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ON THE MEIJER TRANSFORMATION

J. CONLAN and E. L. KOH

Department of Mathematics
University of Regina
Regina, Canada

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ABSTRACT. Recently [8], an operational calculus for the operator $B_\mu = t^{-\mu}Dt^{1+\mu}D$ with $-1 < \mu < \infty$ was developed via the algebraic approach [4], [13], [15]. This paper gives the integral transform version. In particular, a differentiation theorem and a convolution theorem are proved.

1. INTRODUCTION.

Ditkin [4], and later with Prudnikov [6], developed an operational calculus for the operator $\frac{d}{dt}\, t\, \frac{d}{dt}$ similar to the algebraic approach of Mikusinski [15]. Meller [13], [14] generalized Ditkin's calculus to operators $B_\alpha = t^{-\alpha}Dt^{1+\alpha}D$ with $-1 < \alpha < 1$. Krätzel [9], [10], [11], [12] gave an integral transform version to Meller's calculus and also generalized the calculus to operators containing n^{th} order derivatives. He developed a theory of integral transforms of the form

$$\mathcal{L}_v^{(n)}\{f\}(s) = \int_0^\infty w_v^{(n)}(n(st)^{1/n})f(t)dt,$$

where n = 1, 2, ..., $\mathrm{Re}(v) > \frac{1}{n} - 1$, and

$$w_v^{(n)}(z) = \frac{(2\pi)^{\frac{n-1}{2}}\sqrt{n}\,(\frac{z}{n})^{nv}}{\Gamma(v+1-1/n)}\int_1^\infty (y^n-1)^{v-\frac{1}{n}}\exp(-zy)dy.$$

Here, $\mathcal{L}_v^{(1)}$ is the Laplace transform and $\mathcal{L}_v^{(2)}$ is the Meijer transform of the form

$$\mathcal{L}_v^{(2)}\{f\}(s) = 2\int_0^\infty (st)^{v/2} K_v(2\sqrt{st})f(t)dt, \tag{1}$$

where $K_v(z)$ is the MacDonald function of order v. Dimovski [1], [2], [3] developed an operational calculus for the operator

$$B = t^{\alpha_0}\frac{d}{dt}t^{\alpha_1}\frac{d}{dt}\cdots t^{\alpha_{n-1}}\frac{d}{dt}t^{\alpha_n},$$

using an integral transform that for n = 2 reduces to the Meijer transform of the form

$$\tilde{k}_v\{f\}(s) = 2s^{-v}\int_0^\infty (st)^{v/2}K_v(2\sqrt{st})f(t)dt. \tag{2}$$

In [8], Koh reconsidered Meller's operator $B_\mu = t^{-\mu}\frac{d}{dt}t^{1+\mu}\frac{d}{dt}$ but with $\mu\varepsilon(-1,\infty)$. Following Mikusinski, Ditkin, et. al., he constructed an operational calculus through the field extension of a commutative convolution ring without zero divisors. His calculus reduces to Ditkin's when $\mu = 0$ and Meller's when $\mu\varepsilon(-1,1)$.

In this paper, we give an integral transform analogue of [8] via the Meijer transform of the form

$$k_{\mu}\{f\}(p) = \frac{2p}{\Gamma(\mu+1)} \int_0^{\infty} (pt)^{\mu/2} K_{\mu}(2\sqrt{pt}) f(t)dt \tag{3}$$

for $\mathrm{Re}(\mu) > -1$. In particular, we prove a differentiation theorem and a convolution theorem. The presence of a factor $\frac{2p}{\Gamma(\mu+1)}$ in (3), as opposed to those in (1) and (2), is essential in our convolution theorem.

2. THE MAIN THEOREMS.

We will define the convolution, $*$, of two functions, f, g by

$$f*g = \frac{1}{\Gamma(\mu+1)} Dt^{1-\mu} D^{\mu+1} \int_0^t \xi^{\mu}(t-\xi)^{\mu} \int_0^1 f(x\xi) g[(1-x)(t-\xi)]dxd\xi, \tag{4}$$

see Koh [8], where D^{λ} is the Riemann-Liouville derivative of order λ, see Ross [17]. This convolution exists if, for example, f and g are in $C^{\infty}[0,\infty)$, the space of infinitely differentiable complex functions on $[0,\infty)$.

