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On the MHD effects on the force-free monopole outflow

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ABSTRACT

The stationary axisymmetric outflow from a rotating sphere with a (split) monopole magnetic field is considered. The stream equation describing the outflow is linearized in terms of the Michel magnetization parameter $\sigma^{-1} \ll 1$, which allows a self-consistent analysis of the direct problem. It is shown that for a finite σ the fast magnetosonic surface is located at a finite distance $\sim\sigma^{1/3}R_{\rm L}$ ($R_{\rm L} = c/\Omega_{\rm F}$ is the light cylinder). We have also found that the particle energy at the fast surface is just equal to the Michel value $\gamma \sim \sigma^{1/3}$. The particle acceleration and magnetic field collimation are shown to become ineffective outside the fast magnetosonic surface.

Key words: MHD – methods: analytical – stars: neutron – galaxies: jets.

1 INTRODUCTION

Axisymmetric stationary hydrodynamic and magnetohydrodynamic (MHD) flows in the vicinity of a central compact body have long been studied in connection with many astrophysical sources. Spherically symmetric accretion on to ordinary stars and black holes (Bondi 1952; Shapiro & Teukolsky 1983), axisymmetric stellar (solar) wind (Parker 1958; Michalas 1978), jets from young stellar objects (Lada 1985), outflows from axisymmetric magnetosphere of rotating neutron stars (Michel 1991; Beskin, Gurevich & Istomin 1993) – all these are flows of the indicated type.

The MHD models are now developed intensely in connection with the theory of the magnetospheres of rotating supermassive black holes ($\mathcal{M} \sim 10^8$–10^9 M$_\odot$), which are thought of as a 'central engine' in active galactic nuclei (AGNs) and quasars (Begelman, Blandford & Rees 1984; Thorne, Price & Macdonald 1986). In particular, it is the MHD model that is now most popular in connection with the problem of the origin and stability of the jets. Indeed, purely gas-dynamic models require extremely high densities of the thermal energy, which contradicts the observed fluxes of X-ray radiation from AGNs. In turn, within the radiation models it is difficult to explain the existence of Lorentz factors of ejected matter $\gamma > 3$, registered in some jets (Blandford 1992). As for the MHD models, one can easily understand both the activity of the sources (rotating energy of compact objects), and the mechanism of the energy and angular momentum loss. Moreover, within this approach the energy transformation from electromagnetic field to particles may become ineffective which could, in principle, explain the transportation of energy from central engine to the radiating regions. On the other hand, it is well known for young radio pulsars that, even at distances that are small in comparison with the size of a supernova remnant, the greater part of the energy is transferred by particles rather than by an electromagnetic wave, though within the light cylinder the situation is reversed (Rees & Gunn 1974; Kennel & Coroniti 1984).

From the theoretical point of view, the crucial point here is the question of how large the longitudinal electric current circulating in the magnetosphere is. When it exceeds the Goldreich–Julian (1969) current, the pinch effect (the attraction of parallel currents) becomes possible, which results in the collimation of magnetic field lines along the rotational axis. One should remember here that within the force-free approach the longitudinal current is a free parameter. However, in the general MHD approach the number of singular surfaces increases and the smooth transition through them becomes possible only with a definite choice of longitudinal current.

It is well known that for axisymmetric stationary flows the magnetic surfaces are described by the stream equation, which is a non-linear partial differential equation containing (in the general case) five integrals of motion. This equation has been discussed by many authors (Soloviev 1967; Okamoto 1975; Ardavan 1976, 1979; Blandford & Payne 1982; Lovelace et al. 1986; Camenzind 1987; Heyvaerts & Norman 1989; Sakurai 1990; Pelletier & Pudritz 1992; Li, Chiueh & Begelman 1992; Fendt, Camenzind & Appl 1995), and recently it was obtained in the most general case of the Kerr metric (Nitta, Takahashi & Tomimatsu 1991; Beskin & Pariev 1993). Nevertheless, only a few analytical solutions of the stream equation are now known (Michel 1973; Blandford & Znayek 1977; Low & Tsinganos 1986; Tsinganos & Sauty 1992; Bogovalov 1992; Beskin & Pidoprigora 1995; Pariev 1996; Beskin & Malyshkin 1996).

In this paper we consider the relativistic MHD outflow from a magnetized sphere with a (split) monopole magnetic field. This approach may be interesting with regard to the problem of a jet formation from the 'central engine' in AGN and recently discovered jets from radio pulsars (Markwardt & Ögelman 1995; Hester et al. 1995). Henceforth the case of a cold plasma is considered. Such an assumption is quite reasonable because thermal processes play no role in the magnetosphere of radio pulsars. As for the jets from AGN, this approximation is applicable in the rarefied polar regions of magnetosphere, where the density of accreting gas is small enough. Finally, the flow is assumed to be near the Michel (1973) force-free monopole analytical solution. In other words, the Michel (1969) magnetization parameter σ is assumed to be much larger than 1. As a result, the stream equation describing the outflow can be linearized in terms of the value σ^{-1}, and the self-consistent analysis of the flow can be produced.

The goal of this paper is to demonstrate how all characteristics of a flow (magnetic structure, particle energy as well as the position of singular surfaces) can be obtained from a definite number of boundary conditions at the sphere surface. It will be shown that the fast magnetosonic surface placed for the force-free flow at infinity is located in our case of finite σ at finite distance $\sim\sigma^{1/3}R_{\rm L}$ from the rotational axis ($R_{\rm L}$ is the light cylinder), the particle energy at the fast surface being just equal to the Michel value $\gamma \sim \sigma^{1/3}$. It will also be shown that, outside the fast magnetosonic surface, the acceleration of particles becomes ineffective. At the same time, the disturbance of magnetic field lines does not exceed $\sigma^{-2/3}$ here, and grows very slowly at large distances.

2 ALGEBRAIC RELATIONS

Let us consider a stationary axisymmetric MHD outflow of cold plasma in a flat space–time

$$g_{rr} = 1, \quad g_{\theta\theta} = r^2, \quad g_{\varphi\varphi} = \varpi^2 = r^2 \sin^2\theta. \tag{1}$$

Units where $c = 1$ are used throughout the paper.

