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On the Maximum of a Bivariate INMA Model with Integer Innovations

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Abstract
We study the limiting behaviour of the maximum of a bivariate (finite or infinite) moving average model, based on discrete random variables. We assume that the bivariate distribution of the iid innovations belong to the Anderson's class (Anderson, 1970). The innovations have an impact on the random variables of the INMA model by binomial thinning. We show that the limiting distribution of the bivariate maximum is also of Anderson's class, and that the components of the bivariate maximum are asymptotically independent.

Keywords Bivariate maximum · INMA model · Integer random variables · limit distribution

1 Introduction

Hall (2003) studied the limiting distribution of the maximum term $M_n = \max(X_1, \cdots, X_n)$ of stationary sequences $\{X_j\}$ defined by non-negative integer-valued moving average (INMA) sequences of the form

$$X_j = \sum_{i=-\infty}^{+\infty} \alpha_i \circ V_{j-i},$$

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where the innovation sequence $\{V_i\}$ is an iid sequence of non-negative integer-valued random variables (rvs) with exponential type tails of the form

$$1 - F_V(n) \sim n^{\xi} L(n)(1+\lambda)^{-n},\ n \to +\infty, \tag{1}$$

where $\xi \in \mathbb{R}$, $\lambda > 0$, $L(n)$ is slowly varying at $+\infty$ and $\alpha_i \circ$ denotes binomial thinning with probability $\alpha_i \in [0,1]$. Hall (2003) proved that $\{X_j\}$ satisfies Leadbetter's conditions $D(x+b_n)$ and $D'(x+b_n)$, for a suitable real sequence b_n, and then

$$\begin{cases} \limsup_{n\to+\infty} P(M_n \le x+b_n) \le \exp(-(1+\lambda/\alpha_{\max})^{-x}) \\ \liminf_{n\to+\infty} P(M_n \le x+b_n) \ge \exp(-(1+\lambda/\alpha_{\max})^{-(x-1)}), \end{cases}$$

for all real x and $\alpha_{\max} := \max\{\alpha_i,\ i \in \mathbb{Z}\}$. Note that $\alpha_{\max}$ plays an important role in this result. This is an extension of Theorem 2 of Anderson (1970), where it is proved that for sequences of iid rvs with an integer-valued distribution function (df) F with infinite right endpoint, the limit

$$\lim_{n\to+\infty} \frac{1-F(n-1)}{1-F(n)} = r > 1, \tag{2}$$

is equivalent to

$$\exp(-r^{-(x-1)}) \le \liminf_{n\to+\infty} F^n(x+b_n) \le \limsup_{n\to+\infty} F^n(x+b_n) \le \exp(-r^{-x}),$$

for all real x.

The class of dfs satisfying (1), which is a particular case of (2) (see, e.g., Hall and Temido (2007)) is called Anderson's class.

In this paper we extend the result of Hall (2003) for the bivariate case of an INMA model. Concretely, we study the limiting distribution of the maximum term of stationary sequences $\{(X_j, Y_j)\}$ where the two marginals are defined by non-negative integer-valued moving average sequences of the general form

$$(X_j, Y_j) = \left(\sum_{i=-\infty}^{+\infty} \alpha_i \circ V_{j-i},\ \sum_{i=-\infty}^{+\infty} \beta_i \circ W_{j-i} \right),$$

where X_j and Y_j are defined as above with respect to a two-dimensional iid innovation sequence $\{V_i, W_i\}$. The binomial thinning operator $\beta\circ$, due to Steutel and van Harn (1979), is defined by $\beta \circ Z = \sum_{s=1}^{Z} B_s(\beta)$, $\beta \in [0,1]$, where $\{B_s(\beta)\}$ is an iid sequence of Bernoulli rvs independent of the positive integer rv Z. The possible class of bivariate discrete distributions $F_{V,W}$ (see (4)) includes also the bivariate geometric models.

We assume that $X = \alpha \circ V$ and $Y = \beta \circ W$ are conditionally independent given (V, W), because the binomial thinning with $\alpha\circ$ and $\beta\circ$ are independent, X and Y are binomial rv's. with parameters (V, α) respectively (W, β), i.e.

$$\begin{aligned} P(X \in A, Y \in B | V = v, W = w) &= P(X \in A | V = v, W = w) P(Y \in B | V = v, W = w) \\ &= P(X \in A | V = v) P(Y \in B | W = w), \end{aligned}$$

for all events A and B and for all possible values of v and w. We assume that $\alpha_i, \beta_i \in [0,1]$ and

$$\alpha_i, \beta_i = O\left(|i|^{-\delta}\right),\ |i| \to +\infty, \tag{3}$$

for some $\delta > 2$.

We investigate the limiting behaviour of $(M_n^{(1)}, M_n^{(2)}) = (\max_{1\le j\le n} X_j, \max_{1\le j\le n} Y_j)$ and want to find out whether the two maxima components are asymptotically dependent, because of the dependence of the innovations (V_i, W_i). However, we will show that this is not occurring because of the independent thinning, as we believe. We investigate the impact of the dependence of (V_i, W_i) on the limiting distribution and the convergence rate.

Following similar ideas of Hall (2003) for the univariate case, we:

- Define a bivariate model $F_{V,W}$ which contains the bivariate geometric model;
- Characterize the tail of $(\alpha\circ V, \beta\circ W)$ and the tail of (X_j, Y_j), in terms of the model $F_{V,W}$;
- Establish the limiting behaviour of the bivariate maximum $(M_n^{(1)}, M_n^{(2)})$ of the stationary sequence $\{(X_j, Y_j)\}$ which is defined componentwise; and
- Investigate the convergence of the joint distribution of the bivariate maximum to the limiting distribution by simulations.

Examples: 1.) We may consider the $\{(V_i, W_i)\}$ as the number of person newly infected by virus1 (say COVID-19 virus) and virus2 (say the usual seasonal virus) at time i. It is possible that a person is infected by both virus only at the same time point. We count by $\{(X_j, Y_j)\}$ the total number of infected and still contagious persons at time j adding all infected persons before and at time j. After some time these persons are cured (or died) and are no more counted to the number of infected but still contagious persons. Hence the random numbers (V_i, W_i) are thinned at each time point, so the contribution to X_j is $\alpha_{j-i}\circ V_i$, and to Y_j is $\beta_{j-i}\circ W_i$ for $i \le j$.

2.) Another example for bivariate integer valued time series is presented in Pedeli and Karlis (2011) who discuss the bivariate INAR(1) model with negative binomial innovations for the application of road accidents at two different time intervals in Schiphol area. However their bivariate negative binomial innovations are in the case of geometric innovations of a different type herein considered. Similar is the situation in the paper of Silva et al. (2020) who discuss inference of such a bivariate time series with different distribution of the innovations. But their bivariate negative binomial distribution is also not of our type.

3.) A further application of a bivariate time series for count data in finance is given by Quoreshi (2006). He did not specify the bivariate distribution. He derived the mean and variance/covariances of this time series.

2 Preliminaries Results for Bivariate Innovations

Let (V, W) be a non-negative random vector with bivariate df $F_{V,W}$ satisfying

$$\begin{aligned} 1 - F_{V,W}(v,w) \;=\; & \left(1+\lambda_1\right)^{-[v]}[v]^{\xi_1}L_1(v) + \left(1+\lambda_2\right)^{-[w]}[w]^{\xi_2}L_2(w) \\ & - (1+\lambda_1)^{-[v]}(1+\lambda_2)^{-[w]}\theta^{\min([v],[w])}L_3(v)L_4(w)v^{\xi_3}w^{\xi_4}\ell(v,w), \end{aligned} \tag{4}$$

as $v, w \to +\infty$, for positive real constants $\lambda_i > 0$, $i = 1, 2$, $\theta > 0$ such that $\theta < \min\{1+\lambda_1, 1+\lambda_2\}$ and $\theta > 1 - \lambda_1\lambda_2$, some real constants ξ_i and slowly varying functions L_i, $i = 1, 2, 3, 4$, and where $\ell(v,w)$ is a positive bounded (say by ϑ) function which converges to a positive constant L as $v, w \to \infty$. That $\ell(v,w)$ converges to L is for simplicity. It has no impact on the results if the limit L would depend on $v < w, v = w$ or $v > w$. By $[x]$ we denote the greatest integer not greater than x.

Remark 2.1 The marginal tails of $F_{V,W}$ are of the form:

$$1 - F_V(v) = [v]^{\xi_1}(1+\lambda_1)^{-[v]}L_1(v) \quad \text{and} \quad 1 - F_W(w) = [w]^{\xi_2}(1+\lambda_2)^{-[w]}L_2(w), \tag{5}$$

for $v, w \to +\infty$. Hence, both marginal dfs belong to the Anderson's class with

$$\lim_{v\to+\infty} \frac{1-F_V(v)}{1-F_V(v+1)} = 1+\lambda_1 \quad \text{and} \quad \lim_{w\to+\infty} \frac{1-F_W(w)}{1-F_W(w+1)} = 1+\lambda_2.$$

From (4), we can derive the probability function (pf) of (V, W). Because the proofs of the following propositions are technical, we move them to Appendix Proofs.

Proposition 2.1 *The pf of the random vector* (V, W) *with df* (4) *is given by*

$$P(V=v, W=w) \quad = (1+\lambda_1)^{-v}(1+\lambda_2)^{-w}\theta^{\min([v],[w])-1}L_3(v)L_4(w)v^{\xi_3}w^{\xi_4}\ell(v,w)\ell^*(v,w),$$

for v, w *large integers, where*

$$\lim_{v,w\to+\infty} \ell^*(v,w) = \begin{cases} \lambda_2\big(1+\lambda_1-\theta\big)\,, & v<w, \\ \lambda_1\lambda_2+\theta-1\,, & w=v, \\ \lambda_1\big(1+\lambda_2-\theta\big)\,, & w<v, \end{cases} \tag{6}$$

and $\ell(v,w)\ell^*(v,w)$ *is bounded and converges to positive constants.*

Example 2.1 The Bivariate Geometric (BG) distribution is a particular case of the model (4) with margins (5). Consider the bivariate Bernoulli random vector (B_1, B_2) with $P(B_1 = k, B_2 = \ell) = p_{k\ell}, (k,\ell) \in \{0,1\}^2$, and success marginal probabilities $p_{+1} = p_{01} + p_{11}$ and $p_{1+} = p_{10} + p_{11}$. Due to Mitov and Nadarajah (2005), using the construction of a BG, the pf and the df of a random vector (V, W) with BG distribution are given, respectively, by

$$f_{V,W}(v,w) = P(V=v, W=w) = \begin{cases} p_{00}^{v}p_{10}p_{+0}^{w-v-1}p_{+1}\,, & 0 \le v < w, \\ p_{00}^{v}p_{11}\,, & v = w, \\ p_{00}^{w}p_{01}p_{0+}^{v-w-1}p_{1+}\,, & 0 \le w < v, \end{cases} \tag{7}$$

for $v, w \in \mathbb{N}_0$, and

$$\begin{aligned} F_{V,W}(v,w) =& P(V \le v, W \le w) \\ =& 1 - p_{0+}^{[v]+1} - p_{+0}^{[w]+1} + \begin{cases} p_{00}^{[v]+1}p_{+0}^{[w]-[v]}\,, & 0 \le v \le w, \\ p_{00}^{[w]+1}p_{0+}^{[v]-[w]}\,, & 0 \le w < v, \end{cases} \end{aligned} \tag{8}$$

for $v, w \in \mathbb{R}_0^+$, assuming that $0 < p_{0+}, p_{+0} < 1$. Hence, this df satisfies (4) with the constants λ_1, λ_2 given by

$$1+\lambda_1 = \frac{1}{p_{0+}} > 1 \quad \text{and} \quad 1+\lambda_2 = \frac{1}{p_{+0}} > 1$$

and the index θ associated to the dependence structure of (B_1, B_2) is

$$\theta = \frac{p_{00}}{p_{0+}p_{+0}}.$$

The slowly varying functions are constants and $\xi_i = 0$, for $i = 1, 2, 3, 4$. The independence case occurs when $\theta = 1$. For dependence cases, we can have $0 < \theta < 1$ or $\theta > 1$. Finally, we note that $\ell(v, w)$ is a constant. For instance, take $L_1(v) = L_3(v) = 1/(1 + \lambda_1)$, $L_2(v) = L_4(v) = 1/(1 + \lambda_2)$, we have $\ell(v, w) = \theta$ with $\ell^*(v, w)$ as in (6).