The following properties of K_{μ} will be used:

$$K_{\mu}(2\sqrt{pt}) = \frac{1}{2}(t/p)^{\mu/2} \int_0^{\infty} x^{-\mu-1} \exp(-px-\frac{t}{x})\, dx \tag{5.1}$$

$$= \frac{1}{2}(t/p)^{-\mu/2} \int_0^{\infty} x^{\mu-1} \exp(-px-\frac{t}{x})\, dx, \tag{5.2}$$

$\mathrm{Re}(\mu) > -1$, $\mathrm{Re}(p) > 0$, $\mathrm{Re}(t) > 0$.

$$2(pt)^{\mu}K_{\mu}(2\sqrt{pt}) \sim \left\{ \begin{array}{l} -\ln t+0(1),\ \mu = 0 \\ \Gamma(\mu)+0[t^{\min(1,\mu)}],\ \mu > 0 \\ -\frac{\Gamma(1-\mu)}{\mu}(pt)^{\mu}+0(1),\ -1 < \mu < 0 \end{array} \right\},\ t \to 0 \tag{6.1}$$

$$\sim \sqrt{\frac{\pi}{2}}\, t^{\frac{\mu}{2}-\frac{1}{4}} e^{-2\sqrt{pt}}\{1+0(|t|^{-\frac{1}{2}})\},\ t \to \infty\ . \tag{6.2}$$

$$\frac{d}{dt}\{(pt)^{\pm\mu/2}K_{\mu}(2\sqrt{pt})\} = -p(pt)^{\pm\mu-\frac{1}{2}}K_{\mu\mp 1}(2\sqrt{pt}). \tag{7}$$

In order that the Meijer transform (3) converges, it is sufficient for $f(t)$ to be locally Lebesgue integrable on $(0,\infty)$ and $|f(t)| < Ce^{2\gamma\sqrt{t}}$ $(t \to \infty)$ for $\mu > 0$ and for $f(t)$ to remain bounded in the neighbourhood of the origin for $-1 < \mu \leq 0$. The integral then converges absolutely within the parabolic region $\mathrm{Re}\sqrt{p} > \gamma$. This is clear from the asymptotic behaviors (6.1) and (6.2). Indeed,

$$|\int_0^{\infty} f(t)(pt)^{\frac{\mu}{2}}K_{\mu}(2\sqrt{pt})dt| \leq \int_0^{\infty}|f(t)|(pt)^{\frac{\mu}{2}}K_{\mu}(2\sqrt{pt})\, dt$$

$$\leq \sup_{0<t<\varepsilon}|f(t)|\int_0^{\varepsilon}(pt)^{\frac{\mu}{2}}K_{\mu}(2\sqrt{pt})dt + \int_{\varepsilon}^{T}|f(t)|(pt)^{\frac{\mu}{2}}K_{\mu}(2\sqrt{pt})dt$$

$$+ \int_T^{\infty}|f(t)|(pt)^{\frac{\mu}{2}}K_{\mu}(2\sqrt{pt})\, dt, \quad \text{for some } 0 < \varepsilon < T < \infty. \tag{8}$$

The first integral on the right hand side of (8) exists because of (6.1); the second exists because of the local integrability of $f(t)$ and the continuity of $(pt)^{\frac{\mu}{2}}K_{\mu}(2\sqrt{pt})$; and the last integral exists because of (6.2) provided $\mathrm{Re}\sqrt{p} > \gamma$. We state this result in

THEOREM 1. If $f(t)\varepsilon L_{\ell oc}(0,\infty)$, if there are constants C and γ such that $|f(t)| < Ce^{2\gamma\sqrt{t}}$ as $t \to \infty$, and if $\lim_{t\to 0^+} f(t) = f(0^+) < \infty$, then (3) converges absolutely in $\mathrm{Re}\sqrt{p} > \gamma$ for all $\mu\varepsilon(-1,\infty)$. Furthermore, the integral (3) as a function of p is analytic in the region of convergence.

The proof of the analyticity is standard and is omitted. When a function f(t) satisfies the hypothesis of theorem 1, we shall write, for brevity, $f\varepsilon$HypI. Clearly, if a function f has continuous derivative on $[0,\infty)$ and $f'\varepsilon$HypI, then $f\varepsilon$HypI.