In the stationary axisymmetric case, the magnetic field may be written in the form

$$\boldsymbol{B} = \frac{\nabla\Psi\times\boldsymbol{e}_{\hat\varphi}}{2\pi\varpi} - \frac{2I}{\varpi}\boldsymbol{e}_{\hat\varphi}. \tag{2}$$

Here $\Psi(r,\theta)$ is the magnetic flux, and $I(r,\theta)$ is the total electric current flowing inside the region $\Psi < \Psi(r,\theta)$. On the other hand, assuming that the magnetosphere contains enough plasma to screen the longitudinal electric field, one can find

$$\boldsymbol{E} = -\frac{\Omega_{\rm F}}{2\pi}\nabla\Psi, \tag{3}$$

the angular velocity $\Omega_{\rm F} = \Omega_{\rm F}(\Psi)$ being constant on the magnetic surfaces $\Psi(r,\theta) = $ constant.

Further, using continuity of the flow $\nabla\cdot(n\boldsymbol{u}) = 0$, we have

$$\boldsymbol{u} = \frac{\eta}{n}\boldsymbol{B} + \gamma\Omega_{\rm F}\varpi\boldsymbol{e}_{\hat\varphi}. \tag{4}$$

Here n is the concentration of particles in the reference frame comoving with the hydrodynamic flow $\boldsymbol{v}$, $\gamma^{-1} = \sqrt{1-v^2}$, and $\boldsymbol{u} = \boldsymbol{v}/\sqrt{1-v^2}$ is the space component of the four-velocity of the flow. As $\nabla\boldsymbol{B} = 0$, a new quantity η is constant on the magnetic surfaces as well: $\eta = \eta(\Psi)$.

To find two more integrals of motion, one has to write down the energy–momentum conservation law. It immediately follows from this relation that the energy E and the z-component of the angular momentum L must be conserved along a magnetic field line

$$E = E(\Psi) = \frac{\Omega_{\rm F} I}{2\pi} + \mu\eta\gamma, \tag{5}$$

$$L = L(\Psi) = \frac{I}{2\pi} + \mu\eta\varpi u_{\hat\varphi}, \tag{6}$$

where for cold plasma the enthalpy μ is constant. The fifth integral of motion is, in fact, the zero entropy per unit particle $s(\Psi) = 0$.

As a result, all other physical parameters are determined by algebraic relations, e.g.

$$\frac{I}{2\pi} = \frac{L - \Omega_{\rm F}\varpi^2 E}{1 - \Omega_{\rm F}^2\varpi^2 - M^2}, \tag{7}$$

$$\gamma = \frac{1}{\mu\eta}\frac{(E - \Omega_{\rm F}L) - M^2 E}{1 - \Omega_{\rm F}^2\varpi^2 - M^2}, \tag{8}$$

$$u_{\hat\varphi} = \frac{1}{\varpi\mu\eta}\frac{(E - \Omega_{\rm F}L)\Omega_{\rm F}\varpi^2 - LM^2}{1 - \Omega_{\rm F}^2\varpi^2 - M^2}, \tag{9}$$

where

$$M^2 = \frac{4\pi\eta^2\mu}{n}. \tag{10}$$

To determine the Mach number M, one should use an obvious relation $\gamma^2 - \boldsymbol{u}^2 = 1$ which gives

$$\frac{K}{\varpi^2 A^2} = \frac{1}{64\pi^4}\frac{M^4(\nabla\Psi)^2}{\varpi^2} + \mu^2\eta^2, \tag{11}$$

where

$$A = 1 - \Omega_{\rm F}^2\varpi^2 - M^2, \tag{12}$$

and

$$K = \varpi^2(E - \Omega_{\rm F}L)^2(1 - \Omega_{\rm F}^2\varpi^2 - 2M^2) + M^4(\varpi^2E^2 - L^2). \tag{13}$$

Equations (7)–(9) and (11) are the algebraic relations, which for the known stream function Ψ [and, hence, for the known poloidal field $\boldsymbol{B}_{\rm P}$ (2)] and for the given integrals of motion allow us to find all other characteristics of a flow (Weber & Davis 1967; Mestel 1968; Okamoto 1978; Kennel, Fujimura & Okamoto 1983; Camenzind 1986; Bogovalov 1990; Takahashi et al. 1990).

The MHD cold flow described by the algebraic relations (7)–(9) is characterized by the following singular surfaces.

(1) *The Alfvénic surface* defined from the condition of nulling the denominator A (12) in the relations (7)–(9),

$$A = 0. \tag{14}$$

It is necessary to stress that the algebraic relations (7)–(9) themselves contain no singularity, and the regularity conditions (zero nominators for zero denominators) define the position of the Alfvénic surface. At the same time, as shown below, the stream equation itself has a singularity on the Alfvén surface.

(2) *The fast magnetosonic surface* defined as a singularity in the gradient of M^2. Indeed, equation (11) may be expressed in an equivalent form $(\nabla\Psi)^2 = F(M^2, E, L, \eta, \Omega_{\rm F})$, where

$$F = \frac{64\pi^4}{M^4}\frac{K}{A^2} - \frac{64\pi^4}{M^4}\varpi^2\mu^2\eta^2. \tag{15}$$

Taking the gradient of both sides of equation (15), one can obtain the following expression for ∇M^2,

$$\nabla_a M^2 = \frac{N_a}{D} = -\frac{A}{(\nabla\Psi)^2 D}\nabla^b\Psi\nabla_a\nabla_b\Psi + \frac{A}{2}\frac{\nabla'_a F}{(\nabla\Psi)^2 D}, \tag{16}$$

where for a cold flow we have

$$D = \frac{A}{M^2} + \frac{1}{M^2}\frac{B_{\hat\varphi}^2}{B_{\rm P}^2}. \tag{17}$$

Here, the subscripts a, b run through the values r, θ only, the gradient ∇' acts on all the variables except M^2, and everywhere the symbol ∇ means a covariant differentiation in the flat space with metric g_{ik} (1). It is the regularity conditions

$$D = 0;$$
$$N_r = 0; \tag{18}$$
$$N_\theta = 0,$$

together with the regularity condition on the Alfvénic surface, that determine fully the structure of a flow (Heyvaerts 1996). It is obviously enough that the slow magnetosonic surface is absent for the cold flow considered here.