The marginal df of V and W are obviously

$$P(V \leq v) = 1 - p_{0+}^{[v]+1} \text{ and } P(W \leq w) = 1 - p_{+0}^{[w]+1}, \text{ for } v, w \geq 0,$$

which means V and W are geometrically distributed rvs with parameter p_{1+} and p_{+1}, respectively. □

In order to characterize the df of $(X, Y) = (\alpha \circ V, \beta \circ W)$ we start by establishing the relationship between the probability generating function (pgf) of (V, W) and (X, Y), defined e.g. for (V, W) as

$$G_{V,W}(s_1, s_2) := \sum_{k_1=0}^{+\infty} \sum_{k_2=0}^{+\infty} P(V = k_1, W = k_2) s_1^{k_1} s_2^{k_2},$$

which exists for (s_1, s_2) in the following region $\mathcal{R}$ (given in Lemma 2.1).

Taking into account Proposition 2.1, the series $G_{V,W}(s_1, s_2)$ converges obviously for any $s_i \leq 1$. Even for some $s_i > 1$ the series converges because of the assumption (4). By this assumption, we have $E(s_1^V) < +\infty$ if $s_1 < 1 + \lambda_1$ and $E(s_2^W) < +\infty$ if $s_2 < 1 + \lambda_2$. The following lemma gives a condition such that the series $G_{V,W}(s_1, s_2)$ exists.

Lemma 2.1 *The pgf $G_{V,W}(s_1, s_2) = E(s_1^V s_2^W)$ exists for (s_1, s_2) in*

$$\mathcal{R} = \left\{ (s_1, s_2) \in \mathbb{R}_+^2 \ : \ s_1 s_2 < \frac{(1 + \lambda_1)(1 + \lambda_2)}{\theta}, \, s_1 < 1 + \lambda_1, \, s_2 < 1 + \lambda_2 \right\}.$$

Its more technical proof is given also in the appendix. As consequence of this lemma, the pgf $G_{V,W}(s_1, s_2)$ exists for $s_1, s_2 > 1$, if $s_i \leq 1 + \lambda_i, i = 1, 2$ in case $\theta \leq 1$, and if $s_1 \leq 1 + \lambda_1$ and $s_2\theta \leq 1 + \lambda_2$ in case of $\theta > 1$. In the following, we use these convenient conditions for the convergence of $G_{V,W}$.

Now the relationship of the two pgf is the following. It holds as long as the pgf's exist. For our derivations it is convenient to use in the following the given domain $\mathcal{R}$. The proof of this relationship is also given in the appendix.

Proposition 2.2 *The pgf of $(X, Y) = (\alpha \circ V, \beta \circ W)$ is given in terms of the pgf of (V, W):*

$$G_{X,Y}(s_1, s_2) = G_{V,W}(\alpha s_1 + 1 - \alpha, \beta s_2 + 1 - \beta),$$

for all (s_1, s_2) such that $(\alpha s_1 + 1 - \alpha, \beta s_2 + 1 - \beta) \in \mathcal{R}$.

We want to derive an exact relationship of the two distributions $F_{V,W}$ and $F_{X,Y}$ with the help of a suitable transformation, as a modified pgf or a Mellin transform. We define the (bivariate) modified pgf or tail generating function (Sagitov (2017))

$$Q_{V,W}(s_1, s_2) = \sum_{k_1=0}^{+\infty} \sum_{k_2=0}^{+\infty} \left(1 - F_{(V,W)}(k_1, k_2)\right) s_1^{k_1} s_2^{k_2},$$

and analogously for X, Y. The relationship between $Q_{V,W}$ and $G_{V,W}$ is given in the following proposition.

Proposition 2.3 *For* $(s_1, s_2) \in \mathcal{R}$, *we have*

$$(1 - s_1)(1 - s_2) Q_{V,W}(s_1, s_2) = 1 - G_{V,W}(s_1, s_2).$$

Proposition 2.4 *The modified pgf of* (X, Y) *and* (V, W) *satisfy*

$$Q_{X,Y}(s_1, s_2) = \alpha\beta Q_{V,W}(\alpha s_1 + 1 - \alpha, \beta s_2 + 1 - \beta),$$

if the series converge, i.e. $(\alpha s_1 + 1 - \alpha, \beta s_2 + 1 - \beta) \in \mathcal{R}$.

From Propositions 2.2 and 2.4, we can derive now the tail $1 - F_{X,Y}$ in terms of $1 - F_{V,W}$

Proposition 2.5 *The df* $F_{X,Y}$ *is given in terms of the df* $F_{V,W}$ *with* $x, y \in \mathbb{Z}^+$:

$$1 - F_{X,Y}(x, y) = \sum_{k=x}^{+\infty} \sum_{\ell=y}^{+\infty} \binom{k}{x} \binom{\ell}{y} (1 - \alpha)^{k-x} (1 - \beta)^{\ell-y} \alpha^{x+1} \beta^{y+1} \left(1 - F_{V,W}(k, \ell)\right).$$

Hence the tail of $F_{X,Y}$ can be estimated by the assumption (4).

Proposition 2.6 *If the joint df of* (V, W) *satisfies* (4), *then for large integers* x *and* y

$$1 - F_{X,Y}(x, y) = \left(1 + \frac{\lambda_1}{\alpha}\right)^{-x} x^{\xi_1} L_1^*(x) + \left(1 + \frac{\lambda_2}{\beta}\right)^{-y} y^{\xi_2} L_2^*(y) - H(x, y)$$

with

$$0 \le H(x, y) \le \vartheta L_3^*(x) x^{\xi_3} \left(1 + \frac{\lambda_1}{\alpha}\right)^{-x} L_4^*(w) y^{\xi_4} \left(1 + \frac{\lambda_{2\theta}}{\beta}\right)^{-y},$$

where L_i^* *are slowly varying functions, being*

$$L_1^*(x) \sim \alpha \left(\frac{1 + \lambda_1}{\lambda_1 + \alpha}\right)^{\xi_1 + 1} L_1(x), \; L_2^*(y) \sim \beta \left(\frac{1 + \lambda_2}{\lambda_2 + \beta}\right)^{\xi_2 + 1} L_2(y),$$

$$L_3^*(x) \sim \alpha \left(\frac{1 + \lambda_1}{\lambda_1 + \alpha}\right)^{\xi_3 + 1} L_3(x), \; L_4^*(y) \sim \beta \left(\frac{1 + \lambda_{2\theta}}{\lambda_{2\theta} + \beta}\right)^{\xi_4 + 1} L_4(y),$$

with

$$\lambda_{2\theta} = \begin{cases} \lambda_2 \, , & \theta \le 1 \\ \frac{1+\lambda_2}{\theta} - 1 \, , & \theta > 1, \end{cases} \tag{9}$$

and ϑ the bound of $\ell(v, w)$.

Note that $1 < \lambda_{2\theta} < \lambda_2$.

We observe that the stationary bivariate INMA model $\left(X_j, Y_j\right)$ introduced in our work is an extension of the BINAR model of Pedeli and Karlis (2011) defined by

$$\left(\widetilde{X}_j, \widetilde{Y}_j\right) = \left(\alpha \circ \widetilde{X}_{j-1} + R_{1j}, \beta \circ \widetilde{Y}_{j-1} + R_{2j}\right)$$

with an iid innovations sequence $\{(R_{1j}, R_{2j})\}$. In their paper it is stated that it has also the representation

$$\left(\widetilde{X}_j, \widetilde{Y}_j\right) \overset{d}{=} \left(\sum_{i=0}^{+\infty} \alpha^i \circ R_{1,j-i}, \sum_{i=0}^{+\infty} \beta^i \circ R_{2,j-i}\right). \tag{10}$$

Hence, considering $\left(X_j, Y_j\right)$ with $\alpha_i = \beta_i = 0$ for $i < 0$, $\alpha_i = \alpha^i$ and $\beta_i = \beta^i$ for $i \geq 0$ we obtain $\left(\widetilde{X}_j, \widetilde{Y}_j\right)$.

3 The Bivariate Stationary Sequence

We consider now the stationary bivariate INMA model $\{\left(X_j, Y_j\right)\}$ with iid innovations $\{(V_i, W_i)\}$ with df satisfying (4). We establish first the tail behaviour of $\left(X_j, Y_j\right)$. The maximal values of α_i and β_i are most important as in the univariate case. Therefore we write $\alpha_{\max} = \max\{\alpha_i : |i| \geq 0\}$ and $\beta_{\max} = \max\{\beta_i : |i| \geq 0\}$. We assume that they are unique. It may happen in the bivariate case that $\alpha_{\max}$ and $\beta_{\max}$ occurs at the same index or at different ones. We consider both cases. Furthermore, we use that

$$\sum_{i=-\infty}^{+\infty} \alpha_i < +\infty, \sum_{i=-\infty}^{+\infty} \beta_i < +\infty. \tag{11}$$

which holds because of (3).

Suppose first that $\alpha_{\max}$ and $\beta_{\max}$ are occuring at different indexes i_0 and i_1, respectively. We write for any j

$$X_j = \alpha_{\max} \circ V_{j-i_0} + \alpha_{i_1} \circ V_{j-i_1} + \sum_{i \neq i_0, i_1} \alpha_i \circ V_{j-i}$$

and

$$Y_j = \beta_{\max} \circ W_{j-i_1} + \beta_{i_0} \circ W_{j-i_0} + \sum_{i \neq i_0, i_1} \beta_i \circ W_{j-i}\,.$$

Denote $S_1 = \alpha_{\max} \circ V_{j-i_0}$, $S_2 = \alpha_{i_1} \circ V_{j-i_1}$, $S_3 = \sum_{i \neq i_0, i_1} \alpha_i \circ V_{j-i}$, $S = S_2 + S_3$, $T_1 = \beta_{\max} \circ W_{j-i_1}$, $T_2 = \beta_{i_0} \circ W_{j-i_0}$ and $T_3 = \sum_{i \neq i_0, i_1} \beta_i \circ W_{j-i}$, $T = T_2 + T_3$. Hence, $X_j = S_1 + S_2 + S_3 = S_1 + S$ and $Y_j = T_1 + T_2 + T_3 = T_1 + T$. Note that S, S_i, T and T_i depend on j.

For the proof of the main proposition of this section we need the following lemma.

Lemma 3.1

a) *If the rv V belongs to the Anderson's class, then* $E(1+h)^V = 1 + hE(V)(1+o_h(1))$, as $h \to 0^+$.

b) *For any set I of integers with* $\alpha_I = \max\{\alpha_i, i \in I\}$, *consider the rv* $Z = \sum_{i \in I} \alpha_i \circ V_{-i}$. *Then* $E(1+h)^Z$ *is finite for any* $0 < h < \frac{\lambda_1}{\alpha_I}$.

The proof of this lemma is given in the appendix. We deal now with the limiting behaviour of the tail of (X_j, Y_j). Besides of the univariate tail distributions we derive only an appropriate positive upper bound $H^*(x,y)$ for the joint tail which is sufficient for the asymptotic limit distribution of the maxima. We will see that we get asymptotic independence of the components of the bivariate maxima $(M_n^{(1)}, M_n^{(2)})$, since this normalized $H^*(x,y)$ is vanishing, not contributing to the limit.

For the asymptotic behaviour of the tail of the stationary distribution of the sequence $\{(X_j, Y_j)\}$, we write simply (X, Y) for any (X_j, Y_j). As mentioned we deal with the two cases that $\alpha_{\max}$ and $\beta_{\max}$ are occurring at different indexes or at the same one. We start with the first case and the above defined S, S_i, T, T_i.

For this derivation, we use $\psi, \rho \in (0,1)$ and $\lambda > 0$ such that $\frac{\lambda_1}{\alpha_{\max}} < \lambda < \frac{\lambda_1}{\alpha^*}$, with $\alpha^* = \max\{\alpha_i, i \neq i_0\}$, and $\lambda_{2\theta}$ given in (9),

$$1 + \frac{\lambda_1}{\alpha_{\max}} < (1+\lambda)^{\Psi} < 1 + \lambda < 1 + \frac{\lambda_1}{\alpha^*} \tag{12}$$

and

$$\rho < B = \log\left(1 + \frac{\lambda_2}{\beta_{\max}}\right) / \log\left(1 + \frac{\lambda_{2\theta}}{\beta_{i_0}}\right). \tag{13}$$

Proposition 3.1 *If* (V, W) *satisfies* (4) *and* $\alpha_{\max}$ *and* $\beta_{\max}$ *are unique and taken at different indexes, then*

(i) *for the marginal dfs*

$$1 - F_X(x) \sim x^{\xi_1}\left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^{-[x]} L_1^{**}(x),\ x \to +\infty,$$

and

$$1 - F_Y(y) \sim y^{\xi_2}\left(1 + \frac{\lambda_2}{\beta_{\max}}\right)^{-[y]} L_2^{**}(y),\ y \to +\infty,$$

(ii) *for the joint df with* ψ, ρ, λ *satisfying* (12) *and* (13)

$$\begin{aligned}1 - F_{X,Y}(x,y) = {} & x^{\xi_1}\left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^{-[x]} L_1^{**}(x)\big(1 + o_x(1)\big) \\ & + y^{\xi_2}\left(1 + \frac{\lambda_2}{\beta_{\max}}\right)^{-[y]} L_2^{**}(y)(1 + o_y(1)) - H^*(x,y),\end{aligned}$$

as $x, y \to +\infty$, *where*

$$L_1^{**}(x) = L_1^*(x)E\left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^S, \quad L_2^{**}(y) = L_2^*(y)E\left(1 + \frac{\lambda_2}{\beta_{\max}}\right)^T \tag{14}$$

and

$$0 \leq H^*(x,y) \leq o_y(1)\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-x}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-\rho y} x^{\xi_3}L_3^*(x) + Cx^{\xi_1+1}y^{\xi_2}L_1^*(x)L_2^*(y)\times$$
$$\times\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-(1-\psi)x}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-y+(\log y)^2} + O_x(P(S>\psi x)), \tag{15}$$

for some constant $C > 0$.

