On the Mechanics of Economic Development
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The problem of economic development

Mechanics that are consistent with

- sustained growth (per capita)
  - \( y = AK^\beta l^{1-\beta} \), as \( l \to 1 \), growth of capital dies
  - Solow (1957): exogeneous technology change, \( A(t) \)
  - Lucas (1988): human capital \( y = AK^\beta (l h)^{1-\beta} \),
    
    \( K \) and \( l h \) are complementary

- sustained diversity in income levels
Facts

World Bank’s World Development Report 1983

- The diversity across countries in measured per capita income levels is literally too great to be believed
  - (US. $10,000, India $240) (1980)

This is a difference of a factor of 40 in living standards!

- Rates of growth of real per capita GDP are also diverse
  - (Japan 7.1%, India 1.4%)
Motivation

- This is what we need a theory of economic development for
  - to provide some kind of framework for organizing facts like these
  - for judging which represent opportunities (Indian government could do something) and which necessities (nature)
Neoclassical Model (Solow)

Problem

Neoclassical model predicts a strong tendency to income equality and equality in growth rates, inability to account for observed diversity across countries.
Environment

- A closed economy with competitive markets
- Identical agents (rational)
- $N$ workers each endowed with 1 hour that is inelastically devoted to production
- Per capita output $y(t) = f(k(t)) = A(t)k(t)^\beta$
  - $k(t)$ is per capita capital $\frac{K}{N}$, $F(K, N) = AK^\beta N^{1-\beta} = A \left( \frac{K}{N} \right)^\beta \cdot N$
- Exogenous technical change, $\dot{A}/A = \mu$
- A single good, per capita consumption $c(t)$
Preference:

\[
\int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} \, dt
\]  

1. Discount rate \( \rho > 0 \); the larger the \( \rho \), the more impatient the people
2. Coefficient of relative risk aversion \( \sigma > 0 \)

\[
u' = c^{-\sigma}, \quad u'' = -\sigma c^{-\sigma-1}, \quad -\frac{cu''}{u'} = \sigma
\]

The larger the \( \sigma \), the more risk averse the people, the more they prefer consumption smoothing

Growth: it is important to know how people evaluate goods today and tomorrow, \( \rho \) and \( \sigma \) capture how people like to substitute goods across time (inter-temporal substitution)
Social planner’s problem:

Without externality, social planner’s problem coincides with market equilibrium (first welfare theorem)

- Resource constraint

\[ c(t) + \dot{k}(t) = A(t)k(t)^\beta \] (2)

output is divided into consumption and capital accumulation
capital share \(0 < \beta < 1\)

- Price increments to capital

  - Evidently, it is not optimal to choose \(c(t)\) to maximize current period utility \(\frac{c(t)^{1-\sigma}}{1-\sigma}\). For the choice that achieves this is to set \(\dot{k}(t) = 0\). One needs to set some value on increment to physical capital, since this increment to capital will bring more consumption in the future
The planner allocates the resources to increase either the utility from period $t$ consumption or the value of the period $t+1$ capital stock

$$\tilde{H}(k, \theta, c, t) = \frac{c^{1-\sigma}}{1-\sigma} e^{-\rho t} + \theta e^{-\rho t} \left[ A(t)k(t)^\beta - c \right]$$

- the value of increasing consumption
- the value of accumulating capital

**Price**: $\theta$ values the increment to physical capital (asset price in terms of utils)
Optimal Conditions

- One decision variable: $c(t)$. First Order Condition:

$$\tilde{H}_c = 0$$

$$c^{-\sigma} = \theta$$  \hspace{1cm} (3)

On the margin, goods must be equally valuable in their two uses — consumption and capital accumulation.

- Shadow prices:

$$-\frac{d}{dt} \left[ \theta e^{-\rho t} \right] = \tilde{H}_k$$

$$-\dot{\theta} e^{-\rho t} + \rho \theta e^{-\rho t} = \tilde{H}_k$$

The increment of capital decreases its value tomorrow, since its price is higher when the resource is scarce.
Transversality conditions

\[ \lim_{t \to \infty} e^{-\rho t} \theta(t) k(t) = 0 \]

- Price is not too high such that the capital grows explosively to infinity.
- The unique equilibrium is obtained with the highest prices that do not cause too much saving and explosive capital growth, since these highest prices give the fast growth without causing a 'pong' in the price in the future.
- If capital goes explosively such that \( \lim_{t \to \infty} e^{-\rho t} \theta(t) k(t) > 0 \), since \( k(t) \) is already infinity, increment adds no more, the value of the increment should be 0, there is a 'pong' in the prices in the future — capital will lose value in the future. Knowing that, accumulating capital is not rational, there is no equilibrium.

