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ON THE MITTAG-LEFFLER STABILITY OF Q-FRACTIONAL NONLINEAR DYNAMICAL SYSTEMS

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In this article, analogous to the definition of the exponential stability of ordinary dynamical systems and the Mittag-Leffler stability of the fractional dynamical systems, we consider the Mittag-Leffler stability for q-fractional nonlinear dynamical systems. The sufficient conditions for Mittag-Leffler stability of such dynamical systems within the framework of the q-fractional Caputo derivative are studied.

Key Words: q-calculus, q-fractional integral, q-fractional derivative, Mittag-Leffler function, stability.

1. INTRODUCTION

The fractional calculus deals with the generalization of integration and differentiation of integer order to those ones of any order. There has been an increasing interest in this field [1-6], because of its interesting applications in many branches of sciences and engineering. Several authors [7]-[12] have been trying to combine the time scales [13] and fractional calculus looking for a better description of phenomena having both discrete and continuous behaviors.

The q-calculus is thought to be initiated in the early years of the twentieth century and it was a subject of many articles (see Ref. [14] and the references therein). The q-fractional integrals and derivatives was firstly studied by Al-Salam [9,10,15] and then by Agarwal [8]. Their study was improved recently in Refs. [11], [12], and [16].

The stability of fractional order linear and nonlinear dynamic systems was attacked in many articles, see Refs. [17]-[27]. But the stability of q-fractional dynamical systems remains an open issue to be investigated and to the best of our knowledge has not been yet studied.

In Ref. [24], in order to show the advantage of using fractional order derivatives in place of integer order derivatives, the authors considered two nonlinear dynamical systems. The dynamical system with integer order derivative turned out to be unstable. However, the second system, where the integer order derivative was replaced by fractional order derivative turned out to be stable. It turned out that the same argument still holds for q-fractional difference systems.

Being motivated by the above mentioned results we state in this article the Mittag-Leffler stability theorem for nonlinear dynamic systems in the sense of Caputo q-fractional derivatives.

This manuscript is organized as follows:
In section 2, basic definitions of q-calculus and q-fractional calculus are given. Section 3 presents our main results on sufficient conditions for the Mittag-Leffler stability of q-fractional nonlinear dynamical systems. Finally, section 4 is devoted to our conclusions.

2. PRELIMINARIES

In this section we summarize the basic definitions and properties of q-calculus and q-fractional integrals and derivatives. For more details on q-calculus we refer to Ref. [14] and for q-fractional calculus we refer to Refs. [11] and [12].

For $0<q<1$ let T_q be the time scale [13] defined by

$$T_q=\{q^n:\ n\in\mathbb{Z}\}\cup\{0\}.$$

For a function $f:T_q\to\mathbb{R}$, the nabla q-derivative of f is given by

$$\nabla_q f(t)=\frac{f(t)-f(qt)}{(1-q)t},\quad t\in T_q-\{0\}. \tag{1}$$

The nabla q-integral of f on the interval $[0,t]$ is given by

$$\int_0^t f(s)\nabla_q s=(1-q)t\sum_{i=0}^{\infty}q^i f(tq^i) \tag{2}$$

and on the interval $[a,t],\ \ a\in T_q$ it is given by

$$\int_a^t f(s)\nabla_q s=\int_0^t f(s)\nabla_q s-\int_0^a f(s)\nabla_q s. \tag{3}$$

The fundamental theorem of q-calculus gives

$$\nabla_q\int_0^t f(s)\nabla_q s=f(t) \tag{4}$$

and if f is continuous at 0,

$$\int_0^t \nabla_q f(s)\nabla_q s=f(t)-f(0). \tag{5}$$

The q-factorial function for $n\in\mathbb{N}$ is defined by

$$(t-s)_q^n=\prod_{i=0}^{n-1}(t-q^i s). \tag{6}$$

and for $\alpha\neq 1,2,3,\ldots$, the q-factorial function takes the form

$$(t-s)_q^{\alpha}=t^{\alpha}\prod_{i=0}^{\infty}\frac{1-\frac{s}{t}q^i}{1-\frac{s}{t}q^{i+\alpha}}. \tag{7}$$

The q-gamma function $\Gamma_q(\alpha)$ for $\alpha\in\mathbb{R}\setminus\{\ldots,-2,-1,0\}$ is defined by

$$\Gamma_q(\alpha)=\frac{1}{1-q}\int_0^1\left(\frac{t}{1-q}\right)^{\alpha-1}e_q(qt)\nabla t, \tag{8}$$

where $e_q(.)$ is the q-exponential function defined by

$$e_q(t)=\prod_{i=0}^{\infty}(1-q^i t),\ \ e_q(0)=1. \tag{9}$$

The left q-fractional integral of order $\alpha>0$, ${}_qI_a^{\alpha}$ starting from $0<a,\ \ a\in T_q$ is defined

$${}_qI_a^{\alpha}f(t)=\frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}f(s)\nabla_q s. \tag{10}$$