We show also that $P(S > \psi x) = o_x(P(S_1 > x))$.

Proof In fact

$$1 - F_{(X,Y)}(x,y) = 1 - F_X(x) + 1 - F_Y(y) - P(X > x, Y > y). \tag{16}$$

We deal with the three terms in (16), separately.

(i) Since $\frac{\lambda_1}{\alpha_{\max}} < \frac{\lambda_1}{\alpha^*}$, taking the sum $S = Z$ in Lemma 3.1, we conclude that $E\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^S$ is finite. Similarly $E\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^T$ is finite since $\frac{\lambda_2}{\beta_{\max}} < \frac{\lambda_2}{\beta^*}$ with $\beta^* = \max\{\beta_i, i \neq i_1\}$. The tail function of X is given, with $\psi_x = [\psi x]$, by

$$\begin{aligned}1 - F_X(x) &= P\big(S_1 + S > x\big) = \sum_{k=0}^{+\infty} P(S_1 > x-k)P(S=k)\\ &= P\big(S_1 > x\big)\sum_{k=0}^{\psi_x}\frac{P(S_1 > x-k)}{P(S_1>x)}P(S=k) + \\ &\quad + \sum_{k=\psi_x+1}^{+\infty} P(S_1 > x-k)P(S=k).\end{aligned} \tag{17}$$

For the first sum of (17), we get by applying Proposition 2.6 with $\alpha = \alpha_{\max}$ for the marginal distribution

$$\begin{aligned}\sum_{k=0}^{\psi_x}\frac{P(S_1>x-k)}{P(S_1>x)}P(S=k) &= \sum_{k=0}^{\psi_x}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^k(1+o_x(1))P(S=k)\\ &\to \sum_{k=0}^{+\infty}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^k P(S=k)\\ &= E\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^S, \ x \to +\infty,\end{aligned}$$

by dominated convergence. For the second sum in (17), we get for x large

$$\sum_{k=\psi_x+1}^{+\infty} P(S_1 > x-k)P(S=k) \leq P(S > \psi_x)$$
$$= P\big((1+\lambda)^S > (1+\lambda)^{\psi_x}\big) \leq \frac{E(1+\lambda)^S}{(1+\lambda)^{\psi_x}}, \tag{18}$$

using the Markov inequality, since $E(1+\lambda)^S$ is finite for $\lambda < \lambda_1/\alpha^*$. Since $(1+\lambda)^\psi > 1 + \frac{\lambda_1}{\alpha_{\max}}$, we get by Theorem 4 of Hall (2003)

$$\frac{(1+\lambda)^{-\psi_x}}{P(S_1 > x)} \to 0,\ x \to +\infty, \tag{19}$$

and thus together

$$1 - F_X(x) = P(S_1 > x)\left[E\left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^S + O_x\left(\frac{(1+\lambda)^{-\psi_x}}{P(S_1 > x)}\right)\right]$$
$$= P(S_1 > x)E\left(1 + \frac{\lambda_1}{\alpha_{\max}}\right)^S (1 + o_x(1)).$$

With the same arguments we characterize the tail $1 - F_Y$. Hence, the statements on the marginal dfs are shown.

(ii) Now we deal with the third term in (16). Note that (S_1, T_2), (S_2, T_1) and (S_3, T_3) in the representation of X and Y are independent. For any $\psi \in (0, 1)$ and $\lambda > 0$ satisfying (12), we use that (18) and (19) imply

$$P(S_2 + S_3 > \psi x) = P(S > \psi x) = O_x((1+\lambda)^{-\psi x}) \tag{20}$$

and

$$P(S > \psi x) = o_x(P\big(S_1 > x\big)). \tag{21}$$

The probability in the third term of (16) is split into four summands with $\psi < 1$ satisfying (12), $\psi_x = [\psi x]$ and $\delta_y = [y - (\log y)^2]$. We get for x and y large,

$$P(X > x,\ Y > y) = P(S_1 + S_2 + S_3 > x,\ T_1 + T_2 + T_3 > y)$$
$$= \sum_{k=0}^{\psi_x}\sum_{\ell=0}^{\delta_y} P(S_1 > x-k, T_2 > y-\ell)P(S_2 + S_3 = k, T_1 + T_3 = \ell) +$$
$$+ \sum_{k=0}^{\psi_x}\sum_{\ell=\delta_y+1}^{+\infty} P(S_1 > x-k, T_2 > y-\ell)P(S_2 + S_3 = k, T_1 + T_3 = \ell) +$$
$$+ \sum_{k=\psi_x+1}^{+\infty}\sum_{\ell=0}^{\delta_y} P(S_1 > x-k, T_2 > y-\ell)P(S_2 + S_3 = k, T_1 + T_3 = \ell) + \tag{22}$$
$$+ \sum_{k=\psi_x+1}^{+\infty}\sum_{\ell=\delta_y+1}^{+\infty} P(S_1 > x-k, T_2 > y-\ell)P(S_2 + S_3 = k, T_1 + T_3 = \ell)$$
$$=: \sum_{m=1}^{4} S_m(\psi_x, \delta_y)$$

to simplify the proof. The last sum $S_4(\psi_x, \delta_y)$ is bounded by $P(S_2 + S_3 > \psi_x, T_1 + T_3 > \delta_y) \le P(S_2 + S_3 > \psi_x) = O_x((1+\lambda)^{-\psi_x})$ by (20). For the first sum $S_1(\psi_x, \delta_y)$ of (22) we use Proposition 2.6 and obtain with $\rho < 1$ such that (13) holds,

$$\sum_{k=0}^{\psi_x}\sum_{\ell=0}^{\delta_y} P(S_1 > x-k, T_2 > y-\ell)P(S_2+S_3 = k, T_1+T_3 = \ell)$$

$$\le \vartheta \sum_{k=0}^{\psi_x}\sum_{\ell=0}^{\delta_y} ([x]-k)^{\xi_3}([y]-\ell)^{\xi_4}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-([x]-k)}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-([y]-\ell)} \times$$

$$\times L_3^*([x]-k)\, L_4^*([y]-\ell)P\big(S_2+S_3 = k, T_1+T_3 = \ell\big)$$

$$\le \vartheta \sum_{k=0}^{\psi_x}\sum_{\ell=0}^{\delta_y} ([x]-k)^{\xi_3}([y]-\ell)^{\xi_4}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-([x]-k)}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-((1-\rho)+\rho)([y]-\ell)} \times$$

$$\times L_3^*([x]-k)L_4^*([y]-\ell)P\big(S_2+S_3 = k, T_1+T_3 = \ell\big).$$

Note that $([y]-\ell)^{\xi_4}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-(1-\rho)([y]-\ell)} L_4^*[y]-\ell = o_y(1)$ uniformly for $\ell \le \delta_y$, i.e. $y - \ell > (\log y)^2 \to \infty$. Hence the sum is bounded above by

$$o_y(1)x^{\xi_3}L_3^*(x)\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-x}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-\rho y} \times$$

$$\times \sum_{k=0}^{\psi_x}\sum_{\ell=0}^{\delta_y}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{k}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{\rho\ell} P\big(S_2+S_3 = k, T_1+T_3 = \ell\big)$$

$$\le o_y(1)x^{\xi_3}L_3^*(x)\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-x}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-\rho y} \times$$

$$\times E\left[\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{(S_2+S_3)}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{\rho(T_1+T_3)}\right]$$

$$\le o_y(1)x^{\xi_3}L_3^*(x)\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-x}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-\rho y},$$

since the last pgf exists due to Lemma 3.1 and (13) Note that

$$E\left[\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{(S_2+S_3)}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{\rho(T_1+T_3)}\right]$$

$$= E\left[\prod_{i\neq i_0}(1+\frac{\lambda_1}{\alpha_{\max}})^{\alpha_i\circ V_{-i}}\left((1+\frac{\lambda_{2\theta}}{\beta_{i_0}})^{\rho}\right)^{\beta_i\circ W_{-i}}\right]$$

$$= \prod_{i\neq i_0} E\left[\left(1+\frac{\alpha_i\lambda_1}{\alpha_{\max}}\right)^{V_{-i}}\left(1+\beta_i([1+\frac{\lambda_{2\theta}}{\beta_{i_0}}]^{\rho}-1)\right)^{W_{-i}}\right].$$

The expectations exist by assumption (4) since $1+\frac{\alpha_i\lambda_1}{\alpha_{\max}} < 1+\lambda_1$, and also $1+\beta_i([1+\frac{\lambda_{2\theta}}{\beta_{i_0}}]^\rho - 1) \le 1+\beta_{\max}([1+\frac{\lambda_{2\theta}}{\beta_{i_0}}]^\rho - 1) < 1+\lambda_2$, for all i, by the choice of ρ in (13), by using the arguments of Lemma 3.1. We consider now the approximation of the second sum $S_2(\psi_x,\delta_y)$ in (22). We have with some positive constant C

$$\begin{aligned} S_2(\psi_x,\delta_y) \le& \sum_{k=0}^{\psi_x}\sum_{\ell=\delta_y+1}^{+\infty} P(S_1 > x-k)P(T_1+T_3=\ell) \\ \le& Cx^{\xi_1+1}L_1^*(x)\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-(1-\psi)x} P(T_1+T_3>\delta_y). \end{aligned} \tag{23}$$

By the arguments used to approximate $P(X>x)=P(S_1+S_2+S_3>x)$ in (i), we also obtain

$$P(T_1+T_3>\delta_y) \sim Cy^{\xi_2}\, L_2^*(y) E\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{T_3}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-\delta_y},\ y\to+\infty,$$

with some generic constant C. Hence, it implies together with (23)

$$\begin{aligned} &\sum_{k=0}^{\psi_x}\sum_{\ell=\delta_y+1}^{+\infty} P(S_1>x-k, T_2>y-\ell)P(S_2+S_3=k, T_1+T_3=\ell) \\ &\le Cx^{\xi_1+1}y^{\xi_2}L_1^*(x)\, L_2^*(y)\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-(1-\psi)x}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-y+(\log y)^2},\ y\to+\infty. \end{aligned}$$

For the third sum $S_3(\psi_x,\delta_y)$ in (22), we get analogously to the derivation of the second sum

$$\begin{aligned} S_3(\psi_x,\delta_y) &\le \delta_y P(T_2>y-\delta_y)P(S_2+S_3>\psi_x) \\ &\le y(\log y)^{2\xi_2}L_2^*((\log y)^2)(1+\frac{\lambda_2}{\beta_{i_0}})^{-(\log y)^2}P(S_2+S_3>\psi_x) \\ &= o_y(1)P(S_2+S_3>\psi_x) = o_x(P(S>\psi x)). \end{aligned}$$

Combining now the bounds of the four terms $S_i(\psi_x,\delta_y)$, we get the upper bound for $H^*(x,y)$ which shows our statement. □

Suppose now the case that the unique $\alpha_{\max}$ and $\beta_{\max}$ are taken at the same index i_0, say. Write for any j

$$X_j = \alpha_{\max}\circ V_{j-i_0} + \sum_{i\ne i_0}\alpha_i\circ V_{j-i}$$

and

$$Y_j = \beta_{\max}\circ W_{j-i_0} + \sum_{i\ne i_0}\beta_i\circ W_{j-i}.$$

Denote $S_1=\alpha_{\max}\circ V_{j-i_0}$, $S=\sum_{i\ne i_0}\alpha_i\circ V_{j-i}$, $T_1=\beta_{\max}\circ W_{j-i_0}$, and $T=\sum_{i\ne i_0}\beta_i\circ W_{j-i}$, as used for Proposition 3.1. Observe that (S_1,T_1) and (S,T) are independent. Then the corresponding statement of Proposition 3.1 holds for this case (letting $\beta_{i_0} = \beta_{\max}$) which is given in Proposition 3.2. We omit the proof since it is very similar to the given one with a few obvious changes.