So, there is a unique sequence of \( \theta(t) \) that satisfies the transversality condition and the optimal conditions. (one path of \( \theta \) such that people do not accumulate too much or too little)
Valuation change

\[ -\dot{\theta} e^{-\rho t} + \rho \theta e^{-\rho t} = \tilde{H}_k = \theta e^{-\rho t} \beta A(t) k(t)^{\beta - 1} \]

\[ \frac{\dot{\theta}}{\theta} = \rho - \beta A(t) k(t)^{\beta - 1} \]  \hspace{1cm} (4)

Current value Hamiltonian:

\[ H(k, \theta, c, t) = \frac{c^{1-\sigma}}{1-\sigma} + \theta \left[ A(t) k(t)^\beta - c \right] \]

\[ -\dot{\theta} + \rho \theta = H_k \]
Balanced growth path

Solutions on which consumption and capital are growing at constant percentage rates, the prices of the capital are declining at constant rates let

\[ \frac{\dot{c}}{c} = \kappa, \]

\[ \ln c^{-\sigma} = \ln \theta, \quad -\sigma \frac{\dot{c}}{c} = \frac{\dot{\theta}}{\theta}, \quad \text{so} \]

\[ \frac{\dot{\theta}}{\theta} = -\sigma \kappa \]

As capital is accumulated, the price of it decreases.
Balanced growth path con’d

\[ \frac{\dot{\theta}}{\theta} = \rho - \beta A(t) k(t)^{\beta-1}, \text{ so} \]

\[ \beta A(t) k(t)^{\beta-1} = \rho + \sigma \kappa \tag{5} \]

substitute (5) into (2), and divide through by \( k(t) \),

\[ c(t) + \dot{k}(t) = A(t) k(t)^{\beta} \]

\[ \frac{c(t)}{k(t)} + \frac{\dot{k}(t)}{k(t)} = \frac{\rho + \sigma \kappa}{\beta} \tag{6} \]

By definition \( \frac{\dot{k}(t)}{k(t)} \) is constant. log differentiating (6),

\[ \frac{\dot{k}(t)}{k(t)} = \frac{\dot{c}(t)}{c(t)} = \kappa \]
Balanced growth path con’d

- Optimal growth rate

log differentiating (5),

\[ \frac{\dot{A}(t)}{A(t)} + (\beta - 1) \frac{\dot{k}(t)}{k(t)} = 0 \]

\[ \kappa^* = \frac{\mu}{1 - \beta} \]

- Saving rate

\[ s = \frac{\dot{k}(t)}{c(t) + \dot{k}(t)} = \frac{\dot{k}(t)}{k(t)} + \frac{\dot{k}(t)}{k(t)} = \frac{\beta \kappa}{\rho + \sigma \kappa} \]
Remarks on Solow’s Model

- Growth rate is only determined by exogenous growth rate of technology and labor’s share. Technology spillovers across countries — lack a margin for the diversity of growth rate.

- $\rho$ and $\sigma$ have no bearing on growth rate, although lower $\rho$ and $\sigma$ induce a higher saving rate $s$, and high savings rate, in turn, are associated with relatively high output levels on a balanced growth path since $\beta A(t) k(t)^{\beta-1} = \rho + \sigma \kappa$ — a thrifty society will, in the long run, be wealthier, than an impatient one, but it will not grow faster.

- Solow concludes that changes in saving rates are level effects. On the opposite, the influential idea that changes in the tax structure that make savings more attractive can have large sustained effects on an economy’s growth rate sounds so reasonable, and it may even be true.
Human capital: general skill levels

The theory of human capital focuses on the fact that the way an individual allocates his time over various activities in the current period affects its productivity in the future periods.

Introducing human capital into the model, then, involves spelling out both:

- the way human capital levels affect current production
- the way the current time allocation affects the accumulation of human capital
Every worker is endowed with 1 hour that is inelastically devoted to either production or human capital accumulation.

Fraction $l$ to production, $1 - l$ to human capital accumulation.

Skill levels $h$ ranging from 0 to $\infty$.

Output $y(t) = f(k, lh, h_a) = Ak^\beta (lh)^{1-\beta} h_a^\gamma$, $h_a$ captures the external effect of human capital.