THEOREM 2. If $f\varepsilon C^2[0,\infty)$ and $f'\varepsilon$HypI, then

$$k_\mu(B_\mu f) = p(k_\mu f) - pf(0^+).$$

PROOF.

$$k_\mu(B_\mu f) = \frac{2p^{\frac{\mu}{2}+1}}{\Gamma(\mu+1)} \int_0^\infty [t^{-\mu}\frac{d}{dt}t^{1+\mu}\frac{d}{dt}f(t)]t^{\frac{\mu}{2}}K_\mu(2\sqrt{pt})dt$$

$$= \frac{2p^{\frac{\mu}{2}+1}}{\Gamma(\mu+1)} \{t^{-\frac{\mu}{2}}K_\mu(2\sqrt{pt})t^{\mu+1}\frac{df}{dt}\Big|_0^\infty - \int_0^\infty (t^{\mu+1}\frac{df}{dt})\frac{d}{dt}(t^{-\frac{\mu}{2}}K_\mu(2\sqrt{pt})dt\}.$$

The limit terms vanish because $f'\varepsilon$HypI. We now use (7) and another integration by parts to yield

$$k_\mu(B_\mu f) = \frac{2p^{\frac{\mu}{2}+\frac{3}{2}}}{\Gamma(\mu+1)} \int_0^\infty \left(\frac{df}{dt}\right) t^{\frac{\mu+1}{2}} K_{\mu+1}(2\sqrt{pt})dt$$

$$= \frac{2p^{\frac{\mu}{2}+\frac{3}{2}}}{\Gamma(\mu+1)} \{-\lim_{t\to 0^+} [f(t)t^{\frac{\mu+1}{2}} K_{\mu+1}(2\sqrt{pt})] + p^{\frac{1}{2}} \int_0^\infty f t^{\frac{\mu}{2}} K_\mu(2\sqrt{pt})dt\}$$

$$= pk_\mu(f)(p) - pf(0^+). \quad \text{QED}$$

This result immediately generalizes to the next theorem by induction.

THEOREM 3. If $f\varepsilon C^{2k}[0,\infty)$ and $f^{(2k-1)}\varepsilon$HypI, then

$$k_\mu(B_\mu{}^k f) = p^k k_\mu[f] - \sum_{j=1}^{k} p^j B_\mu^{k-j} f(0).$$

Note that this theorem is the integral transform version of Lemma 1 of [8]. The operational calculus for the operator B_μ is now effected through this formula. To solve the initial value problem

$$Q(B_\mu)f(t) = g(t)$$

$$f(0) = C_0,\ B_\mu f(0) = C,\ \ldots,\ B^{k-1}f(0) = C_{k-1} \tag{9}$$

where Q(z) is a polynomial, we transform (9) into

$$Q(p)k_\mu f = P(p) + k_\mu g$$

where P(p) is a polynomial of degree less than or equal to that of Q(p). Therefore

$$k_\mu f = \frac{P(p)}{Q(p)} + \frac{1}{Q(p)} (k_\mu g)(p)$$

and f(t) is retrieved by means of an inversion formula and possibly a convolution theorem.

The following inversion theorem is obtained from Meijer's Theorem [18] through a simple change of variables, viz. $x \to \sqrt{t}$ and $y \to 2\sqrt{p}$.

THEOREM 4. Let μ be a complex number whose real part is not less than $-\frac{1}{2}$. Assume that in $\mathrm{Re}\sqrt{p} > \gamma_0 \geq 0$, $F(p)$ is an analytic function and is bounded according to $|F(p)| < M|p|^{-q}$ where $q > \frac{3}{2}\mathrm{Re}\mu+2$. Then for real $c > \gamma_0$ and for $\mathrm{Re}\sqrt{p} > c$, $F(p) = k_\mu(f)$ where

$$f(t) = \frac{\Gamma(\mu+1)t^{-\frac{\mu}{2}}}{2\pi i} \int_{\mathrm{Re}\sqrt{p}=c} F(p)p^{-\frac{\mu}{2}-1} I_\mu(2\sqrt{pt})dp. \tag{10}$$

The following lemma will be used in proving a convolution theorem for k_μ.

