-χ-color number of G is motivated by a two-stage

How to cite this paper: Zhou, B. (2016) On the Maximum Number of Dominating Classes in Graph Coloring. *Open Journal of Discrete Mathematics*, **6**, 70-73. http://dx.doi.org/10.4236/ojdm.2016.62006

optimization problem. First, we partition the vertex set of G into the minimum number of independent sets; secondly, we maximize the independent sets that are also dominating in G. Clearly, the number of independent sets we use in the first stage will be $\chi(G)$, the chromatic number of G. Among all colorings of G using $\chi(G)$ colors, the maximum number of independent sets that are also dominating is defined to be the dominating-χ-color number of G, denoted by $d_\chi(G)$. Formally, we have

$$d_\chi(G) = \max\{\text{number of coloring classes of } \mathcal{C} \text{ that are dominating in } G : \mathcal{C} \text{ is a } \chi\text{-coloring of } G\}.$$

The dominating-χ-color number of G was first introduced in [2]. More research has been done in this area since then (see for example [1] [3] [4]). However, the two interesting questions posed in [1] and [2] remain unanswered. In this article, we present some more results about the dominating-χ-color number of a graph that are relevant to these two questions.

2. Main Results

The following observation was made in [2].

Theorem 1 *For all graph G,* $1 \le d_\chi(G) \le \chi(G)$.

The following two questions are posed in [1] and [2].

Question 1. Characterize the graphs G for which $d_\chi(G) = 1$.

Question 2. Characterize the graphs G for which $d_\chi(G) = \chi(G)$.

Neither of the two extreme cases is trivial. It is known that if G has an isolated vertex, then $d_\chi(G) = 1$. However, a graph G with $d_\chi(G) = 1$ can be connected and have arbitrarily large minimum degree.

Theorem 2. [1] *For every integer* $k \ge 0$, *there exists a connected graph G with* $\delta(G) = k$ *and* $d_\chi(G) = 1$.

The following lemma may help us understand the relation between the structure of a graph and its dominating-χ-color number. It shows that if a graph G contains a complete bipartite graph as a spanning subgraph, then the dominating-χ-color number of G is the sum of the dominating-χ-color numbers of these two subgraphs.

Lemma 1. *If* $V(G)$ *can be partitioned into two sets* V_1 *and* V_2 *such that every vertex in* V_1 *is adjacent to every vertex in* V_2, *then* $d_\chi(G) = d_\chi(G_1) + d_\chi(G_2)$ *where* G_i *is the subgraph of G induced by* V_i *for* $i = 1, 2$.

Proof. Since in any coloring of G, no vertex in V_1 can share a color with a vertex in V_2, we have $\chi(G) = \chi(G_1) + \chi(G_2)$. Let $\chi(G_1) = k_1$ and $\chi(G_2) = k_2$. Let C_1 be a k_1-coloring of G_1 with $d_\chi(G_1)$ dominating coloring classes using the colors $\{1, 2, \cdots, k_1\}$. Let C_2 be a k_2-coloring of G_2 with $d_\chi(G_2)$ dominating coloring classes using the colors $\{k_1+1, k_1+2, \cdots, k_1+k_2\}$. The combination of C_1 and C_2 is clearly a (k_1+k_2)-coloring of G. A coloring class of C is either a coloring class of C_1 or a coloring class of C_2. Suppose that S is a coloring class of C_1 that dominates G_1. Every vertex in $V_1 \setminus S$ is adjacent to at least one vertex in S. Every vertex in V_2 is adjacent to every vertex in S. Therefore S is a dominating set in G. Similarly, every coloring class of C_2 that dominates G_2 is a dominating set in G. C is a coloring of G with at least $d_\chi(G_1) + d_\chi(G_2)$ coloring classes. We have $d_\chi(G) \ge d_\chi(G_1) + d_\chi(G_2)$.

Suppose that C' is a coloring of G with $\chi(G)$ colors and $d_\chi(G)$ dominating coloring classes. The restriction of C' to G_i is a coloring of G_i with $\chi(G_i)$ colors for $i = 1, 2$. Let S be a dominating coloring class of C'. $S \subset V_1$ or $S \subset V_2$. Suppose that $S \subset V_1$. Then S is a dominating set for G_1. Therefore, every dominating coloring class of C' is either a dominating coloring class of G_1 or a dominating coloring class of G_2. Therefore $d_\chi(G_1) + d_\chi(G_2) \ge d_\chi(G)$.

Using Lemma 1, we have a sufficient condition for the dominating-χ-color number of a graph to be greater than one.

Corollary 1. *If the complement of G is disconnected, then* $d_\chi(G) > 1$.

The join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is defined by

$$V(G_1 \vee G_2) = V(G_1) \cup V(G_2),$$

$$E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}.$$

In other words, we construct $G_1 \vee G_2$ by taking a copy of each of G_1 and G_2 and joining every vertex in

G_1 with every vertex in G_2. It is known that $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$. By Lemma 1, there is a similar relation between the dominating-χ-color numbers.

Theorem 3. $d_\chi(G_1 \vee G_2) = d_\chi(G_1) + d_\chi(G_2)$.

It is shown in [1] that it is possible for a graph with chromatic number k to have dominating-χ-color number l for any k such that $1 \le l \le k$ and $(k,l) \ne (2,1)$. We present a new construction to prove this result using Theorem 3.