Proposition 3.2 *If* (V, W) *satisfies* (4) *and* $\alpha_{\max}$ *and* $\beta_{\max}$ *are unique, occurring at the same index, then the stationary distribution satisfies*

$$\begin{aligned} 1 - F_{X,Y}(x,y) = {} & x^{\xi_1}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-[x]} L_1^{**}(x)\big(1+o_x(1)\big) \\ & + y^{\xi_2}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-[y]} L_2^{**}(y)\big(1+o_y(1)\big) \\ & - H^*(x,y), \end{aligned}$$

as $x, y \to +\infty$, *where*

$$L_1^{**}(x) = L_1^*(x)E\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^S, \quad L_2^{**}(y) = L_2^*(y)E\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^T$$

and

$$\begin{aligned} 0 \le H^*(x,y) \le {} & o_y(1)x^{\xi_3}L_3^*(x)\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-x}\left(1+\frac{\lambda_{2\theta}}{\beta_{\max}}\right)^{-y} + \\ & + Cy^{\xi_2}L_2^*(y)x^{\xi_1+1}L_1^*(x)\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-(1-\psi)x}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-y+(\log y)^2} \\ & + O_x(P\big(S > \psi_x\big)), \end{aligned}$$

for some constant $C > 0$ *and* $\psi \in (0,1)$ *satisfying* (12).

Now we investigate the limiting behaviour for the bivariate maxima, in case of an iid sequence $\{(X_j, Y_j)\}$.

Theorem 3.1 *Let* (V, W) *be such that* (4) *holds and* $\alpha_{\max}$ *and* $\beta_{\max}$ *are unique, occurring either at the same or not the same index. Let*

$$d_1 = 1/\log(1+\frac{\lambda_1}{\alpha_{\max}}), \quad d_2 = 1/\log(1+\frac{\lambda_2}{\beta_{\max}}).$$

Define the normalizations

$$u_n(x) = x + d_1[\log n + \xi_1 \log\log n + \log L_1^{**}(\log n) + \xi_1 \log d_1] \tag{24}$$

and

$$v_n(y) = y + d_2[\log n + \xi_2 \log\log n + \log L_2^{**}(\log n) + \xi_2 \log d_2]. \tag{25}$$

Then, for x, y *real,*

$$\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-x}+\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-y}\le \liminf_{n\to\infty} n(1-F_{(X,Y)})(u_n(x),v_n(y))$$

$$\le \limsup_{n\to\infty} n(1-F_{(X,Y)})(u_n(x),v_n(y))\le \left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-x+1}+\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-y+1}.$$

Proof The convergence for the marginal distributions holds by applying Proposition 3.1 or 3.2 with the chosen normalization sequences. Since $u_n(x)$ and $v_n(y)$ are similar in type, we only show the derivation of the first marginal. Because the normalization $u_n(x)$ is not always an integer, we have to consider lim sup and lim inf. Let us deal with the lim sup case. Note that

$$\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-d_1\log n}=\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-d_2\log n}=\frac{1}{n}$$

and

$$\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-d_1(\xi_1\log\log n+\log L_1^{**}(\log n)+\xi_1\log d_1)}=\frac{(d_1\log n)^{-\xi_1}}{L_1^{**}(\log n)}.$$

For the normalization we get

$$[u_n(x)]\ge x-1+d_1(\log n+\xi_1\log\log n+\log L_1^{**}(\log n)+\xi_1\log d_1)\sim d_1\log n.$$

So

$$\begin{aligned}&n\times[u_n(x)]^{\xi_1}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-[u_n(x)]}L_1^{**}([u_n(x)])\\&\lesssim n\times(d_1\log n)^{\xi_1}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-x+1-d_1(\log n+\xi_1\log\log n+\log L_1^{**}(\log n)+\xi_1\log d_1)}L_1^{**}(\log n)\\&=\left(1+\lambda_1/\alpha_{\max}\right)^{-(x-1)}.\end{aligned}$$

The derivation of the lim inf is similar using $[u_n(x)]\le u_n(x)$.

Now for the joint distribution we use the bounds of $H^*(u_n,v_n)$ of the two propositions. First we consider the case of Proposition 3.1 with α_{max} and β_{max} at different indexes. We have to derive the limits of three boundary terms of $H^*(u_n,v_n)$ given in Proposition 3.1 multiplied by n. The last of these terms tends to 0 because (21) holds and due to the fact that from (14), we get

$$\begin{aligned}nP(S_1>u_n(x))&=n\times(u_n(x))^{\xi_1}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-u_n(x)}L_1^{*}(u_n(x))\\&\sim\left(1+\lambda_1/\alpha_{\max}\right)^{-x}L_1^{*}(\log n)/L_1^{**}(\log n)\\&\sim\left(1+\lambda_1/\alpha_{\max}\right)^{-x}/E\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{S},\end{aligned}$$

which is bounded.

The first of the three boundary terms of $H^*(u_n,v_n)$ is smaller than

$$
\begin{aligned}
& no_n(1)\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-u_n(x)}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-\rho v_n(y)}(u_n(x))^{\xi_3}L_3^*(u_n(x)) \\
& = no_n(1)\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-d_1\log n+o(\log n)}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)^{-\rho d_2\log n+o(\log n)}(d_1\log n)^{\xi_3}L_3^*(\log n) \\
& = o_n(1)(\log n)^{\xi_3}L_3^*(\log n)\exp\left(-\rho d_2\log n\log\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0}}\right)+o(\log n)\right) \\
& = o_n(1)(\log n)^{\xi_3}L_3^*(\log n)\exp\left(-(\rho/B)\log n(1+o_n(1))\right) \\
& = o_n(1),
\end{aligned}
$$

because $\rho/B > 0$ with B given by (13).

The second boundary term of $H^*(u_n, v_n)$ is smaller than

$$
\begin{aligned}
& nC_1(d_1\log n)^{\xi_1+1}(d_2\log n)^{\xi_2}L_1^*(\log n)L_2^*(\log n) \\
& \times\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-(1-\psi)d_1\log n+o(\log n)}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-d_2\log n+(\log(d_2\log n))^2} \\
\leq & C_1(\log n)^{\xi_1+\xi_2+1}L_1^*(\log n)L_2^*(\log n)\exp\left(\log n-(1-\psi)\log n-\log n+o(\log n)\right) \\
= & o_n(1),
\end{aligned}
$$

since $1-\psi > 0$ and where C_1 represents a generic positive constant.

Thus the limiting distribution is proved in case of Proposition 3.1.

Now let us consider the changes of the proof for the case of Proposition 3.2. Again we have to deal with the three boundary terms of $H^*(u_n, v_n)$ where the last two are as in Proposition 3.1. In the first of these terms we have similarly

$$
\begin{aligned}
& no_n(1)(u_n(x))^{\xi_3}L_3^*(u_n(x))\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-u_n(x)}\left(1+\frac{\lambda_{2\theta}}{\beta_{\max}}\right)^{-v_n(y)} \\
& = o_n(1)(\log n)^{\xi_3}L_3^*(\log n)\exp\left(-d_2\log n\log\left(1+\frac{\lambda_{2\theta}}{\beta_{\max}}\right)+o(\log n)\right) \\
& = o_n(1),
\end{aligned}
$$

since $d_2\log\left(1+\frac{\lambda_{2\theta}}{\beta_{\max}}\right) > 0$. Thus the statements are shown. □

4 Main result

We consider now the stationary sequence $\{(X_j, Y_j)\}$. From extreme value theory it is known that the behaviour of their extremes is as in the case of an iid sequence $\{(X_j, Y_j)\}$ if the following two conditions hold: a mixing condition, called $D(u_n, v_n)$, and a local dependence condition, called $D'(u_n, v_n)$. In our bivariate extreme value case we consider the conditions $D(u_n, v_n)$ and $D'(u_n, v_n)$ of Hüsler (1990) (see also Hsing (1989) and Falk et al. (1990)). The condition $D(u_n, v_n)$ is a long range mixing one for extremes and means that extreme values occurring in largely separated (by ℓ_n) intervals of positive integers are asymptotically independent. The condition $D'(u_n, v_n)$ considers the local dependence of extremes and excludes asymptotically the occurrences of local clusters of extreme or large values in each individual margin of $\{(X_j, Y_j)\}$ as well as jointly in the two components. We write u_n, v_n for short because x, y do not play a role in the following proofs.

Definition 4.1 The sequence $\{(X_j, Y_j)\}$ satisfies the condition $D(u_n, v_n)$ if for any integers $1 \leq i_1 < ... < i_p < j_1 < ... < j_q \leq n$, for which $j_1 - i_p > \ell_n$, we have

$$\Big|P\Big(\bigcap_{s=1}^{p}\{X_{i_s} \leq u_n, Y_{i_s} \leq v_n\}, \bigcap_{t=1}^{q}\{X_{j_t} \leq u_n, Y_{j_t} \leq v_n\}\Big)$$
$$-P\Big(\bigcap_{s=1}^{p}\{X_{i_s} \leq u_n, Y_{i_s} \leq v_n\}\Big)P\Big(\bigcap_{t=1}^{q}\{X_{j_t} \leq u_n, Y_{j_t} \leq v_n\}\Big)\Big| \leq \alpha_{n,\ell_n},$$

for some α_{n,ℓ_n} with $\lim_{n\to+\infty} \alpha_{n,\ell_n} = 0$, for some integer sequence $\ell_n = o(n)$.

We use the following $D'(u_n, v_n)$ condition.

Definition 4.2 Let $\{s_n\}$ be a sequence of positive integers such that $s_n \to +\infty$. The sequence $\{(X_j, Y_j)\}$ satisfies the condition $D'(u_n, v_n)$ if

$$n\sum_{j=1}^{[n/s_n]}\Big\{P\big(X_0 > u_n, X_j > u_n\big) + P\big(X_0 > u_n, Y_j > v_n\big)$$
$$+ P\big(Y_0 > v_n, Y_j > v_n\big) + P\big(Y_0 > v_n, X_j > u_n\big)\Big\} \to 0,\ n \to +\infty.$$

In the following we use the sequences $\{s_n\}$, $\{\ell_n\}$ and α_{n,ℓ_n} such that

$$\lim_{n\to+\infty} s_n^{-1} = \lim_{n\to+\infty} \frac{s_n\ell_n}{n} = \lim_{n\to+\infty} s_n\alpha_{n,\ell_n} = 0. \tag{26}$$

Such a sequence $\{s_n\}$ in (26) exists always. Take e.g. for the given ℓ_n and α_{n,ℓ_n} in condition $D(u_n, v_n)$ the sequence $s_n = \min(\sqrt{n/\ell_n}, 1/\sqrt{\alpha_{n,\ell_n}}) \to +\infty$. In our proof we use simpler sequences.

Write $M_n^{(1)} = \max\{X_1, \cdots, X_n\}$ and $M_n^{(2)} = \max\{Y_1, \cdots, Y_n\}$. For the stationary sequence $\{(X_j, Y_j)\}$ satisfying $D(u_n, v_n)$ and $D'(u_n, v_n)$, the limiting behaviour of the bivariate maxima $\big(M_n^{(1)}, M_n^{(2)}\big)$, under linear normalization, is given in Theorem 3.1, as if the sequence $\{(X_j, Y_j)\}$ would be a sequence of independent (X_j, Y_j).

In Theorem 3.1 we derived upper and lower bounds of the limiting distribution of the maximum term of non-negative integer-valued moving average sequences which leads to a "quasi max-stable" limiting behavior of the bivariate maximum in the sense of Anderson's type. So the main result of the maximum of this bivariate discrete random sequence is the following.

Theorem 4.1 *Consider the stationary sequences* $\{(X_j, Y_j)\}$ *defined by*

$$(X_j, Y_j) = \left(\sum_{i=-\infty}^{+\infty} \alpha_i \circ V_{j-i}, \sum_{i=-\infty}^{+\infty} \beta_i \circ W_{j-i}\right).$$

Suppose that the innovation sequence $\{(V_i, W_i)\}$ *is an iid sequence of non-negative integer-valued random vectors with df of the form* (4), *the sequences of* $\{\alpha_i\}$ *and* $\{\beta_i\}$ *satisfy* (3) *and* $\alpha_{\max}$ *and* $\beta_{\max}$ *are unique. Then,*

$$\limsup\,(\liminf)\, P\big(M_n^{(1)} \leq u_n(x), M_n^{(2)} \leq v_n(y)\big) \lesseqgtr$$
$$\lesseqgtr \exp\left(-\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-(x-0(1))} - \left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-(y-0(1))}\right),$$

for all real x and y and where $u_n(x)$ *and* $v_n(y)$ *are defined by* (24) *and* (25).