With externality, optimal solution is different from the market equilibrium. We solve for the market equilibrium.
Households’ Problem

\[
\max \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} \, dt
\] (7)

S.t. budget constraint

\[
c(t) + \dot{k}(t) + k(t) \leq r(t)k(t) + w(t)l(t)h(t) + k(t)
\] (8)

Capital does \textbf{not} depreciate

Technology for accumulating human capital

\[
\dot{h}(t) = h(t)\xi \delta [1 - l(t)]
\]

$1 - l(t)$ time spent on accumulating $h(t)$
\[
\frac{\dot{h}(t)}{h(t)} = h(t)^{\xi-1} \delta [1 - l(t)] \leq h(t)^{\xi-1} \delta
\]

if \( \xi < 1 \), as \( h(t) \to \infty \), \( h(t)^{\xi-1} \to 0 \), eventually, \( \frac{\dot{h}(t)}{h(t)} \) tend to zero no matter how much effort is devoted to accumulating it. \( \Rightarrow \) no sustained growth. Let \( \xi = 1 \),

\[
\dot{h}(t) = h(t) \delta [1 - l(t)] \tag{9}
\]
Evidently, it is also not optimal to choose $l(t)$ to maximize current period output: for the choice that achieves this is to set $\dot{h}(t) = 0$.

One needs to set some value on increment to human capital, since this increment to human capital will bring more output in the future.

\[ \tilde{H}(k, \theta_1, \theta_2, c, l, t) = \frac{c^{1-\sigma}}{1-\sigma} e^{-\rho t} + \theta_1 e^{-\rho t} [rk + whl - c] + \theta_2 e^{-\rho t} [h\delta(1 - l)] \]  

**Prices:** $\theta_1$ values the increment to physical capital  
$\theta_2$ values the increment to human capital
First Order Conditions

Two decision variables: $c(t), l(t)$

$$\tilde{H}_c = 0, \quad c^{-\sigma} = \theta_1$$  \hspace{1cm} (11) $$\tilde{H}_l = 0, \quad (\theta_1 w - \theta_2 \delta)he^{-\rho t} = 0, \quad \theta_1 w = \theta_2 \delta$$  \hspace{1cm} (12)

On the margin,

- goods must be equally valuable in their two uses — consumption and capital accumulation
- time must be equally valuable in its two uses — production and human capital accumulation
Shadow prices

Correct prices:

\[- \frac{d}{dt} [\theta_1 e^{-\rho t}] = \tilde{H}_k, \quad - \frac{d}{dt} [\theta_2 e^{-\rho t}] = \tilde{H}_h\]

\[- \dot{\theta}_1 e^{-\rho t} + \rho \theta_1 e^{-\rho t} = \tilde{H}_k, \quad - \dot{\theta}_2 e^{-\rho t} + \rho \theta_2 e^{-\rho t} = \tilde{H}_h\]

The increment of capital decreases its value tomorrow, since price is higher when resource is scarce.

Transversality conditions:

\[\lim_{t \to \infty} e^{-\rho t} \theta_1(t) k(t) = 0\]

\[\lim_{t \to \infty} e^{-\rho t} \theta_2(t) h(t) = 0\]
Private sector takes $h_a$ as given. Given $h_a(t)$, consider the problem the private sector, consisting of households and firms, would solve if each agent expected the average level of human capital to follow the path $h_a(t)$.

That is, consider the problem of choosing $\{h(t), k(t), c(t), l(t)\}$ so as to maximize utility (7) subject to budget constraint (8) and human capital accumulation constraint (9), taking $h_a(t)$ as exogenously determined. When the solution path $h(t)$ for this problem coincides with the given path $h_a(t)$ — so that the actual and expected behavior are the same — we say the system is in equilibrium.
\[-\dot{\theta}_1 e^{-\rho t} + \rho \theta_1 e^{-\rho t} = \tilde{H}_k = \theta_1 e^{-\rho t} r\]

\[
\frac{\dot{\theta}_1}{\theta_1} = \rho - r \tag{13}
\]

\[-\dot{\theta}_2 e^{-\rho t} + \rho \theta_2 e^{-\rho t} = \tilde{H}_h = \theta_1 e^{-\rho t} w l + \theta_2 e^{-\rho t} \delta (1 - l)\]

\[
\frac{\dot{\theta}_2}{\theta_2} = \rho - \delta(1 - l) - \frac{\theta_1}{\theta_2} w l \tag{14}
\]

from f.o.c. to \( l \), \( \frac{\theta_1}{\theta_2} = \frac{\delta}{w} \),

\[
\frac{\dot{w}}{w} = \frac{\dot{\theta}_2}{\theta_2} - \frac{\dot{\theta}_1}{\theta_1} = r - \delta
\]
Firm’s Problem

