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ON THE MAXIMUM OF SUMS OF RANDOM VARIABLES AND THE SUPREMUM FUNCTIONAL FOR STABLE PROCESSES

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1. Introduction

Let X_i, $i = 1,2,3,\cdots$ be a sequence of independent and identically distributed random variables which belong to the domain of attraction of a stable law of index α. Write $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_i, n \geqq 1$, and $M_n = \max_{0 \leqq k \leqq n} S_k$. In the case where the X_i are such that $\sum_1^\infty n^{-1}\Pr(S_n > 0) < \infty$, we have $\lim_{n\to\infty} M_n = M$ which is finite with probability one, while in the case where $\sum_1^\infty n^{-1}\Pr(S_n < 0) < \infty$, a limit theorem for M_n has been obtained by Heyde [9]. The techniques used in [9], however, break down in the case $\sum_1^\infty n^{-1}\Pr(S_n < 0) = \infty$, $\sum_1^\infty n^{-1}\Pr(S_n > 0) = \infty$ (the case of oscillation of the random walk generated by the S_n) and the only results available deal with the case $\alpha = 2$ (Erdös and Kac [5]) and the case where the X_i themselves have a symmetric stable distribution (Darling [4]). In this paper we obtain a general limit theorem for M_n in the case of oscillation. Specifically, if $\{B_n, n = 1,2,3,\cdots\}$ is a monotone sequence of constants such that $B_n^{-1} S_n$ converges in distribution to the stable law with characteristic function

$$\exp\left\{ -\lambda\,|\,t\,|^{\alpha}\left(1 + i\beta \operatorname{sgn} t \tan\frac{\pi\alpha}{2}\right)\right\}, \tag{1}$$

$\lambda > 0$, $0 < \alpha \leqq 2$, $\beta = 0$ if $\alpha = 1$, $|\beta| < 1$ if $\alpha < 1$, $|\beta| \leqq 1$ if $1 < \alpha \leqq 2$, we shall find

$$H(x) = \lim_{n\to\infty} \Pr(B_n^{-1} M_n \leqq x).$$

In connection with the parameter restrictions, we note that the stable law with characteristic function (1) is one-sided if $\alpha < 1$, $|\beta| = 1$ (e.g., Lukacs [12], page 106) so that the random walk generated by the S_n does not oscillate ([9], Lemma). The case $\alpha = 1$, $\beta \neq 0$ introduces a normalization complication and is not amenable to treatment by the methods of this paper.

It is possible to approach the problem of finding $H(x)$ in various ways. For, if $Y(t)$, $t \geqq 0$, $Y(0) = 0$, is the separable stable process with stationary independent increments which is based on (1), then also,

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$$H(x) = \Pr(\sup_{0 \leqq t \leqq 1} Y(t) < x).$$

The double Laplace transform of $\Pr(\sup_{0 \leqq u \leqq t} Y(u) < x)$ has, in this context, been obtained by Baxter and Donsker [1]. Their results are, however, not in a sufficiently explicit form to enable any of the properties of $H(x)$ to be deduced. In the present work, we make a more detailed investigation leading to considerably more illuminating results.

2. The supremum functional for a stable process

In this section, we shall deal with the case where the X_i themselves have a stable distribution with characteristic function (1) (and, of course, the specified parameter restrictions). This will be followed in the next section by an invariance theorem to establish the generality of the limiting distribution found in the present context.

Under the present circumstances, it is a simple matter to establish the existence of a limiting distribution for $n^{-1/\alpha}M_n$. In fact, if $Y(t)$, $t \geqq 0$, $Y(0) = 0$, is the separable stable process with stationary independent increments which is characterised by

$$E\exp\{iuY(T)\} = \exp\left\{ -T\lambda|u|^\alpha\left(1 + i\beta \operatorname{sgn} u \tan\frac{\pi\alpha}{2}\right)\right\},$$