LEMMA. Letting D_∞^μ denote the Weyl derivative of order μ, we have

$$D_\infty^\mu \{(z/t)^{\frac{\mu}{2}} K_\mu(2\sqrt{zt})\} = (-z/t)^\mu K_{2\mu}(2\sqrt{zt}).$$

PROOF. By definition, $D_\infty^\mu \{f(t)\} = (d/dt)^k W^{k-\mu}\{f(t)\}$, where $k-1 < \mu < k$, and where

$$W^\nu\{f(t)\} = [-1/\Gamma(\nu)] \int_t^\infty f(s)(t-s)^{\nu-1}ds.$$

Since

$$K_\mu(2\sqrt{zt}) = \tfrac{1}{2}(t/z)^{\mu/2} \int_0^\infty \exp\{-(zy+t/y)\}y^{-\mu-1}dy \tag{11}$$

we have

$$W^{k-\mu}\{t^{-\mu/2}K_\mu(2\sqrt{zt})\} =$$

$$= \{-z^{-\mu/2}/2\Gamma(k-\mu)\} \int_t^\infty (t-s)^{k-\mu-1} ds \int_0^\infty \exp\{-(zy+s/y)\} y^{-\mu-1} dy$$

$$= \{-z^{-\mu/2}/2\Gamma(k-\mu)\} \int_0^\infty \exp(-zy) y^{-\mu-1} dy \int_t^\infty \exp(-s/y)$$

$$\cdot (t-s)^{k-\mu-1} ds.$$

On putting $s-t = y\lambda$, $ds = yd\lambda$ and using the definiton of the gamma function, this becomes

$$(-1)^{k-\mu}\{z^{-\mu/2}/2\} \int_0^\infty \exp\{-(zy+t/y)\} y^{k-2\mu-1} dy.$$

Differentiating k times with respect to t, we get

$$D_\infty^\mu\{t^{-\mu/2}K_\mu(2\sqrt{zt})\} =$$

$$= (-1)^{-\mu}\{z^{-\mu/2}/2\} \int_0^\infty \exp\{-(zy+t/y)\} y^{-2\mu-1} dy,$$

and using (11), with μ replaced by 2μ, completes the proof.

THEOREM 5. (Convolution theorem) If f and g belong to $C^\infty[0,\infty)$ and $f^{(n)}$ and $g^{(n)}$ satisfy HypI for every n, then $k_\mu(f*g)$ converges absolutely in $\mathrm{Re}\sqrt{p} > \gamma_f+\gamma_g$ and

$$k_\mu(f*g) = (k_\mu f)(k_\mu g) \text{ for } \mu\varepsilon(-1,\infty).$$

PROOF. We use the two-dimensional Laplace convolution theorem (pp. 26-29, [5]): Let

$$F_i(p,q) \equiv L(f_i) = \int_0^\infty\int_0^\infty e^{-px-qy} f_i(x,y)\,dxdy, \quad i = 1,2$$

and

$$f^*(x,y) = \int_0^x \int_0^y f_1(\xi,\eta) f_2[x-\xi, y-\eta]\, d\eta\, d\xi. \tag{12}$$

If F_i converges absolutely, then so does $F^*(p,q) \equiv L(f^*)$ and

$$F^*(p,q) = F_1(p,q) F_2(p,q). \tag{13}$$

In (12), let $f_1(x,y) = x^{\mu} f(xy)$ and $f_2(x,y) = x^{\mu} g(xy)$. Then $f^*(x,y) = \frac{1}{y^{2\mu}} \zeta(xy)$ where

$$\zeta(t) = \int_0^t \int_0^1 w^{\mu} (t-w)^{\mu} f(wv) g[(t-w)(1-v)]\, dv\, dw, \text{ and}$$

$$L[f^*] = \int_0^{\infty}\int_0^{\infty} e^{-px-qy} \zeta(xy) y^{-2\mu} dx\, dy$$

$$= \int_0^{\infty} \zeta(t) \int_0^{\infty} e^{-qy-pty^{-1}} y^{-2\mu-1} dy\, dt. \tag{14}$$

Similarly,

$$L[x^{\mu} f(xy)] = \int_0^{\infty} f(t)\, dt \int_0^{\infty} \exp(-py-qty^{-1}) y^{\mu-1} dy$$

$$= 2\int_0^{\infty} f(t) p^{-\mu} (pqt)^{\frac{\mu}{2}} K_{\mu}(2\sqrt{pqt})\, dt, \tag{15}$$

on using the integral representation (5.2). Set $p = 1$ in (14) and (15) and invoke (13); we have

$$\int_0^{\infty} \zeta(t)\, dt \int_0^{\infty} \exp(-qy-ty^{-1}) y^{-2\mu-1} dy = \frac{\Gamma^2(\mu+1)}{q^2} (k_{\mu} f)(k_{\mu} g). \tag{16}$$

