

Ladislav Nebeský

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# ON THE MINIMUM NUMBER OF VERTICES AND EDGES IN A GRAPH WITH A GIVEN NUMBER OF SPANNING TREES

LADISLAV NEBESKÝ, Praha

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By a graph we shall mean a finite connected undirected graph without loops and multiple edges (for notions and results of graph theory see, for example, [1] or [2]). If  $p, q$  and  $r$  are integers such that  $1 \leq p \leq q \leq r$  and  $2 \leq q$  then by  $D(p, q, r)$  we shall denote the graph with cyclomatic number 2 and with no separating vertex and such that its two vertices of degree 3 are connected to each other by arcs ([2]) of length  $p, q$  and  $r$ ; the graph  $D(p, q, r)$  has of course  $p + q + r - 1$  vertices,  $p + q + r$  edges and  $pq + qr + pr$  spanning trees.

In the following, by  $x$  we shall denote a positive integer other than 2. By  $\alpha(x)$  we denote the smallest number  $y_1$  such that there is a graph having  $y_1$  vertices and  $x$  spanning trees; by  $\beta(x)$  we denote the smallest number  $y_2$  such that there is a graph having  $y_2$  edges and  $x$  spanning trees. Obviously  $\alpha(x) \leq \beta(x) \leq x$ , for any  $x \geq 3$ . The function  $\alpha$  has been studied by J. SEDLÁČEK [3], who also gave an impulse to the rise of the present paper.

The very simple generalization of one of the procedures used in [3] for the estimate of the function  $\alpha$  leads to the following estimate of the function  $\beta$  which is given by graphs with at least one separating vertex: if  $x_1$  and  $x_2$  are integers and  $x_1, x_2 \geq 3$ , then

$$(1) \quad \beta(x_1 x_2) \leq \beta(x_1) + \beta(x_2).$$

By making use of the graph  $D(1, 2, (x - 2)/3)$  and a graph with no separating edge and with two circuits of length 3 and  $x/3$ , J. Sedláček [3] found an upper estimate of the function  $\alpha$  for almost all  $x \equiv 2, 3 \pmod{3}$ . By using the same graphs it is quite readily possible to find an estimate of the function  $\beta$  for the same values of the argument:

$$(2) \quad \text{if } x \equiv 2 \pmod{3}, \quad x \geq 8, \quad \text{then } \beta(x) \leq (x + 7)/3;$$

$$(3) \quad \text{if } x \equiv 3 \pmod{3}, \quad x \geq 9, \quad \text{then } \beta(x) \leq (x + 9)/3.$$

Estimate (3) of course also follows from estimate (1). Upper estimates of the func-

tion  $\beta$  (and hence also the function  $\alpha$ ) for almost all  $x \equiv 1 \pmod{3}$  are given by the following lemma.

**Lemma.** *It holds that:*

- (4) if  $x \equiv 1 \pmod{30}$ ,  $x \geq 91$ , then  $\beta(x) \leq (x + 269)/30$ ;
- (5) if  $x \equiv 16 \pmod{30}$ ,  $x \geq 106$ , then  $\beta(x) \leq (x + 254)/30$ ;
- (6) if  $x \equiv 4 \pmod{30}$ ,  $x \geq 64$ , then  $\beta(x) \leq (x + 206)/30$ ;
- (7) if  $x \equiv 19 \pmod{30}$ ,  $x \geq 79$ , then  $\beta(x) \leq (x + 221)/30$ ;
- (8) if  $x \equiv 7 \pmod{15}$ ,  $x \geq 37$ , then  $\beta(x) \leq (x + 98)/15$ ;
- (9) if  $x \equiv 10 \pmod{15}$ ,  $x \geq 40$ , then  $\beta(x) \leq (x + 110)/15$ ;
- (10) if  $x \equiv 13 \pmod{15}$ ,  $x \geq 43$ , then  $\beta(x) \leq (x + 92)/15$ .

**Proof.** By  $G_1$  we denote the graph with 10 vertices  $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, b_5, c_0$  and 11 edges  $c_0a_1, a_1a_2, a_2a_3, a_3a_4, a_4c_0, c_0b_1, b_1b_2, b_2b_3, b_3b_4, b_4b_5, b_5c_0$ ;  $G_1$  obviously has 30 spanning trees. By  $G_2$  we denote the graph with 6 vertices  $a_1, a_2, a_3, b_1, b_2, b_3$  and with 8 edges  $a_1a_2, a_2a_3, a_3a_1, b_1b_2, b_2b_3, b_3b_1, a_1b_1, a_3b_3$ ;  $G_2$  obviously has 30 spanning trees. By  $G_3$  we denote the graph with 7 vertices  $a_1, a_2, b_1, b_2, b_3, b_4, c_0$  and 8 edges  $c_0a_1, a_1a_2, a_2c_0, c_0b_1, b_1b_2, b_2b_3, b_3b_4, b_4c_0$ ;  $G_3$  obviously has 15 spanning trees. We now construct graphs  $G_4, \dots, G_{10}$  such that in any one of the graphs  $G_i$ ,  $i = 1, 2, 3$ , we select vertices  $v$  and  $w$ , and then complete the respective graph  $G_i$  by  $j - 1$  vertices and  $j$  edges so that the vertices  $v$  and  $w$  are connected to each other by an arc of length  $j$  of which every inner vertex is different from all vertices of the graph  $G_i$ . We obtain the graph  $G_4, \dots, G_{10}$ , by selecting  $i, v, w$  and  $j$  as follows ( $j$  is, of course, always an integer):