and if we take

$$X_k = \left[Y\left(\frac{k}{2^N}\right) - Y\left(\frac{k-1}{2^N}\right)\right]2^{N\alpha^{-1}}, \qquad k = 1, 2, \cdots, 2^N,$$

then it follows readily from Lemma 2 of Baxter and Donsker [1], that

(2) $$\lim_{n\to\infty} \Pr(n^{-1/\alpha} M_n < x) = H(x),$$

where

(3) $$H(x) = \Pr\left(\sup_{0 \leqq t \leqq 1} Y(t) < x\right)$$

and also

$$\Pr\left(\sup_{0 \leqq t \leqq T} Y(t) < x\right) = H(xT^{-1/\alpha}).$$

We shall proceed, using methods of Darling [4], to obtain an expression for $H(x)$.

For s real and positive, let

$$\begin{aligned}\phi_n(s) &= \Pr(S_n \leqq 0) + E(\exp\{-s Sn\}; S_n > 0)\\ &= \Pr(X \leqq 0) + E(\exp\{-sn^{1/\alpha}X\}; X > 0),\end{aligned}$$

since $n^{-1/\alpha} S_n$ and X_i have the same distribution. Then, using Theorem 4 of Zolotarev [18], we obtain the unilateral Laplace transform

(4) $$E(e^{-sX};\, X>0)=\frac{1}{\pi}\int_0^\infty \exp\{-(sx)^\alpha\lambda_1\}\,\frac{\sin\pi\rho}{x^2+2x\cos\pi\rho+1}\,dx,$$

where

(5) $$\lambda_1=\lambda\left(1+\beta^2\tan^2\frac{\pi\alpha}{2}\right)^{1/2},\ \rho=\tfrac{1}{2}+\frac{1}{\pi\alpha}\arctan\left[-\beta\tan\frac{\pi\alpha}{2}\right]=\Pr(X>0),$$

(the result $\rho=\Pr(X>0)$ was obtained by Chung-Teh [2]), so that for $0\leqq t<1$,

$$\sum_1^\infty n^{-1}t^n\phi_n(s)$$
$$=-(1-\rho)\log(1-t)-\frac{\sin\pi\rho}{\pi}\int_0^\infty \log[1-t\exp\{-(sx)^\alpha\lambda_1\}\,\frac{dx}{x^2+2x\cos\pi\rho+1}.$$

But, using (4),

$$\frac{\sin\pi\rho}{\pi}\int_0^\infty\frac{dx}{x^2+2x\cos\pi\rho+1}=\rho,$$

and therefore,

$$\log(1-t)+\sum_1^\infty n^{-1}t^n\phi_n(s)$$
$$=-\frac{\sin\pi\rho}{\pi}\int_0^\infty\log\left[\frac{1-t\exp\{-(sx)^\alpha\lambda_1\}}{1-t}\right]\frac{dx}{x^2+2x\cos\pi\rho+1}.$$

Consequently, as $t\uparrow 1$,

$$\log(1-t)+\sum_1^\infty n^{-1}t^n\phi_n[(1-t)^{1/\alpha}s]\to-\frac{\sin\pi\rho}{\pi}\int_0^\infty\frac{\log(1+s^\alpha x^\alpha\lambda_1)}{x^2+2x\cos\pi\rho+1}\,dx,$$

and, using the Spitzer-Pollaczek Identity (see for example Prabhu [13], page 218, Theorem 4.1),

(6) $$\lim_{t\uparrow 1}\,(1-t)\sum_0^\infty E(\exp\{-s(1-t)^{1/\alpha}\}\,M_n)=g(s),$$

where

(7) $$g(s)=\exp\left\{-\frac{\sin\pi\rho}{\pi}\int_0^\infty\frac{\log(1+s^\alpha x^\alpha\lambda_1)}{x^2+2x\cos\pi\rho+1}\,dx\right\}.$$

