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# ON THE MEAN-VALUE PROPERTY OF HARMONIC FUNCTIONS

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ABSTRACT. In this note we show that if the areal mean-value theorem holds for a plane domain (subject to a mild regularity condition) for all integrable harmonic functions, then the domain must be a disk. It is also shown that if a plane domain with finite area has at least two boundary components which are continua then the mean-value property cannot hold for the class of all integrable harmonic functions with single-valued harmonic conjugates.

1. In 1962 Epstein[1] [2] proved the following theorem: "Let $D$ be a simply connected domain of finite area and $t$ a point of $D$ such that, for every function $u$ harmonic in $D$ and integrable over $D$, the mean-value of $u$ over the area of $D$ equals $u(t)$. Then $D$ is a disk and $t$ its center." In a later paper Epstein and Schiffer extended the above result to domains in Euclidean space $E^n$ replacing the simple connectivity hypothesis of the earlier paper by the assumption that the complement of $D$ possess a nonempty interior. Nevertheless, the following theorem strongly suggests that, for plane regions at least, the simple connectedness of $D$ is a necessary condition for the mean-value property to hold.

THEOREM 1. *Let $D$ be a plane domain of finite area having at least two boundary components $\gamma_1$ and $\gamma_2$ which are continua. Denote by $\mathfrak{H}$ the class of functions $u$ harmonic in $D$ and integrable over $D$; and by $\mathfrak{F}\subset\mathfrak{H}$ the subclass consisting of functions possessing single-valued harmonic conjugates in $D$. Then $\mathfrak{F}$ (and a fortiori $\mathfrak{H}$) does not satisfy the mean-value property at any point $t\in D$; that is, it is not the case that there exists a $t\in D$ such that, for all $u\in\mathfrak{F}$,*

$$u(t) = \frac{1}{A}\iint_D u(z)\, dxdy,$$

*where $A$ denotes the area of $D$.*

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2. Before proving Theorem 1 we will first establish some preliminary results. Given a point $t$ in a plane region $W$ we denote by $p_0$ and $p_1$ the principal functions (see, for example, [5]) with respect to the normal operators $L_0$ and $L_1$ and the given singularity $t$. Here $p_1$ is defined with respect to the canonical partition $Q$. The functions $P_0$ and $P_1$ are defined by

$$P_\nu(z;t) = p_\nu(z;t) + ip_\nu^*(z;t), \qquad \nu = 0, 1,$$

with

$$\lim_{\tau\to 0} (P_\nu(\tau;t) - 1/\tau) = 0,$$

where $\tau$ is a parameter about $t$ and $p_\nu^*$ is the conjugate harmonic function of $p$.

Let $\tilde{K}(z;t)$ be the kernel function for the class $l_2(D)$ of square integrable analytic functions $f$ in a domain $D$ which possess single-valued integrals in $D$. Then we have

LEMMA 1. $\tilde{K}(z;t) = (1/2\pi)\ (P_0'(z;t) - P_1'(z;t))$.

A proof of this relation may be found in [6]. We also have

LEMMA 2. *Let $W$ be a plane domain of connectivity $n \geqq 2$, possessing at least two boundary components $\gamma_1$, $\gamma_2$, which are disjoint analytic Jordan curves. Then the function $\frac{1}{2}(P_0 - P_1)$ is not univalent in $W$.*

PROOF. The function $\frac{1}{2}(P_0 - P_1)$ is analytic in $W \cup \gamma_1 \cup \gamma_2$. Along $\gamma_i$, $i = 1, 2$,

$$d(P_0 + P_1) - \overline{d(P_0 - P_1)} = 2(dp_1 + i * dp_0) = 0.$$

Therefore, on $\gamma_i$, $i = 1, 2$,

$$\tfrac{1}{2}(P_0 - P_1) = \tfrac{1}{2}\overline{(P_0 + P_1)} + \text{const},$$

where the constant depends on $\gamma_i$. Denote by $\gamma_i'$ the image of $\gamma_i$ under $\frac{1}{2}(P_0 - P_1)$. Due to the mapping properties of $\frac{1}{2}(P_0 + P_1)$, we see that if $\frac{1}{2}(P_0 - P_1)$ is univalent and if the connectivity $n$ is $\geqq 2$, then either $\gamma_1'$ or $\gamma_2'$ is not the outer boundary, the region encircled by it being disjoint from the image region. Since this is a contradiction, we conclude that $\frac{1}{2}(P_0 - P_1)$ is not univalent.