When $\alpha \in \mathbb{N}$, we have

$$\nabla_q^n \, {}_qI_a^\alpha f(t) = f(t) \text{ for } 0 \le a \in T_q . \tag{11}$$

The left (Riemann) q-fractional derivative of order $\alpha > 0$ starting from $0 < a, \quad a \in T_q$ is defined by

$$ {}_q\nabla_a^\alpha f(t) \triangleq \nabla_q^n \, {}_qI_a^{n-\alpha} f(t) = \nabla_q^n \frac{1}{\Gamma_q(n-\alpha)} \int_a^t (t-qs)_q^{n-\alpha-1} f(s) \nabla_q s , \tag{12}$$

where $n = [\alpha] + 1$ ($[\alpha]$ is the greatest integer less than α). While the left q-fractional Caputo derivative is defined by

$$ {}_q^C\nabla_a^\alpha f(t) \triangleq {}_qI_a^{n-\alpha} \nabla_q^n f(t) = \frac{1}{\Gamma_q(n-\alpha)} \int_a^t (t-qs)_q^{n-\alpha-1} \nabla_q^n f(s) \nabla_q s . \tag{13}$$

The relation between the left q-fractional Riemann and Caputo derivatives for $0 < \alpha < 1$ is given the following formula [12]

$$ {}_q^C\nabla_a^\alpha f(t) = {}_q\nabla_a^\alpha f(t) - \frac{(t-a)_q^{-\alpha}}{\Gamma_q(1-\alpha)} f(a) . \tag{14}$$

PROPERTY 1. [12] *Assume that* $0 < \alpha \le 1$ *and* f *is defined for in suitable domains then*

$$ {}_qI_a^\alpha \, {}_q^C\nabla_a^\alpha f(t) = f(t) - f(a) . \tag{15}$$

The q-Mittag-Leffler function was defined in [12] as

$$. \; {}_qE_\alpha(\lambda, z - z_0) = \sum_{k=0}^{\infty} \frac{\lambda^k (z - z_0)_q^{\alpha k}}{\Gamma_q(\alpha k + 1)} , \tag{16}$$

where $z, z_0 \in \mathbb{C}$ and $\Re(\alpha) > 0.$ In a more generalized form it is given by [12]

$$. \; {}_qE_{\alpha,\beta}(\lambda, z - z_0) = \sum_{k=0}^{\infty} \frac{\lambda^k (z - z_0)_q^{\alpha k}}{\Gamma_q(\alpha k + \beta)} . \tag{17}$$

Notice that ${}_qE_{\alpha,1}(\lambda, z - z_0) = {}_qE_\alpha(\lambda, z - z_0)$. It should be mentioned that the solution of the following IVP

$$. \; {}_q^C\nabla_a^\alpha x(t) = \lambda x(t) + f(t), \quad x(t_0) = x_0 , \quad t_0, t \in T_q \tag{18}$$

is given by

$$. \; x(t) = x_0 \, {}_qE_\alpha(\lambda, t - t_0) + \int_{t_0}^t (t-qs)_q^{\alpha-1} \, {}_qE_{\alpha,\alpha}(\lambda, t - q^\alpha s) f(s) \nabla_q s . \tag{19}$$

3. MITTAG-LEFFLER STABILITY FOR Q-FRACTIONAL DIFFERENCE SYSTEMS

Consider the following q-fractional dynamical system

$$. \; {}_q^C\nabla_a^\alpha x(t) = f(t, x(t)), \quad x(t_0) = x_0 , \tag{20}$$

where $t \ge t_0, \;\; t_0, t \in T_q, \;\; 0 < \alpha < 1$ and $f : T_q \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in x. Let $f(t,0) = 0$, so that (20) admits the trivial solution. Analogous of the definition of Mittag-Leffler stability of fractional dynamical systems defined in [24], we define the Mittag-Leffler stability of solutions of (20) as follows:

Definition 1. The trivial solution $x(t)$ of *(20)* is said to be Mittag-Leffler stable if

$$. \| x(t) \| \leq \left[M\left(x(t_0)\right)_q E_\alpha(-\lambda, t-t_0) \right]^b , \tag{21}$$

where $\lambda \geq 0,\ b>0,\ M(0)=0,\ M(x)>0$ and $M(x)$ is locally Lipschitz for $x \in S_\rho = \{x \in \mathbb{R}^n : \|x\| < \rho\} \subset \mathbb{R}^n$. We notice that the condition (21) extends the definition of the classical exponential stability.