Precisely the argument of Darling [4] then yields the relation

(8) $$A(s) = \frac{G(1-s)}{\Gamma(1-s)\Gamma(1-\alpha^{-1}+s\alpha^{-1})},$$

$|\mathscr{R}s|$ sufficiently small, between the Mellin transforms,

$$A(s) = \int_0^\infty x^{s-1} dH(x),$$

$$G(s) = \int_0^\infty x^{s-1}\, g(x)\, dx.$$

As with [4], it follows that $H(x)$ is an absolutely continuous distribution function and upon inverting in (8) we obtain the density function

$$\frac{d}{dx}H(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{G(1-s)x^{-s}}{\Gamma(1-s)\Gamma(1-\alpha^{-1}+s\alpha^{-1})}ds,$$

(see for example Widder [17], pages 246, 247). We have thus established the following theorem.

Theorem 1. If $0 < \alpha \leqq 2$ and $\beta = 0$ if $\alpha = 1$, $|\beta| < 1$ if $\alpha < 1$, $|\beta| \leqq 1$ if $1 < \alpha \leqq 2$, then

(9) $$\frac{d}{dx}H(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{G(1-s)x^{-s}}{\Gamma(1-s)\Gamma(1-\alpha^{-1}+s\alpha^{-1})}ds,$$

where

$$G(s) = \int_0^\infty x^{s-1} g(x) dx,$$

$g(x)$ being given by (7).

The expression in (9) does not in general simplify conveniently. We shall go on to examine an alternative method which is illuminating in its own right and has the advantage of providing more recognisable results in certain particular cases. This is based on the representation of M_n in the form

$$M_n = 0 \text{ if } R_n = 0$$

$$M_n = Z_1 + Z_2 + \cdots + Z_{R_n}, \text{ if } R_n \geqq 1,$$

where the Z_i are successive strong ascending ladder steps for the random walk generated by the S_n (that is, Z_i has the distribution of the minimum S_k with $S_k > 0$) and R_n is the number of strong ascending ladder indices in the first n steps of the random walk.

Firstly, we need to investigate the distribution of R_n. We easily see that the occurrence of a strong ascending ladder index is a recurrent event in the usual sense of Feller. This we shall denote by $\mathscr{E}$, so that R_n is the number of occurrences of $\mathscr{E}$ in n steps. R_n can be expressed as a sum of indicator random variables in the following way:

$$R_n = \sum_{k=1}^{n} \delta_k,$$

where

$$\delta_k = \begin{cases} 1 & \text{if } \mathscr{E} \text{ occurs at the } k\text{th step,} \\ 0 & \text{otherwise,} \end{cases}$$

and we can study R_n via the δ_k. If N_n is the number of positive terms in the sequence $S_1, S_2, \cdots, S_n$,

$$\begin{aligned} \Pr(\delta_k = 1) &= \Pr(S_k > 0, \quad S_k > S_1, \cdots, S_k > S_{k-1}) \\ &= \Pr(S_k > 0, \quad \sum_{2}^{k} X_i > 0, \cdots, X_k > 0) \qquad (10) \\ &= \Pr(N_k = k), \end{aligned}$$

the X_i being identically distributed. We now define sequences $\{u_n, n \geqq 0\}$, $\{f_n, \; n \geqq 1\}$, as follows:

$$\begin{aligned} u_0 &= 1, \quad u_n = \Pr(\delta_n = 1), \quad n \geqq 1, \\ f_n &= \Pr(\delta_1 = 0, \; \delta_2 = 0, \cdots, \delta_{n-1} = 0, \; \delta_n = 1), \qquad n \geqq 1, \end{aligned}$$

and introduce the generating functions

$$U(t) = \sum_{n=0}^{\infty} u_n t^n, \qquad F(t) = \sum_{n=1}^{\infty} f_n t^n, \qquad 0 \leqq t < 1.$$

These, of course, satisfy the standard identity of recurrent event theory,

$$U(t) = [1 - F(t)]^{-1}, \qquad 0 \leqq t < 1. \tag{11}$$

Then, using a well-known result of Sparre-Andersen (see for example Spitzer [16], page 219) together with (5) and (10), we have

$$U(t) = \exp\left\{\sum_{1}^{\infty} t^k k^{-1} \Pr(X > 0)\right\} = (1 - t)^{-\rho}. \tag{12}$$

Lemma. The recurrence time distribution of $\mathscr{E}$ belongs to the domain of attraction of the stable law of index $1 - \rho$.