\[
\max_{k, lh} \left[ f(k, lh, h_a) - wh - rk \right]
\]

capital does not depreciate, \( k \) is returned to lender (household)

\[ r = f_k = \beta Ak^{\beta-1} (lh)^{1-\beta} h_a^\gamma \]

\[ w = f_{lh} = \frac{\partial}{\partial lh} f(k, lh, h_a) = (1 - \beta) Ak^{\beta} (lh)^{-\beta} h_a^\gamma \]

\[
\text{From firm's} \quad \frac{\dot{w}}{w} = \beta \frac{\dot{k}}{k} - \beta \frac{\dot{l}}{l} + \frac{\dot{h}}{h} + \gamma \frac{\dot{h}}{h}
\]

\[
\text{From households'} \quad \frac{\dot{w}}{w} = r - \delta = f_k - \delta
\]

connect the growth rate of wage obtained from firms’ problem and that from households’ problem

\[
\frac{\dot{l}}{l} = \frac{\dot{k}}{k} + \left( \frac{\gamma}{\beta} - 1 \right) \frac{\dot{h}}{h} + \frac{1}{\beta} (\delta - f_k)
\]
A System of Differential Equations

consists of only \( l, k, h \), no prices

\[
\begin{align*}
\dot{l} &= \frac{k}{k} + \left( \frac{\gamma}{\beta} - 1 \right) \frac{\dot{h}}{h} + \frac{1}{\beta} (\delta - f_k) \\
\dot{h} &= \delta (1 - l) \\
\dot{k} &= f_k - c \\
\dot{k} &= \frac{c}{k} - \frac{c}{k}
\end{align*}
\]
Define growth rate \( g_x = \frac{\dot{x}}{x} \) on the balanced growth path, prove that \( g_k = \frac{1 - \beta + \gamma}{1 - \beta} g_h \)

**Proof.**

On the balanced growth path, \( \frac{\dot{l}}{l} = 0 \). \( l \) is constant.

Since \( c^{-\sigma} = \theta_1, \frac{\dot{\theta}_1}{\theta_1} = -\sigma g_c \), and given \( \frac{\dot{\theta}_1}{\theta_1} = \rho - r \), we have \( r = \rho + \sigma g_c \).

\[
r = \rho + \sigma g_c = f_k = \beta Ak^{\beta - 1} (lh)^{1 - \beta} h^\gamma
\]

Clearly, we have \( g_k = \frac{\dot{k}}{k} = (1 + \frac{\gamma}{1 - \beta}) \frac{\dot{h}}{h} = (1 + \frac{\gamma}{1 - \beta}) g_h. \)
Balanced growth path con’d

wage rate:

\[ \frac{\dot{w}}{w} = r - \delta = \rho + \sigma g_c - \delta \]

Since \( f_k = \beta Ak^{\beta-1}(l h)^{1-\beta}h^\gamma = \rho + \sigma g_c \), substitute it into the budget constraint (8), and divide through by \( k(t) \),

\[ c(t) + \dot{k(t)} = r(t)k(t) + w(t)l(t)h(t) = Ak^\beta(l h)^{1-\beta}h^\gamma \]

\[ \frac{c(t)}{k(t)} + \frac{\dot{k(t)}}{k(t)} = \frac{\rho + \sigma g_c}{\beta} \]

(15)

By definition \( \frac{\dot{k(t)}}{k(t)} \) is constant. log differentiating (15),

\[ \frac{\dot{k(t)}}{k(t)} = \frac{\dot{c(t)}}{c(t)} = g_c \]
Balanced growth path con’d

\[
\frac{i}{l} = \frac{k}{k} + (\frac{\gamma}{\beta} - 1)\frac{h}{h} + \frac{1}{\beta}(\delta - f_k)
\]

\[
0 = g_k + (\frac{\gamma}{\beta} - 1)g_h + \frac{1}{\beta}(\delta - \rho - \sigma g_c)
\]

\[
0 = g_c + (\frac{\gamma}{\beta} - 1)\frac{1 - \beta}{1 - \beta + \gamma}g_c + \frac{1}{\beta}(\delta - \rho - \sigma g_c)
\]

\[
0 = \frac{\gamma}{\beta(1 - \beta + \gamma)}g_c + \frac{1 - \beta + \gamma}{\beta(1 - \beta + \gamma)}(\delta - \rho - \sigma g_c)
\]

\[
g_k = g_c = \frac{(\delta - \rho)(1 - \beta + \gamma)}{\sigma(1 - \beta + \gamma) - \gamma}
\]

\[
g_h = \frac{(\delta - \rho)(1 - \beta)}{\sigma(1 - \beta + \gamma) - \gamma}
\]
Comparative statics

\[
g_k = \frac{(\delta - \rho)(1 - \beta + \gamma)}{\sigma(1 - \beta + \gamma) - \gamma} \\
g_h = \frac{(\delta - \rho)(1 - \beta)}{\sigma(1 - \beta + \gamma) - \gamma}
\]

if \(\gamma = 0\), \(g_k = g_h\), while if \(\gamma > 0\), \(g_k > g_h\), so that external effect induces more rapid physical than human capital growth.