It remains to show that the left hand side of (16) is equal to $\frac{\Gamma^2(\mu+1)}{q^2} k_{\mu}(f*g)$. Indeed, letting R_{λ} denote the Riemann-Liouville integral of order λ,

$$k_\mu(f*g) = k_\mu[\frac{1}{\Gamma(\mu+1)} B_\mu t^{-\mu} D^\mu \zeta(t)] \tag{17}$$

$$= \frac{q}{\Gamma(\mu+1)} k_\mu \; [t^{-\mu} D^\mu \zeta(t)] \tag{18}$$

$$= \frac{2q^2}{\Gamma^2(\mu+1)} \int_0^\infty (qt)^{\mu/2} K_\mu(2\sqrt{qt}) t^{-\mu} (\frac{d}{dt})^k R_{k-\mu} \zeta(t) dt \tag{19}$$

$$= \frac{(-1)^k 2q^2}{\Gamma(k-\mu)\Gamma^2(\mu+1)} \int_0^\infty (\frac{d}{dt})^k \{(q/t)^{\mu/2} K_\mu(2\sqrt{qt})\} dt \int_0^t \zeta(s) \cdot (t-s)^{k-\mu-1} ds \tag{20}$$

$$= \frac{(-1)^\mu 2q^2}{\Gamma^2(\mu+1)} \int_0^\infty \zeta(t) D_\infty^\mu \{(q/t)^{\mu/2} K_\mu(2\sqrt{qt})\} dt \tag{21}$$

$$= \frac{2q^2}{\Gamma^2(\mu+1)} \int_0^\infty \zeta(t) (q/t)^\mu K_{2\mu}(2\sqrt{qt}) dt \tag{22}$$

$$= \frac{q^2}{\Gamma^2(\mu+1)} \int_0^\infty \zeta(t) dt \int_0^\infty \exp(-qy-t/y) y^{-2\mu-1} dy \tag{23}$$

which proves our assertion concerning the left hand side of (16). Equation (17) follows from the definition of convolution. Equation (18) follows from theorem 2 since $Dt^{-\mu}D^\mu\zeta(t) \varepsilon \mathrm{HypI}$ for $f^{(n)}$ and $g^{(n)} \varepsilon \mathrm{HypI}$. Furthermore, from a theorem of Ritt [16], we have $f,g = O(1) \Rightarrow \zeta(t) = O(t^{2\mu+1}) \Rightarrow t^{-\mu}D^\mu\zeta(t) = O(t)$ as $t \to 0^+$. Thus $\lim_{t\to 0^+} t^{-\mu}D^\mu\zeta(t) = 0$. Equation (20) follows from (19) by the definition of the Riemann-Liouville integral, $R_{k-\mu}$, and integrating by parts k-times. The integrated terms vanish at $t = 0$ and $t = \infty$ by (6.1), (6.2), and the fact that in the definition of $\zeta(t)$, the functions f and g satisfy the hypotheses of theorem 5. Equation (22) is due to the preceding lemma. That

$k_\mu(f*g)$ converges absolutely follows from the absolute convergence of (13). This completes the proof.

3. SOME OPERATIONAL FORMULAS.

Let $F(y) = \int_0^\infty f(x)K_\mu(xy)^{1/2}dx \equiv \hat{f}$ (24)

If we set $y = 2\sqrt{p}$ and $x = \sqrt{t}$, we get

$$k_\mu[f(\sqrt{t})t^{-1/4-\mu/2}] = \frac{2^{2/3}p^{3/4+\mu/2}}{\Gamma(\mu+1)} F(2\sqrt{p}) \ . \qquad (25)$$

From Erdélyi [7] p. 137 (16),

$$[x^{\beta+\mu-1/2}J_\beta(ax)]^\wedge = 2^{\beta+\mu}a^\beta y^{\mu+1/2}\Gamma(\beta+\mu+1)(y^2+a^2)^{-\beta-\mu-1} \qquad (26)$$

$\mathrm{Re}\beta > |\mathrm{Re}\mu|-1$, $\mathrm{Re}y > |\mathrm{Im}p|$.