3 THE STREAM EQUATION

The general form of the stream equation in the Kerr metric was obtained by Nitta et al. (1991) and Beskin & Pariev (1993). Then for a cold flow $s = 0$ (and neglecting the gravity), we have

$$
\begin{aligned}
&A\left[\nabla_k\left(\frac{1}{\varpi^2}\nabla^k\Psi\right) + \frac{1}{\varpi^2(\nabla\Psi)^2}\frac{\nabla^a\Psi\nabla^b\Psi\nabla_a\nabla_b\Psi}{D}\right] \\
&\quad + \frac{1}{\varpi^2}\nabla_k' A\nabla^k\Psi - \frac{A}{\varpi^2(\nabla\Psi)^2}\frac{1}{2D}\nabla_k' F\nabla^k\Psi \\
&\quad + \Omega_F(\nabla\Psi)^2\frac{d\Omega_F}{d\Psi} \\
&\quad + \frac{64\pi^4}{\varpi^2}\frac{1}{2M^2}\frac{\partial}{\partial\Psi}\left(\frac{G}{A}\right) - 16\pi^3\mu n\frac{1}{\eta}\frac{d\eta}{d\Psi} = 0,
\end{aligned}
\tag{19}
$$

where

$$G = \varpi^2(E - \Omega_F L)^2 + M^2L^2 - M^2\varpi^2E^2, \tag{20}$$

and the derivative $\partial/\partial\Psi$ acts only on the integrals of motion while the other variables are considered as constants. One can see that the solution of the stream equation (19) is to fulfil the regularity condition on the Alfvénic surface $A = 0$

$$
\begin{aligned}
&\frac{1}{\varpi^2}\nabla_k' A\nabla^k\Psi + \Omega_F(\nabla\Psi)^2\frac{d\Omega_F}{d\Psi} \\
&\quad + \frac{64\pi^4}{\varpi^2}\frac{1}{2M^2}\frac{\partial}{\partial\Psi}\left(\frac{G}{A}\right) - 16\pi^3\mu n\frac{1}{\eta}\frac{d\eta}{d\Psi} = 0.
\end{aligned}
\tag{21}
$$

This relation is equivalent to the regularity condition on the light cylinder in the force–free pulsar equation (Michel 1973; Mestel & Wang 1979). In turn, the quantity M^2 is to be considered as a function of the gradient $(\nabla\Psi)^2$ and the four integrals of motion. Hence, one must determine M^2 as $M^2 = M^2\left[(\nabla\Psi)^2, E(\Psi), L(\Psi), \eta(\Psi), \Omega_F(\Psi)\right]$. The latter relation is an implicit form of equation (11).

It is well known that in the force-free limit $M^2 \to 0$, $\eta \to 0$, $(D^{-1} \to 0)$ for specially chosen integrals of motion $\Omega_F =$ constant,

$$
\begin{aligned}
&L_0(\Psi) = \frac{\Omega_F}{8\pi^2}(2\Psi - \Psi^2/\Psi_0); \\
&E_0(\Psi) = \Omega_F L_0(\Psi),
\end{aligned}
\tag{22}
$$

the stream equation has the (split) monopole solution (Michel 1973)

$$\Psi(r,\theta) = \Psi_0(1 - \cos\theta). \tag{23}$$

Here Ψ_0 is the whole flux of the magnetic field through the hemisphere. In accordance with (22), (23) one can rewrite (22) as $E_0 = E_A\sin^2\theta$, where

$$E_A = \frac{\Omega_F^2\Psi_0}{8\pi^2}.$$

The stream equation (19) contains only the magnetic flux $\Psi(r,\theta)$ and four integrals of motion $\Omega_F(\Psi), \eta(\Psi), E(\Psi)$ and $L(\Psi)$. On the other hand, for cold flow it has two singular surfaces (14) and (18). As a result, we have actually three critical conditions, i.e. $D = 0$ and $N_r = 0$ at the fast surface, and (21) at the Alfvénic surface. Together these three conditions determine three values, i.e. the disturbance of current $l(\theta)$, disturbance of stream function f (because it is the second boundary condition to the stream equation), and the position of the fast surface. As to the condition $N_\theta = 0$, it determines the density on the fast surface. The simplest versions of these critical conditions are well-known on the example of the spherically symmetric transonic flows (Bondi 1952; Parker 1958) when $N_\theta \equiv 0$, and two regularity conditions, $D = 0$ and $N_r = 0$, together determine the radius of the sound surface and the sound velocity at this surface. As a result, this problem requires *four* boundary conditions on the star surface $r = R$ (Heyvaerts 1996), say, the angular velocity Ω_F, Lorentz-factor γ_{in}, plasma density in the laboratory frame n_{in}, and the magnetic flux on the star surface $\Psi(R,\theta)$. For simplicity, we consider the case

$$\Omega_F(R,\theta) = \Omega_F = \text{constant}, \tag{24}$$

$$\gamma(R,\theta) = \gamma_{in} = \text{constant}, \tag{25}$$

$$n(R,\theta) = n_{in} = \text{constant}, \tag{26}$$

and put $\Psi(R,\theta) = \Psi_0(1 - \cos\theta)$, i.e. the same as in the monopole solution. Introducing small disturbances to the Michel force-free integrals (22) E_0 and L_0 as

$$
\begin{aligned}
&E(\Psi) = E_0(\Psi) + b(\Psi); \\
&L(\Psi) = L_0(\Psi) + l(\Psi);
\end{aligned}
\tag{27}
$$

one can check that for a star radius $R \ll R_L$ the algebraic relations (7)–(9) give $\eta = n_{in}/B_p$. One can also introduce for convenience a new quantity $e = E - \Omega_F L$. In our case

$$e = E - \Omega_F L = b - \Omega_F l = \frac{B_p}{4\pi}M^2(R) = \gamma_{in}\mu\eta, \tag{28}$$

the integrals of motion η, Ω_F, and e being constant in the whole space. Here $M^2(R)$ is the Mach number on the star surface, $n(r,\theta) = n_{in}/\gamma_{in}$, and $B_p = \Psi_0/2\pi R^2$ is the radial magnetic field on the star surface. According to (28), we can define $b(\Psi) = e + \Omega_F l(\Psi)$. As to $l(\Psi)$ [which for small disturbances of the monopole magnetic field can be considered as a function θ only, i.e. $l = l(\theta)$], it must be determined from the regularity condition at the critical surfaces. As a result, the stream equation (19) has a simpler form

$$
\begin{aligned}
&A\left[\nabla_k\left(\frac{1}{\varpi^2}\nabla^k\Psi\right) + \frac{\nabla^a\Psi\nabla^b\Psi\nabla_a\nabla_b\Psi}{D\varpi^2(\nabla\Psi)^2}\right] \\
&\quad + \frac{\nabla_k' A\nabla^k\Psi}{\varpi^2} - \frac{A}{\varpi^2(\nabla\Psi)^2}\frac{1}{2D}\nabla_k' F\nabla^k\Psi \\
&\quad + \frac{64\pi^4}{\varpi^2 A}\left(L\frac{dL}{d\Psi} - \varpi^2 E\frac{dE}{d\Psi}\right) = 0.
\end{aligned}
\tag{29}
$$