- Effectiveness of investment in human capital \(\delta \uparrow \rightarrow g_k \uparrow\)
- Less patient \(\rho \uparrow \rightarrow g_k \downarrow\), linkage between 'thriftiness' and growth, potential difference in growth across countries, absent in Solow model where \(g_k = \frac{\mu}{1-\beta}\) (\(\mu\) is an exogenous growth rate of technology)
- Relative risk aversion \(\sigma \uparrow \rightarrow g_k \downarrow\)
Growth rate

relative risk aversion $\sigma \uparrow \rightarrow g_k \downarrow$

average the same, prefer less variation

Higher $\sigma$ (risk averse): people prefer slower accumulation of capital but consumption smoothing. (concave utility function: people do not like consumption to vary too much)
Intuitively, stimulating saving can increase growth rate. Subsidy to saving?
Policy analysis

subsidy to saving

\[ c(t) + \dot{k}(t) = r(t)k(t) + w(t)l(t)h(t) + \phi \dot{k}(t) - \tau(t) \]

\( \phi \) subsidy rate on physical capital accumulation
\( \tau \) lump sum tax

\[ \dot{k} = \frac{1}{1 - \phi} \left[ rk + wlh - c - \tau \right] \]

maximization:

\[ \frac{\dot{c}}{c} = \frac{1}{\sigma} \left[ \frac{r}{1 - \phi} - \rho \right] \]

partial equilibrium (taking \( r \) as given) drawback: \( \phi \uparrow \rightarrow \frac{\dot{c}}{c} \uparrow \), this is wrong
\( \phi \) influences \( r \), too
Endogenous Interest Rate

- $\phi \uparrow \rightarrow$ saving (level of physical capital $k$) $\uparrow$, $f_k \downarrow \rightarrow r \downarrow$

\[
\frac{\dot{c}}{c} = \frac{1}{\sigma} \left[ \frac{r}{1 - \phi} - \rho \right]
\]

- on the "new" balanced path,
  \[
  r = (1 - \phi)(\rho + \sigma g_c) = f_k = \beta A k^{\beta - 1} (lh)^{1 - \beta} h^\gamma
  \]
so we still have $g_k = \frac{1 - \beta + \gamma}{1 - \beta} g_h$

- Government budget constraint: $\phi \dot{k}(t) = \tau(t)$,
  \[
  \frac{c(t) + \tau(t)}{k(t)} + (1 - \phi) \frac{\dot{k}(t)}{k(t)} = \frac{(1 - \phi)(\rho + \sigma g_c)}{\beta}, \text{ so we still have } g_c = g_k
  \]

- 
  \[
  \frac{\dot{l}}{l} = \frac{\dot{k}}{k} + \left( \frac{\gamma}{\beta} - 1 \right) \frac{\dot{h}}{h} + \frac{1}{\beta} (\delta - \frac{r}{1 - \phi})
  \]

- $g_c$ is unchanged, subsidy to saving has still only level effect!!
What about subsidy to human capital accumulation? \( \phi \dot{h} \)

\[
\dot{h} = h\delta(1 - l)
\]

\( \delta \) the efficiency parameter, subsidy to human capital accumulation is equivalent to increase the return of human capital accumulation, or similar to increase \( \delta \)

people allocate more time to accumulate human capital, all the growth comes from there
\[ \tilde{H}(k, \theta_1, \theta_2, c, l, t) = \frac{c^{1-\sigma}}{1-\sigma} e^{-\rho t} \]
\[ + \theta_1 e^{-\rho t} [rk + wh + \phi \dot{h} - c - \tau] \]
\[ + \theta_2 e^{-\rho t} [h\delta(1 - l)] \]
\[ \theta_1 e^{-\rho t} [rk + wh - c - \tau] + \theta_2 e^{-\rho t} \left( \frac{\phi \theta_1 e^{-\rho t}}{\theta_2 e^{-\rho t}} + 1 \right) [h\delta(1 - l)] \]