Proof. If $q_n = \sum_{r=n+1}^{\infty} f_r$, we have

$$(1-t)^{-1}[1-F(t)] = \sum_{n=0}^{\infty} q_n t^n = (1-t)^{-(1-\rho)},$$

using (11) and (12), so that

$$(1-t)^{1-\rho} \sum_{n=0}^{\infty} q_n t^n = 1,$$

and applying a standard Tauberian theorem (see for example Feller [7], page 423, Theorem 5),

$$q_n \sim \frac{1}{\Gamma(1-\rho)} n^{-\rho}$$

as $n \to \infty$. The result of the lemma then follows immediately by appeal to Theorem la, page 303 of Feller [7]. Then, using the lemma in conjunction with Theorem 7 of Feller [6], we readily obtain:

Theorem 2. Let G_ρ be the distribution function of the stable law whose characteristic function is given by

$$g_\rho(t) = \exp\left\{-|t|^\rho \left(\cos\frac{\pi\rho}{2} - i\sin\frac{\pi\rho}{2}\operatorname{sgn} t\right)\right\},$$

where $\rho = \Pr(X > 0)$. Then,

$$\lim_{n\to\infty} \Pr(n^{-\rho} R_n \leqq x) = 1 - G_\rho(x^{-1/\rho}), \qquad x \geqq 0. \tag{13}$$

Remark. Only in the case $\rho = \frac{1}{2}$ is it possible to give a convenient alternative form for the limit distribution in (13). We have, making use of a result due to Lévy (see for example Lukacs [12], page 107),

$$\frac{d}{dx} G_{\frac{1}{2}}(x) = \tfrac{1}{2}\pi^{-\frac{1}{2}} x^{-\frac{3}{2}} \exp\{-1/4x\}, \qquad x > 0,$$

so that

$$\frac{d}{dx}[1 - G_{\frac{1}{2}}(x^{-2})] = \pi^{-\frac{1}{2}} \exp\{-1/4x^2\}. \tag{14}$$

Thus, the limit distribution is a truncated normal.

Next, we need to examine the limit properties of the Z_i. We have (see for example Prabhu [13], page 210, Theorem 2.2)

$$1 - Ee^{-sZ} = \exp\left\{ - \sum_1^\infty n^{-1} E(\exp\{-sSn\};\ S_n > 0)\right\}$$

$$= \exp\left\{ - \sum_1^\infty n^{-1} E(\exp\{-sn^{1/\alpha}X\};\ X > 0)\right\}$$

$$= \exp\left\{\frac{\sin \pi\rho}{\pi} \int_0^\infty \log\left[1 - \exp\{-(sx)^\alpha \lambda_1\}\right] \frac{dx}{x^2 + 2x\cos\pi\rho + 1}\right\},$$

using (4). Furthermore, if

$$I(s) = \int_0^\infty \log[1 - \exp\{-(sx)^\alpha \lambda_1\}] \frac{dx}{x^2 + 2x\cos\pi\rho + 1},$$

then we can write $I(s) = I_1(s) + I_2(s)$, where

$$I_1(s) = \int_0^\infty \frac{\log b(sx)}{x^2 + 2x\cos\pi\rho + 1} dx, \quad I_2(s) = \int_0^\infty \frac{\log[(sx)^\alpha \lambda_1]}{x^2 + 2x\cos\pi\rho + 1} dx,$$

and $b(u) = [1 - \exp\{-u^\alpha \lambda_1\}]/(u^\alpha \lambda_1)$. Then, $I_1(s) \to 0$ as $s \to 0$ using dominated convergence. Also,