- $G_4$ :  $i = 1$ ,  $v = a_2$ ,  $w = b_1$ ,  $j = (x - 61)/30 \geq 1$ ;
- $G_5$ :  $i = 1$ ,  $v = a_2$ ,  $w = b_2$ ,  $j = (x - 76)/30 \geq 1$ ;
- $G_6$ :  $i = 2$ ,  $v = a_1$ ,  $w = b_2$ ,  $j = (x - 34)/30 \geq 1$ ;
- $G_7$ :  $i = 2$ ,  $v = a_1$ ,  $w = a_2$ ,  $j = (x - 19)/30 \geq 2$ ;
- $G_8$ :  $i = 3$ ,  $v = a_1$ ,  $w = b_1$ ,  $j = (x - 22)/15 \geq 1$ ;
- $G_9$ :  $i = 3$ ,  $v = a_1$ ,  $w = a_2$ ,  $j = (x - 10)/15 \geq 2$ ;
- $G_{10}$ :  $i = 3$ ,  $v = a_1$ ,  $w = b_2$ ,  $j = (x - 28)/15 \geq 1$ .

There is little difficulty in seeing that the numbers of edges of the graphs  $G_4, \dots, G_{10}$  give successively estimations (4)–(10).

**Theorem 1.** *If  $x = 1$ , then  $\alpha(x) = 1$ ,  $\beta(x) = 0$ ; if  $x$  is one of the numbers 3, 4, 5, 6, 7, 10, 13, 22, then*

$$(11) \quad \alpha(x) = \beta(x) = x;$$

if  $x = 8$ , then  $\alpha(x) = 4$ ,  $\beta(x) = 5$ ; if  $x = 9$ , then  $\alpha(x) = 5$ ,  $\beta(x) = 6$ . Otherwise

$$(12) \quad \alpha(x) < \beta(x) \leq \frac{x+1}{2}.$$

**Proof.** The cases  $x \leq 10$  are easily verifiable; the value of the function  $\alpha$  for  $x \leq 9$  have been given by J. Sedláček [3]. From (2) it follows that (12) holds for  $x = 11$ . The graph  $D(2, 2, 2)$  leads to estimate (12) for  $x = 12$ . There is no graph with cyclomatic number 2 which has 13 spanning trees, and any graph with a greater cyclomatic number has more than 13 spanning trees; hence (11) holds for  $x = 13$ . There is no graph with cyclomatic number 2 or 3 which has 22 spanning trees, and any graph with a greater cyclomatic number has more than 22 spanning trees; hence (11) holds for  $x = 22$ . If  $x \geq 106$  it is possible to use exactly one of the estimates (2)–(10) for it; this one estimate then leads to estimate (12).

Now, let us assume that  $14 \leq x < 106$ ,  $x \neq 22$ . In so far as it is possible to use for such an  $x$  any of estimates (1)–(10), we obtain estimate (12) for it. There remain the cases  $x = 19, 31, 34, 46$  and  $61$ ; for these  $x$  it is possible to obtain estimate (12) by graphs  $D(1, 3, 4)$ ,  $D(1, 3, 7)$ ,  $D(1, 4, 6)$ ,  $D(2, 3, 8)$  and  $D(3, 4, 7)$  in turn. The proof is complete.

Now we shall turn to other relationship between the functions  $\alpha$  and  $\beta$ .

**Theorem 2.** *Let  $z$  be an integer such that  $z \geq 11$  and  $z \neq 13, 22$ . Then there is no graph having simultaneously  $\alpha(z^{z-2})$  vertices,  $\beta(z^{z-2})$  edges and  $z^{z-2}$  spanning trees.*

**Proof.** The only graph having  $\alpha(z^{z-2})$  vertices and  $z^{z-2}$  spanning trees is the complete graph having  $z$  vertices; it has  $z(z-1)/2$  edges. From (1) and (12) it follows that  $\beta(z^{z-2}) \leq (z-2)\beta(z) \leq (z-2)(z+1)/2 < z(z-1)/2$ . The proof is complete.

#### References

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*Author's address:* 116 38 Praha 1, nám. Krasnoarmějců 2 (Filosofická fakulta Karlovy univerzity).