Let us consider the case

$$\sigma = \frac{E_A}{\mu\eta} \gg 1, \quad \gamma_{in} < \sigma^{1/3} \tag{30}$$

(σ is just the Michel magnetization parameter), so that $e/E \ll 1$ as well. Then one can seek the solution of the stream equation (29) in the form

$$\Psi(r,\theta) = \Psi_0[1 - \cos\theta + \varepsilon f(r,\theta)], \tag{31}$$

where $\varepsilon \sim \sigma^{-1} \ll 1$ is a small parameter. Substituting the relation (31) into (29), we find

$$
\begin{aligned}
&\varepsilon A\frac{\partial^2 f}{\partial r^2} + \varepsilon A\frac{D+1}{Dr^2}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\theta}\right) \\
&\quad - 2\varepsilon\Omega_F^2 r\sin^2\theta\frac{\partial f}{\partial r} - 2\varepsilon\Omega_F^2\sin\theta\cos\theta\frac{\partial f}{\partial\theta}
\end{aligned}
$$

$$+ 2\varepsilon\Omega_F^2(3\cos^2\theta - 1)f + \frac{8\pi^2\Omega_F}{\Psi_0}\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin^2\theta l)$$
$$- 2\varepsilon\frac{A}{Dr^2}\frac{\cos\theta}{\sin\theta}\frac{\partial f}{\partial\theta} + 2\varepsilon\frac{A}{Dr^2}\frac{(1+\cos^2\theta)}{\sin^2\theta}f$$
$$- \frac{64\pi^4 A}{\Omega_F^2 D\Psi_0^2}\frac{\cos\theta}{\sin^2\theta}\frac{e^2}{M^4}$$
$$+ 2A\cos\theta\frac{1-M^2}{D\Omega_F^2 r^4\sin^2\theta} + \frac{16\pi^2\cos\theta A}{\Omega_F^2 r^2\sin^2\theta}\frac{e}{D\Psi_0} \quad (32)$$
$$- \frac{8\pi^2\sin\theta A}{D\Omega_F r^2\Psi_0}\frac{d}{d\theta}\left(\frac{l}{\sin^2\theta}\right) - \frac{1}{Dr^2}\frac{A\cos\theta}{\Omega_F^2 r^2\sin^2\theta}$$
$$+ 2\Omega_F^2\sin^2\theta\cos\theta\frac{M^2}{A} - 16\pi^2\Omega_F^2 r^2\sin^2\theta\cos\theta\frac{e}{A\Psi_0} = 0,$$

the Mach number $M^2[(\nabla\Psi)^2, E, L, \Omega_F, \eta]$ being the physical root of the algebraic equation (11).

4 FAST MAGNETOSONIC SURFACE

To determine the position of the fast magnetosonic surface $r = r_F$ (and to determine the physical root M^2) it is necessary to solve the fourth-order algebraic equation (11). Let us define a new quantity

$$g = \frac{M^2}{\Omega_F^2\varpi^2}. \quad (33)$$

According to (8) and (33), $g \ll 1$ for $\sigma \gg 1$, and

$$\gamma = \frac{E}{\mu\eta}g. \quad (34)$$

Hence, g is actually the ratio of the particle energy flux to total flux of energy. For $g \ll 1$, particles play no role in the total energy losses.

As a result one can rewrite the algebraic equation (11) in the form (cf. Tomimatsu 1994)

$$2g^3 - \left(\xi + \frac{1}{\Omega_F^2 r^2\sin^2\theta}\right)g^2 + \frac{\mu^2\eta^2}{E^2} + \frac{e^2}{\Omega_F^2 r^2\sin^2\theta E^2} = 0, \quad (35)$$

where we omit the term g^4 (which gives the unphysical root $g < 0$), and introduce

$$\xi = 1 - \frac{\Omega_F^4 r^2\sin^2\theta(\nabla\Psi)^2}{64\pi^4 E^2}. \quad (36)$$

One can check that $\xi = 0$ for the force-free Michel solution (22)–(23), and $\xi \ll 1$ for $\sigma^{-1} \ll 1$. As a result, we find

$$\xi(r,\theta) = 2\frac{e + \Omega_F l}{E} - \frac{2\varepsilon}{\sin\theta}\frac{\partial f}{\partial\theta} + 4\varepsilon\frac{\cos\theta}{\sin^2\theta}f. \quad (37)$$

It is the dependence of ξ on εf that allows us to analyse the problem self-consistently.

The fast magnetosonic surface corresponds to the intersection of two roots of equation (35) at one point (see Fig. 1). On the other hand, equation (35) has two real positive roots if $Q \le 0$, where Q is the discriminant of the third-order algebraic equation (35). For r near r_F, where the last term in (35) can be neglected, we have

$$Q = \frac{1}{16}\frac{\mu^4\eta^4}{E^4} - \frac{1}{16\times 27}\frac{\mu^2\eta^2}{E^2}\left(\xi + \frac{1}{\Omega_F^2 r^2\sin^2\theta}\right)^3, \quad (38)$$

the regularity conditions at the fast magnetosonic surface $r = r_F$ being

$$Q = 0,$$
$$\partial Q/\partial r = 0, \quad (39)$$
$$\partial Q/\partial\theta = 0.$$

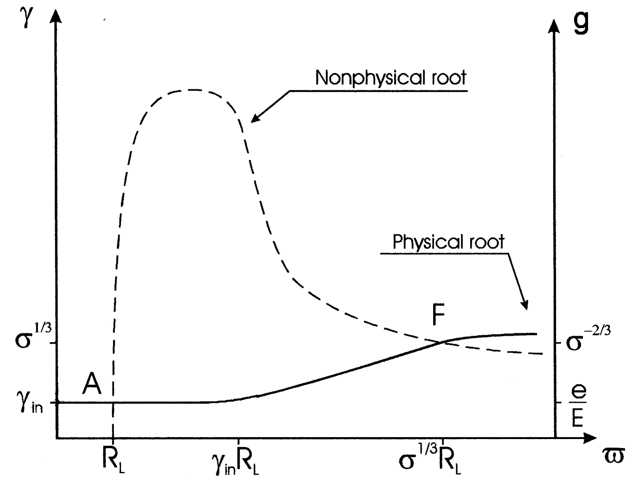

Figure 1. The behaviour of physical and non-physical roots of algebraic equation (35).