$$\int_0^\infty \frac{dx}{x^2 + 2x\cos\pi\rho + 1} = \frac{\pi\rho}{\sin\pi\rho}, \quad \int_0^\infty \frac{\log x\, dx}{x^2 + 2x\cos\pi\rho + 1} = 0,$$

using (4) and Gradshteyn and Ryzhik [8], page 533, respectively, so that

$$I_2(s) = \frac{\pi\rho}{\sin\pi\rho}(\log\lambda_1 + \alpha\log s)$$

and hence

$$s^{-\alpha\rho}(1 - Ee^{-sZ}) = s^{-\alpha\rho}\exp\left\{\frac{\sin\pi\rho}{\pi}[I_1(s) + I_2(s)]\right\}$$

$$(15) \qquad = \lambda_1^\rho \exp\left\{\frac{\sin\pi\rho}{\pi} I_1(s)\right\}$$

$$\to \lambda_1^\rho = \left[\lambda\left(1 + \beta^2\tan^2\frac{\pi\alpha}{2}\right)^{\frac{1}{2}}\right]^{\Pr(X>0)}$$

as $s \to 0$. Now it is easily checked from (5) that $\alpha\rho \leqq 1$, with equality if and only if $1 < \alpha \leqq 2$ and $\beta = 1$ when $\alpha < 2$. Thus, if $0 < \alpha \leqq 1$ or $1 < \alpha < 2$ and $-1 \leqq \beta < 1$, $n^{-1/\alpha\rho}(Z_1 + \cdots + Z_n)$ converges in distribution to the stable law with characteristic function

$$\exp\left\{-\lambda_1^\rho |t|^{\alpha\rho}\left(1 - i\,\mathrm{sgn}\, t\tan\frac{\pi\alpha\rho}{2}\right)\right\},$$

while if $1 < \alpha \leqq 2$ and $\beta = 1$ when $\alpha < 2$, we have $\alpha\rho = 1$ so that

$$s^{-1}(1-Ee^{-sZ})\to\left[\lambda\left(1+\tan^2\frac{\pi\alpha}{2}\right)^{\frac{1}{2}}\right]^{1/\alpha}=\left[\lambda\sec\left(\pi-\frac{\pi\alpha}{2}\right)\right]^{1/\alpha},$$

and therefore,

$$EZ=\left[\lambda\sec\left(\pi-\frac{\pi\alpha}{2}\right)\right]^{1/\alpha}<\infty. \tag{16}$$

These results clarify those of Rogozin [15], where it is not possible to calculate specifically the constant in (15). We are now in a position to establish the following limiting result.

Theorem 3. If $1<\alpha\leqq 2$ and $\beta=1$ when $\alpha<2$, $|\beta|\leqq 1$ when $\alpha=2$, then

$$H(x)=1-G_{1/\alpha}\left(-\lambda\sec\frac{\pi\alpha}{2}x^{-\alpha}\right),\qquad x\geqq 0, \tag{17}$$

where $G_{1/\alpha}$ is the distribution function of the stable law whose characteristic function is given by

$$g_{1/\alpha}(t)=\exp\left\{-|t|^{1/\alpha}\left(\cos\frac{\pi}{2\alpha}-i\sin\frac{\pi}{2\alpha}\operatorname{sgn}t\right)\right\}.$$

Proof. We may write

$$\frac{M_n}{n^{1/\alpha}}=\frac{Z_1+\cdots+Z_{R_n}}{n^{1/\alpha}}=\frac{R_n}{n^{1/\alpha}}\cdot\frac{Z_1+\cdots+Z_{R_n}}{R_n}. \tag{18}$$

Now, in view of (16) and using Theorem 2 of Richter [14],

$$\frac{Z_1+\cdots+Z_{R_n}}{R_n}\xrightarrow{p}EZ=\left[\lambda\sec\left(\pi-\frac{\pi\alpha}{2}\right)\right]^{1/\alpha}, \tag{19}$$

("p" stands for convergence in probability). Also, we have from Theorem 1 that

$$\lim_{n\to\infty}\Pr(n^{-1/\alpha}R_n\leqq x)=1-G_{1/\alpha}(x^{-\alpha}). \tag{20}$$