To prove this theorem, it remains to show that the conditions $D(u_n, v_n)$ and $D'(u_n, v_n)$ hold with u_n and v_n given by (24) and (25).

Proof of $D(u_n, v_n)$**:**

Let $1 \leq i_1 \leq \cdots \leq i_p < j_1 \leq \cdots \leq j_q \leq n$ with $j_1 - i_p > 2\ell_n$, with separation $\ell_n = n^\phi$, where $\phi < 1$. We select ϕ later. We use the following notation:

$$X_j^\star = \sum_{k=\ell_n}^{+\infty} \alpha_k \circ V_{j-k}\,, \quad X_j^{\star\star} = \sum_{k=-\infty}^{-\ell_n} \alpha_k \circ V_{j-k}$$

and

$$Y_j^\star = \sum_{k=-\infty}^{-\ell_n} \beta_k \circ W_{j-k}\,, \quad Y_j^{\star\star} = \sum_{k=\ell_n}^{+\infty} \beta_k \circ W_{j-k}.$$

Note that

$$\left\{X_i - X_i^{\star\star}, Y_i - Y_i^\star\,, i \leq i_p\right\} = \left\{\sum_{k=-\ell_n+1}^{+\infty} \alpha_k \circ V_{i-k}, \sum_{k=-\ell_n+1}^{+\infty} \beta_k \circ W_{i-k}\,, i \leq i_p\right\}$$

and

$$\left\{X_j - X_j^\star, Y_j - Y_j^{\star\star}\,, j \geq j_1\right\} = \left\{\sum_{k=-\infty}^{\ell_n-1} \alpha_k \circ V_{j-k}, \sum_{k=-\infty}^{\ell_n-1} \beta_k \circ W_{j-k}\,, j \geq j_1\right\} \tag{27}$$

are independent.

a) We have as upper bound

$$\begin{aligned}
&P\left(\bigcap_{s=1}^{p}\{X_{i_s} \leq u_n, Y_{i_s} \leq v_n\}, \bigcap_{t=1}^{q}\{X_{j_t} \leq u_n, Y_{j_t} \leq v_n\}\right)\\
&\leq P\left(\bigcap_{s=1}^{p}\{X_{i_s} - X_{i_s}^{\star\star} \leq u_n, Y_{i_s} - Y_{i_s}^{\star} \leq v_n\}\right) P\left(\bigcap_{t=1}^{q}\{X_{j_t} - X_{j_t}^{\star} \leq u_n, Y_{j_t} - Y_{j_t}^{\star\star} \leq v_n\}\right)\\
&\leq P\left(\bigcap_{s=1}^{p}\{X_{i_s} \leq u_n + M_n^{(1,1)}, Y_{i_s} \leq v_n + M_n^{(1,2)}\}\right)\times\\
&\times P\left(\bigcap_{t=1}^{q}\{X_{j_t} \leq u_n + M_n^{(2,1)}, Y_{j_t} \leq v_n + M_n^{(2,2)}\}\right),
\end{aligned} \tag{28}$$

where $M_n^{(1,1)} = \max_{0\le j\le n} X_j^{\star\star}$, $M_n^{(1,2)} = \max_{0\le j\le n} Y_j^{\star}$, $M_n^{(2,1)} = \max_{0\le j\le n} X_j^{\star}$, and $M_n^{(2,2)} = \max_{0\le j\le n} Y_j^{\star\star}$.
We split furthermore this upper bound.

$$\begin{aligned} &P\Big(\bigcap_{s=1}^{p}\{X_{i_s}\le u_n, Y_{i_s}\le v_n\}, \bigcap_{t=1}^{q}\{X_{j_t}\le u_n, Y_{j_t}\le v_n\}\Big)\\ &\le \Big[P\Big(\bigcap_{s=1}^{p}\{X_{i_s}\le u_n+M_n^{(1,1)}, Y_{i_s}\le v_n+M_n^{(1,2)}\}, M_n^{(1,1)}=0, M_n^{(1,2)}=0\Big)\\ &+P\Big(M_n^{(1,1)}\ge 1\vee M_n^{(1,2)}\ge 1\Big)\Big]\\ &\times\Big[P\Big(\bigcap_{t=1}^{q}\{X_{j_t}\le u_n+M_n^{(2,1)}, Y_{j_t}\le v_n+M_n^{(2,2)}\}, M_n^{(2,1)}=0, M_n^{(2,2)}=0\Big)\\ &+P\Big(M_n^{(2,1)}\ge 1\vee M_n^{(2,2)}\ge 1\Big)\Big]\\ &\le\Big[P\big(\bigcap_{s=1}^{p}\{X_{i_s}\le u_n, Y_{i_s}\le v_n\}\big)+P\big(M_n^{(1,1)}\ge 1\vee M_n^{(1,2)}\ge 1\big)\Big]\\ &\times\Big[P\big(\bigcap_{t=1}^{q}\{X_{j_t}\le u_n, Y_{j_t}\le v_n\}\big)+P\big(M_n^{(2,1)}\ge 1\vee M_n^{(2,2)}\ge 1\big)\Big]\\ &\le P\Big(\bigcap_{s=1}^{p}\{X_{i_s}\le u_n, Y_{i_s}\le v_n\}\Big)\times P\Big(\bigcap_{t=1}^{q}\{X_{j_t}\le u_n, Y_{j_t}\le v_n\}\Big)\\ &+2P\big(M_n^{(1,1)}\ge 1\big)+2P\big(M_n^{(1,2)}\ge 1\big)+2P\big(M_n^{(2,1)}\ge 1\big)+2P\big(M_n^{(2,2)}\ge 1\big). \end{aligned} \tag{29}$$

The last four terms in (29) tend to 0 as it is proved in Hall (2003) depending on ℓ_n. We show it for one term.

$$\begin{aligned} P\big(M_n^{(1,1)}\ge 1\big) &\le (n+1)P\Big(\sum_{k=-\infty}^{-\ell_n}\alpha_k\circ V_{-k}\ge 1\Big)\\ &\le (n+1)\sum_{k=-\infty}^{-\ell_n}E(\alpha_k\circ V_{-k}) = (n+1)\sum_{k=-\infty}^{-\ell_n}\alpha_k E(V_{-k})\\ &\le Cn\sum_{k=\ell_n}^{+\infty}\frac{1}{k^{\delta}}\\ &\le Cn\ell_n^{1-\delta}, \end{aligned} \tag{30}$$

for some generic constant C and $\{\alpha_k\}$ satisfying (3) with $\delta > 2$. Selecting $\phi > 1/(\delta-1)$, this bound tends to 0. The sum of the bounds of the last four terms in (29) gives the bound $\alpha_{n,\ell_n} = Cn\ell_n^{1-\delta}$, which tends to 0.

b) In the same way we establish the lower bound of (28). In fact, using again the independence mentioned in (27), we get

$$
\begin{aligned}
&P\big(\bigcap_{s=1}^{p}\{X_{i_s}\le u_n, Y_{i_s}\le v_n\}\big)P\big(\bigcap_{t=1}^{q}\{X_{j_t}\le u_n, Y_{j_t}\le v_n\}\big)\\
\le &P\big(\bigcap_{s=1}^{p}\{X_{i_s}-X^{\star\star}_{i_s}\le u_n, Y_{i_s}-Y^{\star}_{i_s}\le v_n\}\big)P\big(\bigcap_{t=1}^{q}\{X_{j_t}-X^{\star}_{j_t}\le u_n, Y_{j_t}-Y^{\star\star}_{j_t}\le v_n\}\big)\\
= &P\big(\bigcap_{s=1}^{p}\{X_{i_s}-X^{\star\star}_{i_s}\le u_n, Y_{i_s}-Y^{\star}_{i_s}\le v_n\},\bigcap_{t=1}^{q}\{X_{j_t}-X^{\star}_{j_t}\le u_n, Y_{j_t}-Y^{\star\star}_{j_t}\le v_n\}\big)\\
\le &P\Bigg(\bigcap_{s=1}^{p}\{X_{i_s}\le u_n+M_n^{(1,1)}, Y_{i_s}\le v_n+M_n^{(1,2)}\},\\
&\bigcap_{t=1}^{q}\{X_{j_t}\le u_n+M_n^{(2,1)}, Y_{j_t}\le v_n+M_n^{(2,2)}\}\Bigg)\\
\le &P\Bigg(\bigcap_{s=1}^{p}\{X_{i_s}\le u_n, Y_{i_s}\le v_n\},\bigcap_{t=1}^{q}\{X_{j_t}\le u_n, Y_{j_t}\le v_n\}\Bigg)+Cn\ell_n^{1-\delta}
\end{aligned}
$$

using (29) and (30). Hence the condition $D(u_n, v_n)$ holds.

In the proof of $D'(u_n, v_n)$, we need also that $s_n\alpha_{n,\ell_n}\to 0$. With $s_n = n^{\zeta}$ we select ζ such that $s_n n\ell_n^{1-\delta} = n^{1+\zeta-\phi(\delta-1)}\to 0$, which holds for $1+\zeta<\phi(\delta-1)$.

Proof of $D'(u_n, v_n)$:

We have to consider first the sums on the terms $P\big(X_0 > u_n, Y_j > v_n\big)$ and on the terms $P\big(Y_0 > v_n, X_j > u_n\big)$.

We show it for the sum of the first terms, since for the second one the proof follows in the same way. Let $\gamma_n = n^{\nu}$ with $\nu < 1-\zeta$, which implies that $\gamma_n = o(n/s_n) = o(n^{1-\zeta})$. For $j < 2\gamma_n$, we write

$$
(X_0, Y_j) = \left(\sum_{i=-\infty}^{+\infty}\alpha_i\circ V_{-i}\,,\ \sum_{i=-\infty}^{+\infty}\beta_{i+j}\circ W_{-i}\right).
$$

Note that $\alpha_{i_0} = \alpha_{\max}$ for some i_0 and $\beta_{j_0} = \beta_{\max}$ for some j_0. For one j we have $i_0 + j = j_0$, i.e. $j = j_0 - i_0$. Hence the maximum terms occur at the same index for V_{-i_0} and W_{-i_0} if $j = j_0 - i_0$. If $j_0 = i_0$, hence $j = 0$, but this case does not occur in the sum. For all other j's the maxima is occurring at different indexes. We consider the bound established in Proposition 3.1 and 3.2 for H^*.

For $j = j_0 - i_0$, we showed in the proof of Theorem 3.1 that $nH^*(u_n, v_n)\to 0$.

For $j \ne j_0 - i_0$, we have $\beta_{i_0+j} < \beta_{\max}$ for the terms $P(X_0 > u_n, Y_j > v_n)$ and deduce from Proposition 3.1 the following upper bound for $H^*(u_n, v_n)$

$$
\begin{aligned}
&o_n(1)\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-u_n}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0+j}}\right)^{-\rho v_n} u_n^{\xi_3}L_3^*(u_n) + Cu_n^{\xi_1+1}v_n^{\xi_2}L_1^*(u_n)L_2^*(v_n)\times\\
&\times\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-(1-\psi)u_n}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-v_n+(\log v_n)^2} + O\big(P(S>\psi u_n)\big),
\end{aligned}
\tag{31}
$$

with $\rho,\psi\in(0,1)$ defined in (12) and (13). Note that $\rho=\rho(j)$ should be such that $\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0+j}}\right)^{\rho(j)} < 1+\frac{\lambda_2}{\beta_{\max}}$, for all $j\ne j_0-i_0$ that (13) is satisfied. It means that the term B in (13) depends on j, i.e. $B = B_j$. Note that B_j may be larger or smaller than 1, but is bounded above by $\log(1+\lambda_2/\beta_{\max})/\log(1+\lambda_{2\theta}/\beta_{\max}) = B^*$. For $B_j > 1$, we select $\epsilon < 1$ large such that $(1-\epsilon)B^* < 1$, which implies that $(1-\epsilon)B_j < 1$, thus we select $\rho(j) > (1-\epsilon)B_j$. In case $B_j \le 1$, we select also $\rho(j) > (1-\epsilon)B_j$.