One can check that equations (39) just correspond to the regularity conditions $D = 0$, $N_r = 0$, and $N_\theta = 0$ (18). As a result, we can rewrite the condition $Q = 0$ as

$$\xi(r_F,\theta) + \frac{1}{\Omega_F^2 r_F^2\sin^2\theta} = 3\left(\frac{\mu\eta}{E}\right)^{2/3}, \quad (40)$$

and $\partial Q/\partial r = 0$ as

$$r_F\left(\frac{\partial\xi}{\partial r}\right)_{r=r_F} - \frac{2}{\Omega_F^2 r_F^2\sin^2\theta} = 0. \quad (41)$$

As for the third condition $N_\theta = 0$, as was already stressed, it determines the density n on the fast surface. As a result, using the approximation $r(d\xi/dr) \approx \xi$, we see that

$$\xi(r_F) \approx \sigma^{-2/3}; \quad \Omega_F r_F \approx \sigma^{1/3}. \quad (42)$$

It means that the disturbance ξ remains small on any way up to the fast surface r_F, where

$$r_F(\theta) \approx R_L\sigma^{1/3}\sin^{-1/3}\theta, \quad (43)$$

when $\theta > \sigma^{-1/2}$, and

$$r_F \approx R_L(\sigma/\gamma_{in})^{1/2}, \quad (44)$$

near the axis. The positions of Alfvénic and fast magnetosonic surfaces are shown in Fig. 2. On the other hand, for $\gamma_{in} > \sigma^{1/3}$ the fast magnetosonic surface is a sphere with the radius (44), and there is no particle acceleration for $r < r_F$ (cf. Bogovalov 1996).

Moreover, as the root $g(r_F)$ at $r = r_F$ does not depend on the second term in (35), we have exactly

$$g(r_F,\theta) = \left(\frac{\mu\eta}{E}\right)^{2/3}, \quad (45)$$

and, hence,

$$\gamma(r_F,\theta) = \left(\frac{E}{\mu\eta}\right)^{1/3} = \sigma^{1/3}\sin^{2/3}\theta, \quad (46)$$

which just corresponds to Michel's (1969) result. The only difference is that this energy is achieved at a finite distance r_F (43). As we see from (41), it takes place because we took into consideration the dependence of ξ on the field disturbance εf. Indeed, according to (28) and (42), it is the disturbance of the magnetic surfaces εf that plays the main role in (37) at the fast magnetosonic surface.

Our conclusions are also in full agreement with results obtained by Begelman & Li (1994). In their consideration of the flux tubes geometry diverging from a precisely radial flow, it was found that

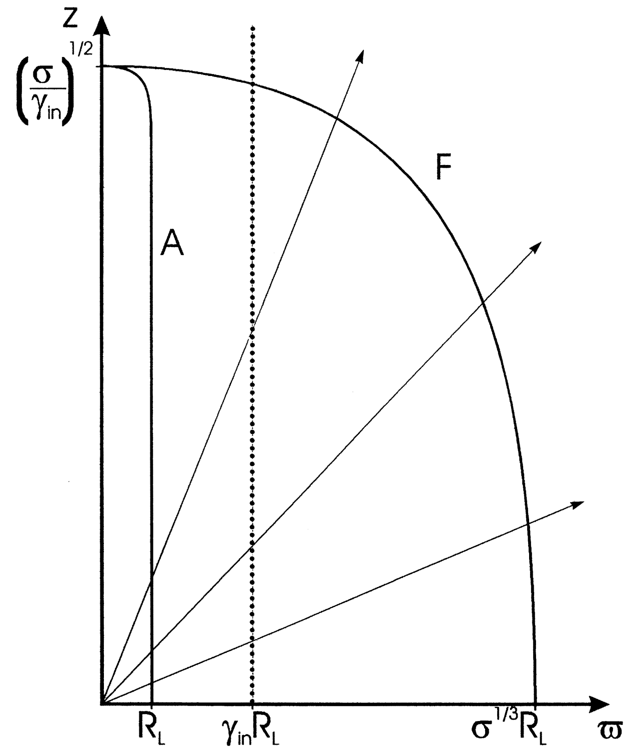

Figure 2. Spatial structure of Alfvénic and fast magnetosonic surfaces for $\gamma_{\rm in} \ll \sigma^{1/3}$.

the fast magnetosonic surface moves inward from infinity and the kinetic energy at the fast point is close to the cube root of the total energy, which confirms the result (46).

5 INNER REGION

In the region $r\sin\theta \ll r_{\rm F}$ we can essentially simplify our problem. It can be found, for the physical root of equation (35) (i.e. the root corresponding to a subsonic flow $D > 0$ for $r < r_{\rm F}$ and a supersonic flow $D < 0$ for $r > r_{\rm F}$, see Fig. 1) at distances much smaller than $r_{\rm F}$, that

$$g(r,\theta) = \frac{1}{\sigma \sin^2\theta}\sqrt{\gamma_{\rm in}^2 + r^2\Omega_{\rm F}^2\sin^2\theta}. \quad (47)$$

In accordance with (34), we have for particle energy

$$\gamma(r,\theta) = \sqrt{\gamma_{\rm in}^2 + r^2\Omega_{\rm F}^2\sin^2\theta}. \quad (48)$$

In the inner region $r\sin\theta \ll \gamma_{\rm in}$ we have then $\gamma(r,\theta) = \gamma_{\rm in}$ and, hence, the particle energy here remains the same as on the star surface.

In the intermediate region $\gamma_{\rm in} < r\sin\theta$, $r < r_{\rm F}$ we have for the physical root $D > 0$

$$g = \frac{\Omega_{\rm F} r}{\sigma\sin\theta}, \quad (49)$$

and for the non-physical root, $D < 0$,

$$g_n = \frac{1}{2\Omega_{\rm F}^2 r^2\sin^2\theta}. \quad (50)$$

Accordingly, here

$$\gamma(r,\theta) = \Omega_{\rm F} r\sin\theta, \quad (51)$$

that is, the energy grows linearly with the distance in this range and $D = M^{-2}$ while

$$M^2(r,\theta) = \frac{\Omega_{\rm F}^3 r^3}{\sigma}\sin\theta.$$

As we can see, in this region all the physical characteristics of a flow *do not depend* on the field disturbance $\varepsilon f(r,\theta)$.