Then, using a standard convergence result (see for example Cramér [3], page 254), it follows from (18), (19) and (20) that

$$H(x)=\lim_{n\to\infty}\Pr(n^{-1/\alpha}M_n\leqq x)=\lim_{n\to\infty}\Pr(n^{-1/\alpha}R_nEZ\leqq x)$$
$$=1-G_{1/\alpha}\left(-\lambda\sec\frac{\pi\alpha}{2}x^{-\alpha}\right).$$

This completes the proof of the theorem. In the particular case $\alpha=2$, we see from (14) that

$$H(x)=\frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{\lambda}}e^{-\frac{1}{4}u^2}du=\frac{1}{\sqrt{(\pi\lambda)}}\int_0^x e^{-u^2/4\lambda}du. \tag{21}$$

The result (21) is well known and has a long history. Different methods of derivation can be seen in Erdös and Kac [5], Lévy [11], page 85, and Darling [4].

3. Limit theorem for maxima of partial sums

We shall establish the following invariance theorem.

Theorem 4. Let X_i, $i = 1,2,3,\cdots$ be a sequence of independent and identically distributed random variables and write $S_n = \sum_{i=1}^{n} X_i$, $n \geqq 1$, $S_0 = 0$, and $M_n = \max_{0 \leqq k \leqq n} S_k$. Suppose that there exists a monotone sequence of constants $\{B_n, n=1,2,3,\cdots\}$ such that $B_n^{-1} S_n$ converges in distribution to the stable law with characteristic function (1). Then,

$$\lim_{n\to\infty} \Pr(M_n \leqq B_n x) = H(x),$$

the distribution function $H(x)$ being given by (9), (17) or (21).

Proof. Let Y_i, $i = 1,2,3,\cdots$ be independent and identically distributed random variables with characteristic function (1) and write $W_n = \sum_{i=1}^{n} Y_i$. Write also, $H_n(x) = \Pr(M_n \leqq B_n x)$ and for $n_j = [jk^{-1}n]$, $j = 1,\cdots,k$, ($[u]$ here refers to the integer part of u) $H_{n,k}(x) = \Pr[\max(S_{n_1}, S_{n_2},\cdots,S_{n_k}) \leqq B_n x]$. Then, since B_n is of the form $n^{1/\alpha}L(n)$ where $L(n)$ is slowly varying as $n \to \infty$ (Ibragimov and Linnik [10], Theorem 2.1.1), we have

$$H_{n,k}(x) = \Pr\left(\operatorname{Max}(S_{n_1}, S_{n_2},\cdots,S_{n_k}) \leqq n_1^{1/\alpha}L(n_1) \left(\frac{n}{n_1}\right)^{1/\alpha} \frac{L(n)}{L(n_1)} x \right).$$

Consequently, it follows from the multidimensional central limit theorem and Equation (2) that for $x > 0$,

$$\lim_{k\to\infty}\lim_{n\to\infty} H_{n,k}(x) = \lim_{k\to\infty} \Pr\left(\max_{1\leqq j\leqq k} W_j \leqq k^{1/\alpha}x\right) = H(x). \tag{22}$$

Now, for arbitrary $\varepsilon > 0$, define

$$X_{kn} = \begin{cases} X_k & \text{if } -K\varepsilon B_n \leqq X_k \leqq \varepsilon B_n, \\ 0 & \text{otherwise,} \end{cases}$$

where $K = K(\varepsilon, n)$ is chosen such that $EX_{kn} = 0$. It is easily seen that $K(\varepsilon,n) \to K_0$, a positive constant, as $n \to \infty$. Write $S_{n,m} = \sum_{k=1}^{m} X_{kn}$ where $m = n_{i+1} - r$ and $n_i < r \leqq n_{i+1}$. Then, if $N = n_{i+1} - n_i$, we have