By (25),

$$k_\mu[t^{\beta/2}J_\beta(2\sqrt{at})] = \frac{a^{\beta/2}p^{\mu+1}}{(p+a)^{\beta+\mu+1}} \cdot \frac{\Gamma(\beta+\mu+1)}{\Gamma(\mu+1)} \ , \qquad (27)$$

$\mathrm{Re}\sqrt{p} > |\mathrm{Im}\sqrt{a}|$.

Letting $\beta = v-\mu$, this becomes, for $\mathrm{Re}v > \mathrm{Re}\mu$,

$$k_\mu[t^{(v-\mu)/2}J_{v-\mu}(2\sqrt{at})] = \frac{a^{(v-\mu)/2}p^{\mu+1}\Gamma(v+1)}{(p+a)^{v+1}\Gamma(\mu+1)} \ . \qquad (28)$$

Since $K_\mu(\cdot) = K_{-\mu}(\cdot)$, we have from (26), for $\mathrm{Re}\sqrt{p} > |\mathrm{Im}a|$,

$$[x^{\beta-\mu+1/2}J_\beta(ax)]^\wedge = \frac{2^{\beta-\mu}a^\beta y^{-\mu+1/2}\Gamma(\beta-\mu+1)}{(y^2+a^2)^{-\mu+\beta+1}}. \qquad (29)$$

From (25),

$$k_\mu[t^{\beta/2-\mu}J_\beta(2\sqrt{at})] = \frac{a^{\beta/2}\Gamma(\beta-\mu+1)p}{\Gamma(\mu+1)(p+a)^{\beta-\mu+1}} \qquad (30)$$

Setting $\beta = v+\mu$,

$$k_\mu[t^{(v-\mu)/2}J_{v+\mu}(2\sqrt{at})] = \frac{\Gamma(v+1)a^{(v+\mu)/2}p}{\Gamma(\mu+1)(p+a)^{v+1}} . \tag{31}$$

If $v = 0$ in (31),

$$k_\mu[\Gamma(\mu+1)(at)^{-\mu/2}J_\mu(2\sqrt{at})] = \frac{p}{p+a} . \tag{32}$$

Letting $a \to -a$, and using $I_\mu(z) = e^{-\mu\pi i/2}J_\mu(iz)$,

$$k_\mu[\Gamma(\mu+1)(at)^{-\mu/2}I_\mu(2\sqrt{at})] = \frac{p}{p-a} . \tag{33}$$

Equation (31) can be written as

$$k_\mu[\frac{\Gamma(\mu+1)}{\Gamma(v+1)}t^v(at)^{-(v+\mu)/2}J_{v+\mu}(2\sqrt{at})] = \frac{p}{(p+a)^{v+1}} . \tag{34}$$

Again letting $a \to -a$, and $I_\mu(z) = e^{-\mu\pi i/2}J_\mu(iz)$, this gives

$$k_\mu[\frac{\Gamma(\mu+1)}{\Gamma(v+1)}t^v(at)^{-(v+\mu)/2}I_{v+\mu}(2\sqrt{at})] = \frac{p}{(p-a)^{v+1}}. \tag{35}$$

These expressions are useful in inverting rational functions. As an application, consider the problem of solving

$$(B_\mu^2+3B_\mu+2)\phi(t) = f(t),$$

$$\phi(0) = \phi_0,$$

$$(B_\mu\phi)(0) = \phi_1.$$

One gets

$$(p^2+3p+2)k_\mu(\phi) = \phi_0 p^2+(3\phi_0+\phi_1)p+k_\mu f,$$

whence

$$k_\mu(\phi) = -\frac{(\phi_0+\phi_1)p}{p+2} + \frac{(2\phi_0+\phi_1)p}{p+1} + (\frac{1}{2} + \frac{p}{2(p+2)} - \frac{p}{p+1})k_\mu f.$$

Therefore

$$\phi = -(\phi_0+\phi_1)\Gamma(\mu+1)(2t)^{-\mu/2}J_\mu(2\sqrt{2t})$$
$$+(2\phi_0+\phi_1)\Gamma(\mu+1)t^{-\mu/2}J_\mu(2\sqrt{t})+\frac{1}{2}f(t)$$
$$+\{\frac{1}{2}\Gamma(\mu+1)(2t)^{-\mu/2}J_\mu(2\sqrt{2t})-\Gamma(\mu+1)t^{-\mu/2}J_\mu(2\sqrt{t})\}*f(t).$$

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