It is also quite straightforward that at distances $r \ll r_{\rm F}$ we have $D = M^{-2} = 1/(g\Omega_{\rm F}^2 r^2\sin^2\theta) \gg 1$. Substituting this expression together with (47) in equation (32), one obtains that in the limit $\Omega_{\rm F} r\sin\theta \gg 1$ the non-uniform part N of (32) takes the form

$$N \approx \frac{\Omega_{\rm F}^2}{\sin\theta}\frac{\rm d}{{\rm d}\theta}\left(\frac{l\Omega_{\rm F}}{E_A}\sin^2\theta\right) - \frac{2\Omega_{\rm F}^4 r^2\sin^2\theta\cos\theta}{\sigma(1-\Omega_{\rm F}^2 r^2\sin^2\theta)}\left(\sqrt{\gamma_{\rm in}^2 + r^2\Omega_{\rm F}^2\sin^2\theta} - \gamma_{\rm in}\right) - \frac{\Omega_{\rm F}^4 r^2\sin^2\theta\cos\theta}{\sigma\sqrt{\gamma_{\rm in}^2 + \Omega_{\rm F}^2 r^2\sin^2\theta}}. \quad (52)$$

The first term in (52) is of the order of l/L_0 everywhere, while the value of the others grows as $r^2\Omega_{\rm F}^2/(\sigma\gamma_{\rm in})$ at the distances $r\sin\theta \ll \gamma_{\rm in}R_{\rm L}$, and grows as $r\Omega_{\rm F}/\sigma$ when $\gamma_{\rm in}R_{\rm L} \ll r\sin\theta \ll r_{\rm F}$. It will be shown further that for all $r\sin\theta > R_{\rm L}$, the latter two terms are higher in magnitude than the former one. Owing to the linearity of equation (32), we can seek its particular solution as the sum of solutions found for each term in (52).

If we take only the first term into account, the stream equation coincides with the force-free equation

$$\varepsilon(1 - x^2\sin^2\theta)\frac{\partial^2 f}{\partial x^2} + \varepsilon(1 - x^2\sin^2\theta)\frac{\sin\theta}{x^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\theta}\right) - 2\varepsilon x\sin^2\theta\frac{\partial f}{\partial x} - 2\varepsilon\sin\theta\cos\theta\frac{\partial f}{\partial\theta} + 2\varepsilon(3\cos^2\theta - 1)f + I_A\frac{1}{\sin\theta}\frac{1}{\sin\theta}\frac{\rm d}{{\rm d}\theta}(l\sin^2\theta) = 0. \quad (53)$$

Here $x = \Omega_{\rm F} r$ and the disturbance of current is written as $I_M + \delta I(\theta) = I_A\sin^2\theta + l(\theta)$.

Equation (53) has a singularity on the Alfvénic surface, coinciding in the force-free limit with the light cylinder $x\sin\theta = 1$. In view of the simple power-like dependence on the variable x, one can seek the solution of this equation as a series,

$$f(x,\theta) = \sum_{m=1}^{\infty} Z_m(x,\theta), \quad (54)$$

where

$$Z_m(x,\theta) = \sum_{n=1}^{m} a_{m,n}(\theta)x^{2n}. \quad (55)$$

Now it is readily seen that for each set of harmonics n, the coefficient corresponding to x^{2n} will contain only the function $a_{n,n}(\theta)$ and its derivatives, the one corresponding to x^{2n-2} will contain functions $a_{n,n}(\theta)$ and $a_{n,n-1}(\theta)$ only, the coefficient corresponding to x^{2n-4} will contain the functions $a_{n,n-1}(\theta)$ and $a_{n,n-2}(\theta)$, and so on. As for the coefficient of a free term, it is an equation giving us the eigenfunctions $l_m(\theta)$, and we must use them in the expansion of a current disturbance function $l(\theta)$.

Unfortunately in the general case of an arbitrary $l(\theta)$ both the coefficients $a_{m,n}$ and the eigenfunctions $l_m(\theta)$ are too complicated. However, for the specially chosen disturbance $l(\theta) = (E_A/\Omega_F)h\sin^2\theta$, $h = \text{constant} \ll 1$ (and for $R \ll R_L$), we have the following *exact* solution of equation (53) which has no singularity on the Alfvénic surface:

$$\varepsilon f(r,\theta) = h\Omega_F^2 r^2 \sin^2\theta\cos\theta. \tag{56}$$

Of course, one can use (56) if $\varepsilon f \ll 1$ only. Because of the simplicity of this solution we shall henceforth consider only this special form of $l(\theta)$.

The solution (56) demonstrates that in the force-free limit the current $l(\theta)$ on the star surface is a free parameter, and the solution in the region $r \ll r_F$ can be constructed for arbitrary values of $h \ll 1$. The disturbance of current $l(\theta)$, as was already stressed, can be determined only within the full MHD equation together with the regularity conditions on the fast surface. Unfortunately, the exact analytical determination of $l(\theta)$ is impossible. On the other hand, comparing the solution (56) with the regularity condition (42) on the fast surface, one can restrict the disturbance of current

$$l/L_0 \sim \sigma^{-4/3}. \tag{57}$$

As we see, the disturbance of current must be very small to have no singularity on the fast magnetosonic surface. It confirms our choice of monopole solution (22)–(23) with the Michel current $I = 2\pi L_0(\Psi)$ as a zero approximation.

Evaluation (57) explains as well what was said earlier about the magnitude of the terms in (52). Indeed, we see now that the ratio of the first term in (52) and the second and third is of the order of $\gamma_{\rm in}/(\sigma^{1/3}x^2)$ in the region $1 \ll x \ll \gamma_{\rm in}$, and $\sim 1/(\sigma^{1/3}x)$ in the region$\gamma_{\rm in} \ll x \ll r_F\Omega_F \sim \sigma^{1/3}$, which is much less than unity in both ranges. However, we cannot neglect the first term of (52) because the solution (56) corresponding to it is not negligible in the vicinity of a fast magnetosonic surface because of the regularity condition.

The expression (56) also shows us that for $I < I_M$ (and hence $h < 0$), the field lines are slightly deviated to form a disc ($f < 0$), light surface $|\boldsymbol{E}| = |\boldsymbol{B}|$ being located at a finite distance $\varpi_C = |2h|^{-1/4}R_L$, where the disturbance of the monopole field is still small. Accordingly, for $I > I_M$ ($h > 0$), the magnetic field lines are stretched along the rotational axis ($f > 0$), and in this case light surface is achieved only at infinity.