$$\begin{aligned} \Pr(|S_{n_{i+1}} - S_r| > \varepsilon B_n) &= \Pr(|S_m| > \varepsilon B_n) \\ &\leqq m[\Pr(X < -K\varepsilon B_n) + \Pr(X > \varepsilon B_n)] + \Pr(|S_{n,m}| > \varepsilon B_n) \\ &\leqq N[\Pr(X < -K\varepsilon B_n) + \Pr(X > \varepsilon B_n)] + \varepsilon^{-2} B_n^{-2} ES_{n,m}^2 \qquad (23) \\ &\leqq N[\Pr(X < -K\varepsilon B_n) + \Pr(X > \varepsilon B_n)] + N\varepsilon^{-2} B_n^{-2} EX_{kn}^2, \end{aligned}$$

where we have used Chebyshev's inequality in obtaining the second last inequality. Now, $nB_n^{-2}\, EX_{kn}^2 \leqq c_1\varepsilon^{2-\alpha}$, $n\Pr(X < -K\varepsilon B_n) \leqq c_2\varepsilon^{-\alpha}$, and $n\Pr(X>\varepsilon B_n) \leqq c_3\varepsilon^{-\alpha}$ as $n \to \infty$, where c_1, c_2, c_3 are positive constants (not depending on ε) (Feller [7], pages 304, 544–546) and, furthermore, $N \sim k^{-1}n$. It follows then, from (23), that we can choose a positive constant c such that for all i, r, n,

$$\Pr(|S_{n_i+1} - S_r| > \varepsilon B_n) \leqq ck^{-1}\varepsilon^{-\alpha}. \tag{24}$$

Consequently for $x > 0$,

$$\begin{aligned}
\Pr(M_n > B_n x) &= \sum_{r=1}^{n} \Pr\Big(\max_{1\leqq j\leqq r-1} S_j \leqq B_n x, S_r > B_n x\Big)\\
&= \sum_{i=0}^{k-1} \sum_{n_i<r\leqq n_{i+1}} \Pr\Big(\max_{1\leqq j\leqq r-1} S_j \leqq B_n x, S_r > B_n x, |S_{n_i+1} - S_r| > \varepsilon B_n\Big)\\
&\quad + \sum_{i=0}^{k-1} \sum_{n_i<r\leqq n_{i+1}} \Pr\Big(\max_{1\leqq j\leqq r-1} S_j \leqq B_n x, S_r > B_n x, |S_{n_i+1} - S_r| \leqq \varepsilon B_n\Big)\\
&\leqq ck^{-1}\varepsilon^{-\alpha} \sum_{r=1}^{n} \Pr\Big(\max_{1\leqq j\leqq r-1} S_j \leqq B_n x, S_r > B_n x\Big)\\
&\quad + \sum_{i=0}^{k-1} \sum_{n_i<r\leqq n_{i+1}} \Pr\Big(\max_{1\leqq j\leqq r-1} S_j \leqq B_n x, S_r > B_n x, |S_{n_i+1} - S_r| \leqq \varepsilon B_n\Big)\\
&\leqq ck^{-1}\varepsilon^{-\alpha} + 1 - H_{n,k}(x-\varepsilon),
\end{aligned} \tag{25}$$

using (24). Finally, since $H_n(x) \leqq H_{n,k}(x)$, and with the aid of (25), we deduce that

$$H_{n,k}(x-\varepsilon) - ck^{-1}\varepsilon^{-\alpha} \leqq H_n(x) \leqq H_{n,k}(x). \tag{26}$$

In view of (22), the result of the theorem is then immediate if we let $n \to \infty$ and then $k \to \infty$ in (26) since $H(x)$ is an absolutely continuous distribution function.

Addendum

The author has been informed by a referee that Theorems 3 and 4 of this paper are derivable from results of A. V. Skorokhod in "Random Processes with Independent Increments" Izdat. Nauk (Moscow) 1964 (in Russian). It has not been possible to compare methods due to the unobtainability of this book. A number of improvements to the exposition of this paper have been suggested by the referees and these the author gratefully acknowledges.

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