It implies that there exists an $\epsilon > 0$ to select $\rho(j)$ for every $j \neq j_0 - i_0$ such that

$$\log\left(1+\frac{\lambda_2}{\beta_{\max}}\right) > \rho(j)\log\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0+j}}\right) > (1-\epsilon)\log\left(1+\frac{\lambda_2}{\beta_{\max}}\right).$$

a) Now the sum of the first term in the bound (31) of $H^*(u_n, v_n)$ multiplied by n, for $\{j \le 2\gamma_n, j \neq j_0\}$, is bounded by

$$\begin{aligned}
& o_n(1)n^{1+\nu}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-d_1\log n+o(\log n)}\left(1+\frac{\lambda_{2\theta}}{\beta_{i_0+j}}\right)^{-\rho(j)d_2\log n+o(\log n)}(d_1\log n)^{\xi_3}L_3^*(\log n)\\
& < o_n(1)\exp\left\{(\log n)\left(1+\nu-d_1\log\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)-d_2(1-\epsilon)\log\left(1+\frac{\lambda_2}{\beta_{\max}}\right)\right)+o(\log n)\right\}\\
& = o_n(1)\exp\left\{-(\log n)(1-\epsilon-\nu+o_n(1))\right\} \to 0,\ n\to+\infty,
\end{aligned}$$

if also ν is such that $\nu < 1-\epsilon$.

The sum of the second term in (31) multiplied by n, for $\{j \le 2\gamma_n, j \neq j_0\}$, tends to 0 because

$$\begin{aligned}
& n^{1+\nu}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-(1-\psi)u_n}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-v_n+(\log v_n)^2}\exp(o(\log n))\\
& = n^{1+\nu}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-(1-\psi)d_1\log n+o(\log n)}\left(1+\frac{\lambda_2}{\beta_{\max}}\right)^{-d_2\log n+o(\log n)}\exp(o(\log n))\\
& = \exp\{(\log n)[1+\nu-(1-\psi)-1]+o(\log n)\}\\
& = \exp\left\{(\log n)[\nu+\psi-1+o_n(1)]\right\} \to 0,\quad \text{as } n\to+\infty,
\end{aligned}$$

if also $\nu < 1-\psi$. Hence we choose $\nu < \min\{1-\epsilon, 1-\zeta, 1-\psi\}$.

It remains to deal with the sum of the third terms in (31) for $\{j \le 2\gamma_n, j \neq j_0\}$. We showed that $P(S > \psi x) = O((1+\lambda)^{-\psi x})$ in (20) with $(1+\lambda)^{\psi} > 1 + \frac{\lambda_1}{\alpha_{\max}}$ in (12). Let $\tilde{\psi} > 1$ such that $(1+\lambda)^{\psi/\tilde{\psi}} = 1 + \frac{\lambda_1}{\alpha_{\max}}$. This sum on $\{j \le 2\gamma_n, j \neq j_0\}$ multiplied with n is bounded by

$$Cn^{1+\nu}\left(1+\frac{\lambda_1}{\alpha_{\max}}\right)^{-\tilde{\psi}u_n} = C\exp\left\{(\log n)\left[1+\nu-\tilde{\psi}+o_n(1)\right]\right\} \to 0,\ n\to+\infty,$$

if also $\nu < \tilde{\psi} - 1$ and C is a generic positive constant.

Thus combining these three bounds it shows that

$$n\sum_{j\le 2\gamma_n} P\left(X_0 > u_n, Y_j > v_n\right) \to 0,\ n\to+\infty,$$

if $\nu < \min\{1-\epsilon, 1-\zeta, 1-\psi, \tilde{\psi}-1\}$.

b) We consider now the sum on j with $2\gamma_n < j \le n/s_n$ and write

$$X_0' = \sum_{i=-\gamma_n}^{+\infty} \alpha_i \circ V_{-i}\ , \quad X_0'' = \sum_{i=-\infty}^{-\gamma_n-1} \alpha_i \circ V_{-i},$$

$$X_j' = \sum_{i=-\infty}^{\gamma_n} \alpha_i \circ V_{j-i}\ , \quad X_j'' = \sum_{i=\gamma_n+1}^{+\infty} \alpha_i \circ V_{j-i}$$

and

$$Y_j' = \sum_{i=-\infty}^{\gamma_n} \beta_i \circ W_{j-i}\ , \quad Y_j'' = \sum_{i=\gamma_n+1}^{+\infty} \beta_i \circ W_{j-i}.$$

Note that X_0' and Y_j' are independent. We have, for $j > 2\gamma_n$ and some $k > 1$ (chosen later, not depending on n),

$$\begin{aligned}
&P(X_0 > u_n, Y_j > v_n) = P\Big(X_0' + X_0'' > u_n, Y_j' + Y_j'' > v_n\Big) \\
\leq &P\Big(X_0' > u_n - X_0'', Y_j' > v_n - Y_j'', X_0'' < k, Y_j'' < k\Big) \\
&+ P(X_0'' \geq k) + P\Big(Y_j'' \geq k\Big) \\
\leq &P\Big(X_0' > u_n - k, Y_j' > v_n - k\Big) + P(X_0'' \geq k) + P\Big(Y_j'' \geq k\Big) \\
\leq &P(X_0 > u_n - k)P(Y_j > v_n - k) + P(X_0'' \geq k) + P\Big(Y_j'' \geq k\Big) \\
= &O\Big(\frac{1}{n}\Big)O\Big(\frac{1}{n}\Big) + P(X_0'' \geq k) + P\Big(Y_j'' \geq k\Big).
\end{aligned}$$

Similar to Hall (2003), the last two probabilities are sufficiently fast tending to 0. For, we have

$$\begin{aligned}
P(X_0'' \geq k) = &P\left(\sum_{i=-\infty}^{-\gamma_n-1} \alpha_i \circ V_{-i} \geq k\right) \\
= &P\Big((1+h_n)^{\sum_{i=-\infty}^{-\gamma_n-1} \alpha_i \circ V_{-i}} > (1+h_n)^k\Big) \\
\leq &\frac{E\Big((1+h_n)^{\sum_{i=-\infty}^{-\gamma_n-1} \alpha_i \circ V_{-i}}\Big)}{(1+h_n)^k}.
\end{aligned}$$

We select h_n such that $h_n \gamma_n^{1-\delta} = C > 0$, for some constant C. For $i \leq -\gamma_n - 1$ and $\delta > 2$ and some positive constant C^*, it follows that

$$0 < \alpha_i h_n \leq C^* |i|^{-\delta} h_n \leq C^* (\gamma_n + 1)^{-\delta} h_n = O(1/\gamma_n) \to 0, n \to +\infty,$$

by the assumption (3) on the sequence $\{\alpha_i\}$. It implies again that

$$E\Big((1+h_n)^{\sum_{i=-\infty}^{-\gamma_n-1} \alpha_i \circ V_{-i}}\Big) = \prod_{i=-\infty}^{-\gamma_n-1} E\big((1+\alpha_i h_n)^{V_{-i}}\big)$$

where the expectations exist, and, due to Lemma 3.1,

$$
\begin{aligned}
E\Big((1+h_n)^{\sum_{i=-\infty}^{-\gamma_n-1}\alpha_i\circ V_{-i}}\Big) &\le \prod_{i=-\infty}^{-\gamma_n-1}\left(1+\alpha_i h_n E(V_0)(1+o_n(1))\right)\\
&= \exp\left(E(V_0)h_n O(1)\sum_{i=-\infty}^{-\gamma_n-1}|i|^{-\delta}\right)\\
&= \exp\left(O(1)h_n\gamma_n^{1-\delta}\right) = O(1),\ n\to+\infty,
\end{aligned}
$$

by the choice of h_n. Note that $h_n = C\gamma_n^{\delta-1} = Cn^{\nu(\delta-1)} \to +\infty$. Now, select k depending on δ, ν and ζ such that $n^2/((1+h_n)^k s_n) \sim n^2/(C^k n^{k\nu(\delta-1)} n^{\zeta}) = o(1)$ which holds for $k > (2-\zeta)/(\nu(\delta-1))$. This choice implies that $(n^2/s_n)P\big(X_0'' \ge k\big) \to 0$. In the same way we can show that also $n\sum_{j\le n/s_n} P\Big(Y_j'' \ge k\Big) \to 0$ for such a k, since also $\beta_i \le C\,|i|^{-\delta}$ for $|i| \ge \gamma_n$ and some constant $C > 0$.

c) In order to deduce

$$
n\sum_{j=1}^{[n/s_n]} P\big(X_0 > u_n, X_j > u_n\big) \to 0, \quad n\to+\infty,
$$

we use the same arguments as for $P\big(X_0 > u_n, Y_j > v_n\big)$. In this case, since X_0' and X_j' are independent, we get for some positive k

$$
\begin{aligned}
&P\big(X_0 > u_n, X_j > u_n\big)\\
\le &P\Big(X_0' > u_n - k, X_j' > u_n - k\Big) + P\big(X_0'' \ge k\big) + P\Big(X_j'' \ge k\Big)\\
= &O\Big(\frac{1}{n^2}\Big) + P\big(X_0'' \ge k\big) + P\Big(X_j'' \ge k\Big).
\end{aligned}
$$

As above we can show that $n\sum_{j\le n/s_n} P\Big(X_j'' \ge k\Big) \to 0$ and $(n^2/s_n)P\big(X_0'' \ge k\big) \to 0$. In the same way it follows also that

$$
n\sum_{j=1}^{[n/s_n]} P\big(Y_0 > v_n, Y_j > v_n\big) \to 0,\ n\to+\infty.
$$

Hence condition $D'(u_n, v_n)$ holds.

5 Simulations

We investigate the convergence of the distribution of the bivariate maxima $(M_n^{(1)}, M_n^{(2)})$ to the limiting distribution as given in Theorem 4.1. We notice that the thinning coefficients α_i and β_i have an impact on the norming values of the bivariate maxima, besides of the distribution of the (V_i, W_i).

Let us consider the bivariate geometric distribution for (V_i, W_i) mentioned in Example 2.1 and a finite number of positive values α_i and β_i. As mentioned, the bivariate geometric distribution satisfies the conditions of the general assumptions of the joint distribution of (V_i, W_i). We assumed a strong dependence with $p_{00} = 0.85, p_{01} = 0.03, p_{10} = 0.02$ and $p_{11} = 0.1$.

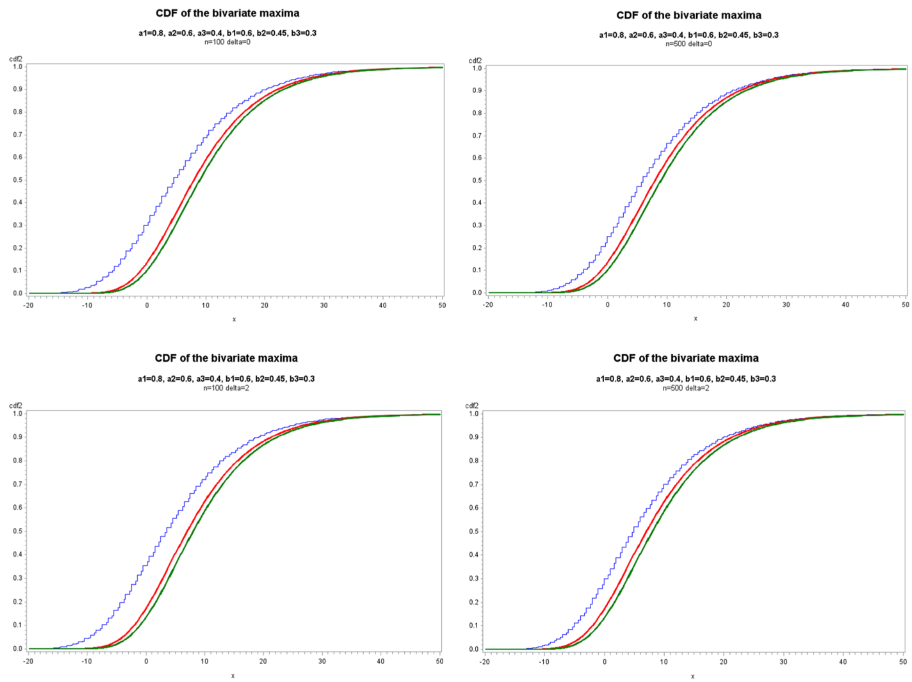

Fig. 1 Simulated cdf with upper and lower asymptotic cdf, first case, where $ai = \alpha_i, bi = \beta_i, i = 1, 2, 3$

We consider quite different models with different α_i and β_i to investigate the convergence rate. Let in the first case $\alpha_1 = 0.8, \alpha_2 = 0.6, \alpha_3 = 0.4, \beta_1 = 0.6, \beta_2 = 0.45, \beta_3 = 0.3$ and $\alpha_i = 0 = \beta_i$ for $i > 3$, and in the second case $\alpha_1 = 0.6, \alpha_2 = 0.35, \alpha_3 = 0.1, \beta_1 = 0.5, \beta_2 = 0.3, \beta_3 = 0.1$ and $\alpha_i = 0 = \beta_i$ for $i > 3$.