Now let us turn to the equation corresponding to the second and third terms in (52). Because the latter shows complex behaviour in the region $x \sim 1$, we shall consider this equation only at distances $x \gg 1$ and neglect all the terms of the order of $1/(x^2\sin^2\theta)$ in it. Then equation (32) looks like

$$\varepsilon\frac{\partial^2 f}{\partial x^2}x^2 + 2\frac{\partial f}{\partial x}x + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\theta}\right) + 2\frac{\cos\theta}{\sin\theta}\frac{\partial f}{\partial\theta} - 2\frac{(3\cos^2\theta - 1)}{\sin^2\theta}f + 2\frac{\cos\theta}{\sigma\sin^2\theta}\left(\sqrt{\gamma_{\rm in}^2 + x^2\sin^2\theta} - \gamma_{\rm in}\right) + \frac{x^2\sin^2\theta\cos\theta}{\sigma\sqrt{\gamma_{\rm in}^2 + x^2\sin^2\theta}} = 0. \tag{58}$$

We seek the solution of (58) as a series

$$\varepsilon f(x,\theta) = \sum_{k=1}^{\infty} P_k^2(\cos\theta)T_k(x), \tag{59}$$

and we find that there exists an explicit expression of the functions $T_k(x)$ as a series of powers of x with coefficients depending upon x, but it is too complex to be written here. Nevertheless, from this exact expression one can obtain some asymptotic describing the behaviour of εf far from the point $\gamma_{\rm in} r_L$, namely

$$\varepsilon f = \alpha_3 P_3^2(\cos\theta)\frac{x^2\ln x}{\sigma\gamma_{\rm in}} + \sum_{k=5}^{\infty}\alpha_k P_k^2(\cos\theta)\frac{x^2}{\sigma\gamma_{\rm in}}, \quad R_L \ll r \ll \gamma_{\rm in}R_L \tag{60}$$

$$\varepsilon f = \sum_{k=3}^{\infty}\beta_k P_k^2(\cos\theta)\frac{x}{\sigma}, \qquad \gamma_{\rm in}R_L \ll r \ll r_F, \tag{61}$$

where α_k and β_k are some constants. This result could also be obtained as an exact solution of equation (58), where we substitute for the last terms of (58) their asymptotic expressions in the ranges indicated in (60) and (61). The first way of dealing with this problem, however, enables us to obtain an *exact* solution at distances $\sim\gamma_{\rm in}$.

Solution (61) satisfies directly the regularity conditions (42) because all β_k are approximately equal to unity. Finally, one can see from (56), (60) and (61) that it is the field disturbance εf, rather than the change of integrals of motion e and $l(\theta)$, that plays the main role in the regularity conditions (42)–(44) at the fast magnetosonic surface. It gives us $\varepsilon f(r_F) \sim \sigma^{-2/3}$. Hence, the disturbance of the magnetic surfaces remains small on any way up to the fast surface $r = r_F$. On the other hand, we can conclude that it is impossible to analyse the regularity conditions in a given magnetic field $\varepsilon f = 0$. Indeed, for $\varepsilon f = 0$ we have $(\partial\xi/\partial r) = 0$, and hence $r_F \to \infty$ (cf. Michel 1969). In other words, the finite radius of the fast magnetosonic surface results from the self-consistency of our analysis.

6 FAR ASYMPTOTIC REGION

In the far asymptotic region $r \gg r_F$, the physical root $D < 0$ is

$$g = \frac{\xi}{2}\left(1 - 4\frac{\mu^2\eta^2}{\xi^3E^2}\right). \tag{62}$$

One can check that (62) corresponds to the drift velocity

$$U_{\rm dr} = \frac{|\boldsymbol{E}|}{|\boldsymbol{B}|} \approx \frac{|\boldsymbol{E}|}{B_\varphi} \tag{63}$$

with the energy $\gamma = (1 - U_{\rm dr}^2)^{-1/2}$ just equal to (34). Indeed, using the definition (2), (3) and relations (7)–(9), equation (11) in the limit $r \gg r_F$ can be rewritten in a form

$$B_\varphi^2 = |\boldsymbol{E}|^2 + \frac{(4\pi)^2\Omega_F^2\varpi^2\mu^2\eta^2}{M^4}. \tag{64}$$

As a result, we have

$$1 - U_{\rm dr}^2 \approx \frac{B_\varphi^2 - |\boldsymbol{E}|^2}{B_\varphi^2} \approx \frac{1}{g^2}\left(\frac{\mu\eta}{E}\right)^2, \tag{65}$$

which corresponds to particle energy (34). On the other hand, in this region the stream equation can be rewritten in the simple form

$$\varepsilon r^2\frac{\partial^2 f}{\partial r^2} + 2\varepsilon r\frac{\partial f}{\partial r} - \sin\theta\frac{D+1}{D}\frac{\partial g}{\partial\theta} = 0, \tag{66}$$

where $g \approx \xi/2$, and $D + 1 = 8\mu^2\eta^2/(E^2\xi^3) \ll 1$. For a pure radial flow $\partial f/\partial r = 0$ we have

$$\frac{{\rm d}g}{{\rm d}\theta} = 0, \tag{67}$$

in agreement with previous consideration (Heyvaerts & Norman 1989; Nitta 1995). However, as we see, in the full stream equation (66) the derivative $\partial g/\partial\theta$ has a small coefficient $D+1 \ll 1$ (which is actually a non-linear term). Hence, this term actually plays no role in equation (66). As a result, omitting the last term in (66), we have at large distances

$$\varepsilon f(r,\theta) = \sigma^{-2/3} a(\theta), \qquad (68)$$

where $a(\theta) \sim 1$ [formally taking the last term in (66) into consideration one can obtain $\varepsilon f \propto (\ln r)^{1/3}$ (Tomimatsu 1994), i.e. very weak collimation]. We can see that the disturbance of magnetic field lines does not actually exceed $\sigma^{-2/3}$. According to (34), the particle energy at large distances changes very slowly as well. Hence, outside the fast magnetosonic surface the particle acceleration and the magnetic field collimation become ineffective, and the flow remains magnetically dominated up to infinity.

7 CONCLUSIONS

Thus, studying analytically a model of a stationary axisymmetric outflow from a rotating sphere with a (split) monopole magnetic field, we have found that in our approach the fast magnetosonic surface is located at a finite distance $\sim\sigma^{1/3}R_{\rm L}$. In the inner region $\gamma_{\rm in}$ it is shown that the particle energy retains its initial value $r < \gamma_{\rm in}R_{\rm L}$ and then grows linearly with distance within the range $\gamma_{\rm in}R_{\rm L} < r < r_{\rm F}$. We have also obtained that on the fast magnetosonic surface this energy is just equal to the Michel value $\gamma \sim \sigma^{1/3}$ and that outside this surface the acceleration of particles becomes ineffective – the energy here does not actually exceed $\sigma^{1/3}$.