Theorem 4. *For all integers* k,l *such that* $1 \le l \le k$ *and* $(k,l) \ne (2,1)$, *there exists a connected graph* G *with* $\chi(G) = k$ *and* $d_\chi(G) = l$.

Proof. We prove by induction on l. If $l = 1$, the existence of such graphs is guaranteed by Theorem 2. For $(k,l) = (3,2)$, it is easy to check that $\chi(C_5) = 3$ and $d_\chi(C_5) = 2$. Therefore the theorem is true for $(k,l) = (3,2)$. Suppose that $l > 1$ and $(k,l) \ne (3,2)$. Let $k' = k-1$ and $l' = l-1$. $(k',l') \ne (2,1)$. By inductive hypothesis, there is a connected graph H with $\chi(H) = k'$ and $d_\chi(H) = l'$. Let $G = H \vee K_1$. Since $\chi(K_1) = d_\chi(K_1) = 1$, by Theorem 3 we have

$$\chi(G) = \chi(H) + 1 = k' + 1 = k$$

and

$$d_\chi(G) = d_\chi(H) + 1 = l' + 1 = l.$$

This proves the theorem.

Next we turn our attention to Question 2. Arumugam *et al*. [2] showed that if G is uniquely χ-colorable, then $d_\chi(G) = \chi(G)$. Therefore if G contains a subgraph that is uniquely $\chi(G)$-colorable, then $d_\chi(G) = \chi(G)$. It is natural to ask whether there are any other kind of such graph, that is, whether there are any graph G such that $d_\chi(G) = \chi(G) = k$ and G does not contain a uniquely k-colorable subgraph. For $k = 2$, the answer is no since every edge is a uniquely 2-colorable subgraph. For $k = 3$, the answer is yes. Arumugam *et al*. [1] showed that $d_\chi(C_{6i+3}) = \chi(C_{6i+3}) = 3$ for any nonnegative integer i. C_{6i+3} was not uniquely 3-colorable for $i > 0$. Using this fact and Theorem 3, we can show that the answer of our question is yes for all $k \ge 3$.

First, we need a technical lemma.

Lemma 2. *The graph* $G = G_1 \vee G_2$ *is uniquely* $(\chi(G_1) + \chi(G_2))$*-colorable if and only if* G_1 *is uniquely* $\chi(G_1)$*-colorable and* G_2 *is uniquely* $\chi(G_2)$*-colorable.*

The proof is easy and omitted.

Theorem 5. *Let* k *be an integer greater than* 3. *There is a graph* G_k *such that* $d_\chi(G_k) = \chi(G_k) = k$ *and* G_k *do not contain a uniquely* k*-colorable subgraph.*

Proof. We prove by induction on k. We have shown that the statement is true for $k = 3$. Suppose that $k \ge 4$ and the statement is true for $k-1$. Let $G_k = G_{k-1} \vee K_1$. Since $d_\chi(K_1) = \chi(K_1) = 1$, $d_\chi(G_k) = d_\chi(G_{k-1}) + d_\chi(K_1) = k$ by Theorem 3 and the inductive hypothesis. Every k-chromatic subgraph H of G_k must have the form $H = H_{k-1} \vee K_1$ where H_{k-1} is a subgraph of G_{k-1}. By Lemma 2, H is uniquely k-colorable if and only if H_{k-1} is uniquely $(k-1)$-colorable. Since G_{k-1} does not contain a uniquely $(k-1)$-colorable subgraph, G_k does not contain any uniquely k-colorable subgraph. This proves the theorem.

The graphs constructed in Theorem 5 contain large cliques. In fact, G_k contains many copies of K_{k-1}. If $k = 3l + j$ for some integers l and j, we may reduce the size of the largest clique in G_k by taking the join of copies of C_9 in the first l steps and then taking the join with K_1 afterwards. Thus, we have the following result.

Theorem 6. *Let* j,l *be nonnegative integers and* $k = 3l + j$. *There is a graph* G_k *such that* $d_\chi(G_k) = \chi(G_k) = k$. G_k *does not contain a uniquely* k*-colorable subgraph and the largest clique in* G_k *has size* $2l + j$.

3. Remarks

It is well known that there are uniquely k-colorable graphs with arbitrarily large girth. Therefore, there are graphs G such that $d_\chi(G) = \chi(G)$ and G has arbitrarily large girth. In light of Theorems 5 and 6, we would like to ask the following question.

Question 3. Are there triangle-free graphs G such that $d_\chi(G) = \chi(G) = k$, and does G not contain a uniquely k-colorable graph? Furthermore, are there such graphs with arbitrarily large girth?

References

[1] Arumugam, S., Haynes, T.W., Henning, M.A. and Nigussie, Y. (2011) Maximal Independent Sets in Minimum Colorings. *Discrete Mathematics*, **311**, 1158-1163. http://dx.doi.org/10.1016/j.disc.2010.06.045

[2] Arumugam, A., Hamid, I.S. and Muthukamatchi, A. (2008) Independent Domination and Graph Clorings. *Ramanujan Mathematical Society Lecture Notes Series*, **7**, 195-203.

[3] Arumugam, S. and Chandrasekar, K.R. (2012) Minimal Dominating Sets in Maximum Domatic Partitions. *Australasian Journal of Combinatorics*, **52**, 281-292.

[4] Li, S., Zhang, H. and Zhang, X. (2013) Maximal Independent Sets in Bipartite Graphs with at Least One Cycle. *Discrete Mathematics and Theoretical Computer Science*, **15**, 243-258.