For each of these first two models we simulated 10’000 time series, selected $n = 100$ and 500 and derived the bivariate maxima $(M_n^{(1)}, M_n^{(2)})$. Thus we compared the empirical (simulated) distribution functions (cdf) with the asymptotic cdf.

We plotted two cases with $P(M_n^{(1)} - \tilde{u}_n \le x, M_n^{(2)} - \tilde{v}_n \le x + \delta)$ where $\tilde{u}_n = u_n - x$ and $\tilde{v}_n = v_n - y$ with u_n, v_n given in (24) and (25), respectively, using $\delta = 0$ and 2 (see Figs. 1 and 2).

We notice from these simulations that the convergence rate is quite good, but it depends on the dependence, which is given by the thinning factors α_i and β_i. We find that the convergence rate is slower for the more dependent time series (the first case, Fig. 1) and that the factor δ has a negligible impact. This is even more clear in the second cases shown in Fig. 2.

In some additional models we considered larger and more thinning factors different from 0. We show the simulations of the cases with $\alpha_i = (0.7)^i, \beta_i = (0.6)^i$, for $i \le 25$, and also with $\alpha_i = (0.9)^i, \beta_i = (0.8)^i$, for $i \le 40$. These cases are close to a infinite MA series, since α_i, β_i are very small for $i > 26$ or $i > 41$, respectively. It means that such small values have an impact on the maxima. We figured out that the number of positive values is not so important. However, in these cases the second largest value of α_i or β_i is closer to the maximal value (=1), in particular in the second of these additional models. Considering the results of again 10’000 simulations (Fig. 3), we show that the convergence rates are quite slower than in the first two models (Figs. 1 and 2). We show the results of the two cases with $n = 100$ and 500 with $\delta = 0$ only. We also figured out from the simulations of other models and distributions that if the correlation of the two components of the sequence is stronger, then the convergence to the limiting distribution (with asymptotic independence) is slower.

Proofs

Proof of Proposition 2.1:

Observe first that

$$P(V = v, W = w) = P(V \le v, W \le w) - P(V \le v - 1, W \le w) \\ - P(V \le v, W \le w - 1) + P(V \le v - 1, W \le w - 1).$$

By using the representation (4) for v and w large, and $w > v > A$ (some large constant A) we deduce, for $v \le w - 1$,

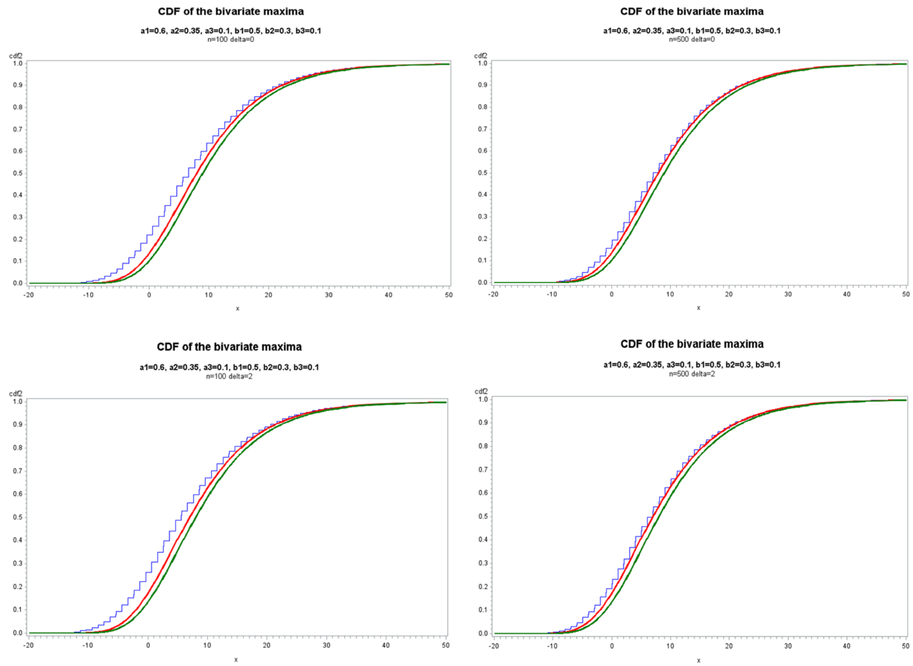

Fig. 2 Simulated cdf with upper and lower asymptotic cdf, second case, where $ai = \alpha_i, bi = \beta_i, i = 1, 2, 3$

$$P(V=v,W=w)=\left(1+\lambda_1\right)^{-v}\left(1+\lambda_2\right)^{-w}\theta^{v-1}v^{\xi_3}w^{\xi_4}L_3(v)L_4(w)\ell(v,w)\times$$
$$\left[\theta-(1+\lambda_1)\left(1-\frac{1}{v}\right)^{\xi_3}\frac{L_3(v-1)}{L_3(v)}\frac{\ell(v-1,w)}{\ell(v,w)}\right.$$
$$-(1+\lambda_2)\theta\left(1-\frac{1}{w}\right)^{\xi_4}\frac{L_4(w-1)}{L_4(w)}\frac{\ell(v,w-1)}{\ell(v,w)}$$
$$\left.+(1+\lambda_1)(1+\lambda_2)\left(1-\frac{1}{v}\right)^{\xi_3}\left(1-\frac{1}{w}\right)^{\xi_4}\frac{L_3(v-1)L_4(w-1)}{L_3(v)L_4(w)}\frac{\ell(v-1,w-1)}{\ell(v,w)}\right]$$
$$=(1+\lambda_1)^{-v}(1+\lambda_2)^{-w}\theta^{v-1}v^{\xi_3}w^{\xi_4}L_3(v)L_4(w)L(1+o(1))\ell^*(v,w),$$

where $\ell^*(v,w)\longrightarrow\lambda_2(1+\lambda_1-\theta)$, as $v,w\to+\infty$.

For $v\geq w+1$, with $v>w>A$ (some large constant A), the steps are similar, with $\ell^*(v,w)$ such that

$$\begin{aligned}\lim_{v,w}\ell^*(v,w)=&\theta-\theta(1+\lambda_1)-(1+\lambda_2)+(1+\lambda_1)(1+\lambda_2)\\=&\lambda_1(1+\lambda_2-\theta).\end{aligned}$$

For $v=w>A,\ A$ large, we get with similar steps as above

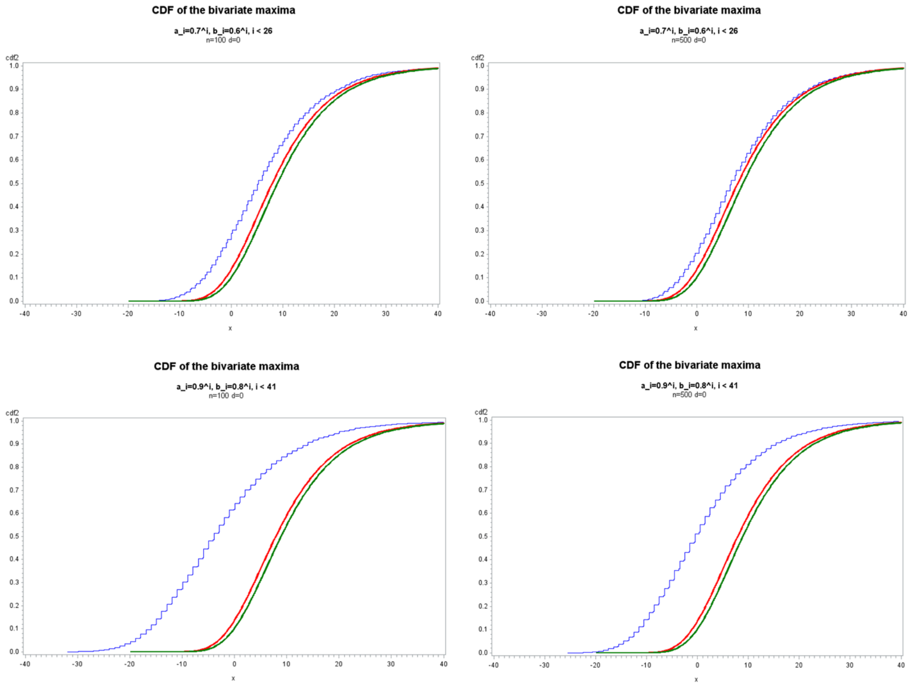

Fig. 3 Simulated cdf with upper and lower asymptotic cdf, third and fourth model where $ai=\alpha_i, bi=\beta_i$ for $i\leq 25$ (third model), and $i\leq 40$ (fourth model), respectively, with $n=100$, and 500 and $\delta=0$

$$
\begin{aligned}
&P(V=v,W=v)=\\
&=\left(1+\lambda_1\right)^{-v}\left(1+\lambda_2\right)^{-v}\theta^{v-1}v^{\xi_3+\xi_4}L_3(v)L_4(v)\ell(v,v)\times\\
&\left[\theta-(1+\lambda_1)\left(1-\frac{1}{v}\right)^{\xi_3}\frac{L_3(v-1)}{L_3(v)}\frac{\ell(v-1,v)}{\ell(v,v)}-(1+\lambda_2)\left(1-\frac{1}{v}\right)^{\xi_4}\frac{L_4(v-1)}{L_4(v)}\frac{\ell(v,v-1)}{\ell(v,v)}\right.\\
&\left.+(1+\lambda_1)(1+\lambda_2)\left(1-\frac{1}{v}\right)^{\xi_3+\xi_4}\frac{L_3(v-1)L_4(v-1)}{L_3(v)L_4(v)}\frac{\ell(v-1,v-1)}{\ell(v,v)}\right]\\
&=(1+\lambda_1)^{-v}(1+\lambda_2)^{-v}\theta^{v-1}v^{\xi_3+\xi_4}L_3(v)L_4(v)L(1+o(1))\ell^*(v,v),
\end{aligned}
$$

where $\ell^*(v,v)\longrightarrow\lambda_1\lambda_2+\theta-1>0$ as $v\to+\infty$, a positive constant by assumption. □

Proof of Proposition 2.2:

Denoting $B(n,p)$ a binomially distributed random variable with parameters n and p, we have for the pgf of (X,Y)

$$
\begin{aligned}
&G_{X,Y}(s_1,s_2)=\\
&=\sum_{j_1=0}^{+\infty}\sum_{j_2=0}^{+\infty}\sum_{k_1=j_1}^{+\infty}\sum_{k_2=j_2}^{+\infty}P(X=j_1,Y=j_2|V=k_1,W=k_2)P(V=k_1,W=k_2)s_1^{j_1}s_2^{j_2}\\
&=\sum_{j_1=0}^{+\infty}\sum_{j_2=0}^{+\infty}\sum_{k_1=j_1}^{+\infty}\sum_{k_2=j_2}^{+\infty}P(X=j_1|V=k_1)P(Y=j_2|W=k_2)P(V=k_1,W=k_2)s_1^{j_1}s_2^{j_2}\\
&=\sum_{j_1=0}^{+\infty}\sum_{j_2=0}^{+\infty}\sum_{k_1=j_1}^{+\infty}\sum_{k_2=j_2}^{+\infty}P(B(k_1,\alpha)=j_1)P(B(k_2,\beta)=j_2)P(V=k_1,W=k_2)s_1^{j_1}s_2^{j_2}\\
&=\sum_{k_1=0}^{+\infty}\sum_{k_2=0}^{+\infty}\left(\sum_{j_1=0}^{k_1}P(B(k_1,\alpha)=j_1)s_1^{j_1}\right)\left(\sum_{j_2=0}^{k_2}P(B(k_2,\beta)=j_2)s_2^{j_2}\right)P(V=k_1,W=k_2)\\
&=\sum_{k_1=0}^{+\infty}\sum_{k_2=0}^{+\infty}\left(\alpha s_1+1-\alpha\right)^{k_1}\left(\beta s_2+1-\beta\right)^{k_2}P(V=k_1,W=k_2)\\
&=G_{V,W}(\alpha s_1+1-\alpha,\beta s_2+1-\beta).\square
\end{aligned}
$$

Proof of Lemma 2.1:

We have

$$
\begin{aligned}
E(s_1^Vs_2^W)=&\sum_{k=0}^{m}\sum_{\ell=0}^{n}s_1^ks_2^\ell P(V=k,W=\ell)+\sum_{k=0}^{m}\sum_{\ell=n+1}^{+\infty}s_1^ks_2^\ell P(V=k,W=\ell)\\
&+\sum_{k=m+1}^{+\infty}\sum_{\ell=0}^{n}s_1^ks_2^\ell P(V=k,W=\ell)+\sum_{k=m+1}^{+\infty}\sum_{\ell=n+1}^{+\infty}s_1^ks_2^\ell P(V=k,W=\ell).
\end{aligned}
$$

The first partial sum is finite. The second one is bounded by $\frac{(s_1^{m+1}-1)}{s_1-1}\sum_{\ell=n+1}^{+\infty}s_2^\ell P(W=\ell)$ which is finite for $s_2<1+\lambda_2$ due to (5). Analogously, the third one is bounded by $\frac{(s_2^{n+1}-1)}{s_2-1}\sum_{k=m+1}^{+\infty}s_1^kP(V=k)$ which is finite for $s_1<1+\lambda_1$.