Solving the linearized version of the stream equation in the vicinity of Michel force-free analytical solution, we found the first-order correction to this solution in the analytical form. We showed that it behaves mainly as r^2 in the range $R_{\rm L} < r < \gamma_{\rm in}R_{\rm L}$ but then at a distances $\gamma_{\rm in}R_{\rm L} < r < r_{\rm F}$ the linear branch of the solution arises giving a contribution of the same order of magnitude. Far outside the fast magnetosonic surface, the disturbance of magnetic field lines grows with the distance from the rotational axis but so slowly that can be treated as a constant. Finally, it was shown that the magnitude of the disturbance does not practically exceed $\sigma^{-2/3}$, which justifies our use of the linearized theory.

Negligible small divergence of magnetic flux tubes from purely radial geometry can explain an ineffectiveness of particle acceleration in our model in view of the results obtained by Begelman & Li (1994). They pointed out that sufficient acceleration occurs through the so-called 'magnetic nozzle' effect, which takes place when the quantity B_pR^2 decreases significantly from its value at the fast point to its asymptotic value at infinity once the fast point is passed. In our consideration, however, it turned out that the disturbance of a flux function is small in the whole space and the quantity B_pR^2 is practically the same both at the fast point and at infinity, which implies an ineffectiveness of 'magnetic nozzle' effect and inefficiency of particle acceleration far from the fast magnetosonic surface.

In other words, we have constructed an example in which the regularity conditions limit the longitudinal electric current, so the collimation of the magnetic surfaces is ineffective. In our opinion, the absence of any sufficient magnetic field collimation and the ineffectiveness of the particle acceleration imply that one can discuss the asymptotic behaviour of a flow only including by taking the surroundings into consideration.

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REFERENCES

Ardavan H., 1976, ApJ, 204, 889
Ardavan H., 1979, MNRAS, 189, 397
Begelman M. C., Blandford R. D., Rees M. J., 1984, Rev. Mod. Phys., 56, 255
Begelman M. C., Li Z.-Y., 1994, ApJ, 426, 269
Beskin V. S., Gureich A. V., Istomin Ya. N., 1993, Physics of the Pulsar Magnetospere. Cambridge Univ. Press, Cambridge
Beskin V. S., Malyshkin L. M., 1996, Astron. Lett., 22, 475
Beskin V. S., Pariev V. I., 1993, Phys.-Usp., 36, 529
Beskin V. S., Pidoprigora Yu. M., 1995, J. Exp. Theor. Phys., 80, 575
Blandford R.D, 1992, in Blandford R., Netzer H., Woltjer L., eds, Active Galactic Nuclei. Springer, Berlin p.1
Blandford R. D., Payne D. G., 1982, MNRAS, 199, 883
Blandford R. D., Znajek R. L., 1977, MNRAS, 179, 433
Bogovalov S. V., 1990, Sov. Astron. Lett., 16, 362
Bogovalov S. V., 1992, Sov. Astron. Lett., 18, 337
Bogovalov S. V., 1996, in Dremin I. M., Semikhatov A. M., eds, Proc. Second International Sakharov Conference on Physics. World Scientific, Singapore p. 150
Bondi H., 1952, MNRAS, 112, 195
Camenzind M., 1986, A&A, 162, 32
Camenzind M., 1987, A&A, 184, 341
Fendt C., Camenzind M., Appl S., 1995, A&A, 300, 791
Goldreich P., Julian W. H., 1969, ApJ, 157, 869
Hester J. J. et al., 1995, ApJ, 448, 240
Heyvaerts J., 1996, in Chiuderi C., Einaudi G., eds, Plasma Astrophysics. Springer, Berlin p. 31
Heyvaerts J., Norman J., 1989, ApJ, 347, 1055
Kennel C. F., Coroniti F. V., 1984, ApJ, 283, 694
Kennel C. F., Fujimura F. S., Okamoto I., 1983, Geophys. Astrophys. Fluid Dyn., 26, 147
Lada C. J., 1985, ARA&A, 23, 267
Li Zh.–Yu., Chiueh T., Begelman M. C., 1992, ApJ, 394, 459
Lovelace R. V. E., Mehanian C., Mobarry C. M., Sulkanen M. E., 1986, ApJS., 62, 1
Low B. C., Tsinganos K., 1986, ApJ, 302, 163
Markwardt C. B., Ögelman H., 1995, Nat, 375, 40
Mestel L., 1968, MNRAS, 138, 359
Mestel L., Wang Y., 1979, MNRAS, 188, 799
Michalas D., 1978, Stellar Atmospheres. Freeman Press, San Francisco
Michel F. C., 1969, ApJ, 158, 727
Michel F. C., 1973, ApJ, 180, L133
Michel F. C., 1991, Theory of Neutron Star Magnetosphere. Univ. Chicago Press, Chicago
Nitta S., 1995, MNRAS, 276, 825
Nitta S., Takahashi M., Tomimatsu A., 1991, Phys. Rev. D, 44, 2295
Okamoto I., 1975, MNRAS, 173, 357
Okamoto I., 1978, MNRAS, 185, 69
Pariev V. N., 1996, MNRAS, 283, 1264
Parker E. N., 1958, ApJ, 128, 664
Pelletier G., Pudritz R. E., 1992, ApJ, 394, 117
Rees M. J., Gunn J. E., 1974, MNRAS, 167, 1
Sakurai T., 1990, Computer Phys. Rep., 12, 247
Shapiro S. L., Teukolsky S. A., 1983, Black Holes, White Dwarfs and Neutron Stars. Wiley–Interscience Publication, New York
Soloviev L. S., 1967, in Leontovich M. A., ed., Reviews of Plasma Physics. Consultants Bureau, New York, p. 284
Takahashi M., Nitta S., Tatematsu Ya., Tomimatsu A., 1990, ApJ, 363, 206

Thorne K. S., Price R. N., Macdonald D. A., 1986, Black holes: The Membrane Paradigm. Yale Univ. Press, New Haven, CT
Tomimatsu A., 1994, PASJ, 46, 123
Tsinganos K., Sauty C., 1992, A&A, 255, 405
Weber E. J., Davis L., Jr, 1967, ApJ, 148, 217

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