3. **Proof of Theorem 1.** Assume to the contrary that $\mathfrak{F}$ satisfies the mean-value property at a point $t \in D$. By the Schwarz inequality we observe that for each $f(z) \in l_2(D)$, the real and imaginary parts of $f(z)$ are integrable over $D$, and so

$$f(t) = \iint_D f(z) \cdot A^{-1}\, dxdy.$$

On the other hand the equality $f(t) = \iint_D f(z)\overline{\tilde{K}(z;\, t)}dxdy$ holds for each $f \in l_2(D)$. Since $\tilde{K}(z; t)$ is uniquely determined by its reproducing property we conclude that

$$\tilde{K}(z; t) = A^{-1}.$$

This last relation together with Lemma 1 imply that $\frac{1}{2}(P_0 - P_1)$ is a linear function of $z$ and hence univalent. If $\gamma_1$ and $\gamma_2$ are not both disjoint analytic Jordan curves to begin with we can by repeated applications of the Riemann mapping theorem map $D$ conformally onto a domain $D$ with finite area such that $\gamma_1$ and $\gamma_2$ are mapped onto disjoint analytic Jordan curves. We note that if $\phi$ is the one-to-one conformal map of $D$ onto $\hat{D}$ then $\hat{P}_0 = P_0 \circ \phi^{-1}$ and $\hat{P}_1 = P_1 \circ \phi^{-1}$ are the corresponding principal functions for $\hat{D}$. Since $\frac{1}{2}(P_0 - P_1)$ is univalent on $D$ it follows that $\frac{1}{2}(\hat{P}_0 - \hat{P}_1)$ is univalent on $\hat{D}$. But by Lemma 2 this is a contradiction. Hence $\mathfrak{F}$ does not satisfy the mean-value property at the point $t$ as claimed.

4. We now turn to the theorem of Epstein and Schiffer [**3**] which is as follows:

THEOREM 2. *Let $D$ be any domain in Euclidean space $E^n$, possessing finite measure, and let the complement of $D$ possess nonempty interior. Suppose that there exits a point $t$ in $D$ such that, for every function $u$ harmonic in $D$ and integrable over $D$, the mean-value of $u$ over $D$ equals $u(t)$. Then $D$ is a sphere with center at $t$.*

Epstein and Schiffer remark at the end of their paper that due to the assumption made about the complement of $D$, Theorem 2 leaves open the possibility that there exists a domain $D$ which has finite area, is dense in $E^n$, and for which the mean-value property stated in their theorem holds. We now show that for plane regions we can give a sharper formulation of Theorem 2 in that the assumption that the complement of $D$ possess nonempty interior may be replaced by a weaker assumption. Because of this result we are able partially to answer the question posed by Epstein and Schiffer above. Note that the class of functions considered in Theorem 2 corresponds to the class $\mathfrak{H}$ considered in Theorem 1 when $n = 2$.

A plane point set $E$ which is compact is said to belong to the class $N_B$ if the unbounded component of the complement of $E$ belongs to the class $O_{AB}$ of Riemann surfaces which possess no nonconstant bounded analytic functions (cf. Ahlfors-Beurling [1]). Sets in the class $N_B$ are totally disconnected. We now state

THEOREM 3. *Let D be a plane domain with finite area having at least one boundary component $\gamma$ which is a continuum. If $\mathcal{H}$ satisfies the mean-value property at a point $t\in D$ then D is a disk.*

PROOF. Since the mean-value property holds at the point $t$ for $\mathcal{H}$ it holds for the subclass $\mathcal{F}$ of functions possessing single-valued harmonic conjugates in $D$. As in the proof of Theorem 1 we see that this implies that $\frac{1}{2}(P_0-P_1)$ is univalent. Because of Theorem 2 we may assume without loss of generality that $D$ is dense in the entire plane. When this is so there are two possibilities to consider:

*Case* 1. Suppose that the complement of $D$ with respect to the extended plane is the union $\gamma\cup A$, where $A$ is the union of an at most countable number of compact sets of class $N_B$. Since a set of type $N_B$ has 2-dimensional Hausdorff measure zero (cf. Sario-Oikawa [5]), and hence zero area, $A$ has zero area. Also $D$ has finite area by assumption and so $\gamma$ must have infinite 2-dimensional Hausdorff measure. But this is impossible. Hence Case 1 does not in fact occur.

*Case* 2. Suppose now that the complement of $D$ with respect to the extended plane is the union $\gamma\cup B$, where $B$ is not an at most countable union of sets of type $N_B$. By the Riemann mapping theorem the extended plane less $\gamma$ can be mapped conformally onto a disk with $\gamma$ being mapped onto the bounding circle. The image of $B$ under this mapping is a set $\hat{B}$ which is again not an at most countable union of sets of type $N_B$. But by Theorems 1, 2 of Sakai [4] this implies that $\frac{1}{2}(P_0-P_1)$ is not univalent. It follows that Case 2 also does not occur, and this concludes the proof.

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