Finally, for the last partial sum we use Proposition 2.1. For simplicity we write $g_i(k)=s_i^k(1+\lambda_i)^{-k}k^{\xi_{i+2}}L_{i+2}(k)$, for $k\in\mathbb{N}$, $i=1,2$. Then, for large k,ℓ and some positive constants C_1,C_2, we get that this sum is bounded by

$$C_1 \sum_{k=m+1}^{+\infty} g_1(k) \sum_{\ell \geq \max(n+1,k)} g_2(\ell)\theta^k + C_2 \sum_{\ell=n+1}^{+\infty} g_2(\ell) \sum_{k \geq \max(m+1,\ell)} g_1(k)\theta^\ell$$
$$\leq C_1 \sum_{k=m+1}^{+\infty} g_1(k) \sum_{\ell \geq k} g_2(\ell)\theta^k + C_2 \sum_{\ell=n+1}^{+\infty} g_2(\ell) \sum_{k \geq \ell} g_1(k)\theta^\ell$$

The sums $\sum_{\ell \geq k} g_2(\ell)$ and $\sum_{k \geq \ell} g_1(k)$ are finite by applying the ratio criterium for $g_2(\ell)$ and $g_1(k)$. These sums are bounded by $Cg_2(k)$ and $Cg_1(\ell)$, respectively, with C a generic constant if $s_i < (1 + \lambda_i)$. Then the convergence of the last sum is obtained for $s_1 s_2 < \frac{(1+\lambda_1)(1+\lambda_2)}{\theta}$.

Proof of Proposition 2.3:
Let $q(k, \ell) = 1 - F_{(V,W)}(k, \ell)$ and $p(k, \ell) = P(V = k, W = \ell)$. Then,

$$(1 - s_1)(1 - s_2) \sum_{k=0}^{+\infty} \sum_{\ell=0}^{+\infty} \left(1 - F_{(V,W)}(k, \ell)\right) s_1^k s_2^\ell$$
$$= \sum_{k=1}^{+\infty} \sum_{\ell=1}^{+\infty} q(k, \ell) s_1^k s_2^\ell + \sum_{\ell=1}^{+\infty} q(0, \ell) s_2^\ell + \sum_{k=1}^{+\infty} q(k, 0) s_1^k + q(0, 0)$$
$$- \sum_{k=1}^{+\infty} \sum_{\ell=0}^{+\infty} q(k-1, \ell) s_1^k s_2^\ell - \sum_{k=0}^{+\infty} \sum_{\ell=1}^{+\infty} q(k, \ell-1) s_1^k s_2^\ell + \sum_{k=1}^{+\infty} \sum_{\ell=1}^{+\infty} q(k-1, \ell-1) s_1^k s_2^\ell$$
$$= \sum_{k=1}^{+\infty} \sum_{\ell=1}^{+\infty} (q(k, \ell) + q(k-1, \ell-1)) s_1^{k_1} s_2^\ell + q(0, 0) + \sum_{\ell=1}^{+\infty} q(0, \ell) s_2^\ell + \sum_{k=1}^{+\infty} q(k, 0) s_1^k$$
$$- \sum_{k=1}^{+\infty} \sum_{\ell=1}^{+\infty} q(k-1, \ell) s_1^k s_2^\ell - \sum_{k=1}^{+\infty} q(k-1, 0) s_1^k$$
$$- \sum_{k=1}^{+\infty} \sum_{\ell=1}^{+\infty} q(k, \ell-1) s_1^k s_2^\ell - \sum_{\ell=1}^{+\infty} q(0, \ell-1) s_2^\ell$$
$$= \sum_{k=1}^{+\infty} \sum_{\ell=1}^{+\infty} (q(k, \ell) + q(k-1, \ell-1) - q(k-1, \ell) - q(k, \ell-1)) s_1^k s_2^\ell$$
$$+ \sum_{\ell=1}^{+\infty} q(0, \ell) s_2^\ell + \sum_{k=1}^{+\infty} q(k, 0) s_1^k - \sum_{k=1}^{+\infty} q(k-1, 0) s_1^k - \sum_{\ell=1}^{+\infty} q(0, \ell-1) s_2^\ell + q(0, 0)$$

$$= \sum_{k=1}^{+\infty} \sum_{\ell=1}^{+\infty} (-p(k, \ell)) s_1^k s_2^\ell + \sum_{k=1}^{+\infty} (-p(k, 0)) s_1^k + \sum_{\ell=1}^{+\infty} (-p(0, \ell)) s_2^\ell + 1 - p(0, 0)$$
$$= 1 - \sum_{k=0}^{+\infty} \sum_{\ell=0}^{+\infty} p(k, \ell) s_1^k s_2^\ell = 1 - G_{(V,W)}(s_1, s_2). \square$$

Proof of Proposition 2.4:
Write $a_1 = \alpha s_1 + 1 - \alpha$ and $a_2 = \beta s_2 + 1 - \beta$. By Proposition 2.2, for $s_i \neq 1, i = 1, 2,$ we have

$$Q_{X,Y}(s_1,s_2)=\frac{1-G_{X,Y}(s_1,s_2)}{(1-s_1)(1-s_2)}=\frac{1-G_{V,W}(\alpha s_1+1-\alpha,\beta s_2+1-\beta)}{(1-s_1)(1-s_2)}$$
$$=\frac{1-G_{V,W}(a_1,a_2)}{\frac{1-a_1}{\alpha}\frac{1-a_2}{\beta}}=\alpha\beta\frac{1-G_{V,W}(a_1,a_2)}{(1-a_1)(1-a_2)}=\alpha\beta Q_{V,W}(a_1,a_2).$$

□

Proof of Proposition 2.5:
By Proposition 2.4, and the definition of $Q_{V,W}$ we have

$$Q_{X,Y}(s_1,s_2)=\alpha\beta Q_{V,W}(\alpha s_1+1-\alpha,\beta s_2+1-\beta)$$
$$=\alpha\beta\sum_{k=0}^{+\infty}\sum_{\ell=0}^{+\infty}\big(1-F_{V,W}(k,\ell)\big)\big(\alpha s_1+1-\alpha\big)^k\big(\beta s_2+1-\beta\big)^\ell$$
$$=\alpha\beta\sum_{k=0}^{+\infty}\sum_{\ell=0}^{+\infty}\big(1-F_{V,W}(k,\ell)\big)\sum_{i=0}^{k}\binom{k}{i}(1-\alpha)^{k-i}(s_1\alpha)^i\sum_{j=0}^{\ell}\binom{\ell}{j}(1-\beta)^{\ell-j}(s_2\beta)^j$$
$$=\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\left\{\sum_{k=i}^{+\infty}\sum_{\ell=j}^{+\infty}\binom{k}{i}\binom{\ell}{j}(1-\alpha)^{k-i}(1-\beta)^{\ell-j}\alpha^{i+1}\beta^{j+1}\big(1-F_{V,W}(k,\ell)\big)\right\}s_1^is_2^j$$
$$=\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\big(1-F_{X,Y}(i,j)\big)s_1^is_2^j.\square$$

Proof of Proposition 2.6:
Using Proposition 2.5, $1-F_{X,Y}(x,y)$ is given by the sum of three terms due to the assumption (4). Each term, defined by double sums, can be determined or bounded by (unique) sums associated to univariate tail functions satisfying Theorem 4 of Hall (2003), see also Hall and Temido (2007). The first sum can be approximated for x and y large, as

$$\sum_{k=x}^{+\infty}\sum_{\ell=y}^{+\infty}\binom{k}{x}\binom{\ell}{y}(1-\alpha)^{k-x}(1-\beta)^{\ell-y}\alpha^{x+1}\beta^{y+1}\big(1+\lambda_1\big)^{-k}k^{\xi_1}L_1(k)$$
$$=\sum_{k=x}^{+\infty}\binom{k}{x}(1-\alpha)^{k-x}\big(1+\lambda_1\big)^{-k}k^{\xi_1}L_1(k)\alpha^{x+1}\sum_{\ell=y}^{+\infty}\binom{\ell}{y}(1-\beta)^{\ell-y}\beta^{y+1}$$
$$=\sum_{k=0}^{+\infty}\binom{k+x}{x}(1-\alpha)^k\big(1+\lambda_1\big)^{-k-x}(k+x)^{\xi_1}L_1(k+x)\alpha^{x+1}$$
$$\sim\alpha\left(\frac{1+\lambda_1}{\lambda_1+\alpha}\right)^{\xi_1+1}x^{\xi_1}L_1(x)\left(1+\frac{\lambda_1}{\alpha}\right)^{-x}$$
$$=\left(1+\frac{\lambda_1}{\alpha}\right)^{-x}x^{\xi_1}L_1^*(x).$$

The second sum can be dealt with in the same way.

For the third term observe that due to the fact $\ell(v,w)$ is a bounded function, with bound ϑ, we get, for large integers x and y,

$$\begin{aligned} H(x,y) =& P(X > x, Y > y) = \\ & \leq \vartheta \sum_{k=x}^{+\infty} \sum_{\ell=y}^{+\infty} \binom{k}{x} \binom{\ell}{y} (1-\alpha)^{k-x} (1-\beta)^{\ell-y} \alpha^{x+1} \beta^{y+1} \left(1+\lambda_1\right)^{-k} \left(1+\lambda_2\right)^{-\ell} \\ & \times L_3(k) L_4(\ell) \max(1, \theta^{\min(k,\ell)}) k^{\xi_3} \ell^{\xi_4} \\ \leq & \vartheta \sum_{k=x}^{+\infty} \binom{k}{x} (1-\alpha)^{k-x} \alpha^{x+1} \left(1+\lambda_1\right)^{-k} k^{\xi_3} L_3(k) \\ & \times \sum_{\ell=y}^{+\infty} \binom{\ell}{y} (1-\beta)^{\ell-y} \beta^{y+1} \left(1+\lambda_2\right)^{-\ell} \max(1, \theta^{\ell}) \ell^{\xi_4} L_4(\ell) \\ \sim & \vartheta L_3^*(x) x^{\xi_3} \left(1 + \frac{\lambda_1}{\alpha}\right)^{-x} L_4^*(y) y^{\xi_4} \left(1 + \frac{\lambda_{2\theta}}{\beta}\right)^{-y} . \square \end{aligned}$$

Proof of Lemma 3.1:

a.) All moments of V exist, since the moment generating function of V exists for small positive values. Applying Taylor's expansion to the function $f(1+h) = (1+h)^k$, $h > 0$, we get, for $k \geq 2$,

$$(1+h)^k = 1 + kh + \frac{k(k-1)}{2} h^2 (1+h_1)^{k-2} \leq 1 + kh + \frac{k(k-1)}{2} h^2 (1+h^*)^{k-2},$$

for some $0 < h_1 < h < h^*$, h^* not depending on k. The expectation $E(V^2(1+h^*)^V)$ is finite for $h^* < \lambda$. Thus $E(1+h)^V \leq 1 + hE(V) + h^2 E(V^2(1+h^*)^V)$. Due to the fact that $(1+h)^k > 1 + hk$ the proof of the first claim is complete.

b.) Similarly we have

$$E(1+\alpha_i h)^V \leq 1 + \alpha_i h E(V) + (\alpha_i h)^2 E(V^2 (1 + \alpha_I h^*)^V),$$

for some h^* such that $0 < h < h^* < \lambda_1 / \alpha_I$. Then

$$\begin{aligned} E(1+h)^Z = & \prod_{i \in I} E(1+h)^{\alpha_i \circ V_{n-i}} = \exp\left(\sum_{i \in I} E(1+\alpha_i h)^{V_{n-i}} \right) \\ \leq & \exp\left(1 + \alpha_i h E(V) + (\alpha_i h)^2 E(V^2 (1+\alpha_I h^*)^V)\right) \\ \leq & \exp(C_1 h + C_2 h^2) \end{aligned}$$

where $C_1 = E(V) \sum_{i \in I} \alpha_i$ and $C_2 = E(V^2 (1 + \alpha_I h^*)^V) \sum_{i \in I} \alpha_i^2$. $\square$

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