THEOREM 1. *If there exist a scalar function* $V(t,x) \in C[T_q \times S_\rho, \mathbb{R}_+]$ *and positive constants* c_1, c_2 *and* c_3 *such that*

$$. c_1 \|x\|^2 \leq V(t,x) \leq c_2 \|x\|^2 \tag{22}$$

and

$$. {}_q^C\nabla_{t_0}^\alpha V(t,x) \leq -c_3 \|x\|^2 , \tag{23}$$

for all $(t,x) \in T_q \times S_\rho$, $t \geq t_0$, *then the trivial solution of (20) is Mittag-Leffler stable.*

Proof. Let $x(t) = x(t,t_0,x_0)$ be any solution of (20). From conditions (22) and (23) it follows that ${}_q^C\nabla_{t_0}^\alpha V(t,x) \leq -\frac{c_3}{c_2} V(t,x(t))$. Thus, there exists a function $g(t) \geq 0$ defined for $t \in T_q$, $t \geq t_0$ such that ${}_q^C\nabla_{t_0}^\alpha V(t,x) = -g(t) - \frac{c_3}{c_2} V(t,x(t))$. This is an equation of the form of (18) and therefore it has the solution

$$. V(t,x(t)) = V(t_0,x_0)\,{}_q E_\alpha(-\frac{c_3}{c_2}, t-t_0) - \int_{t_0}^t (t-qs)_q^{\alpha-1}\,{}_q E_{\alpha,\alpha}(\lambda, t-q^\alpha s) g(s) \nabla_q s. \tag{24}$$

Consequently, $V(t,x(t)) \leq V(t_0,x_0)\,{}_q E_\alpha(-\frac{c_3}{c_2}, t-t_0)$. Again from (22), we have $\|x(t)\|^2 \leq \frac{1}{c_1} V(t,x)$ .and $V(t_0,x_0) \leq c_2 \|x_0\|^2$ Therefore we have $\|x(t)\|^2 \leq \frac{1}{c_1} V(t,x) \leq \frac{1}{c_1} V(t_0,x_0)\,{}_q E_\alpha(-\frac{c_3}{c_2}, t-t_0) \leq$ $\leq \frac{c_2}{c_1} \|x_0\|^2\,{}_q E_\alpha(-\frac{c_3}{c_2}, t-t_0)$. That is $\|x(t)\| \leq \sqrt{\frac{c_2}{c_1}} \|x_0\| \left[{}_q E_\alpha(-\frac{c_3}{c_2}, t-t_0) \right]^{\frac{1}{2}}$. Hence $x(t)$ is Mittag-Leffler stable.

We remark that this theorem can be applied to some complex systems considered in q-calculus together with fractional operators.

THEOREM 2. *If there exist a scalar function* $V(t,x) \in C[T_q \times \mathbb{R}^n, \mathbb{R}_+]$ *and positive constants* c_1, c_2 *and* c_3 *such that*

$$. c_1 \|x\|^2 \leq V(t,x) \leq c_2 \|x\|^2 \tag{25}$$

and

$$. {}_q^C\nabla_{t_0}^\alpha V(t,x) \leq -c_3 \|x\|^2 , \tag{26}$$

for all $(t,x) \in T_q \times \mathbb{R}^n$ $t \geq t_0$, *then the trivial solution of (20) is globally Mittag-Leffler stable.*

Proof. Similarly as in the proof of Theorem 1, we still have the estimate

$\| x(t) \| \leq \sqrt{\frac{c_2}{c_1}} \| x_0 \| \left[{}_qE_\alpha(-\frac{c_3}{c_2}, t-t_0) \right]^{\frac{1}{2}}$ for all $(t,x) \in T_q \times \mathbb{R}^n$ $t \geq t_0$. Thus $x(t)$ is globally Mittag-Leffler stable. We stress on the fact that (26) is formulated taking into account the definition mentioned in (13).

LEMMA 1. *If* $V(t_0, x(t_0)) \geq 0$*, then for* $0 < \alpha \leq 1$ *we have* ${}_q^C\nabla_{t_0}^\alpha V(t,x) \leq {}_q\nabla_{t_0}^\alpha V(t,x)$ for $t \geq t_0$.

Proof. The result can be easily noticed from equation (14). We notice that is similar with the one from the classical case.

Taking into account (14) we present below the corresponding theorems for the left (Riemann) q-fractional derivative.

THEOREM 3. *If there exist a scalar function* $V(t,x) \in C[T_q \times S_\rho, \mathbb{R}_+]$ *and positive constants* c_1, c_2 *and* c_3 *such that*

$$. \ c_1 \| x \|^2 \leq V(t,x) \leq c_2 \| x \|^2 \tag{27}$$

and

$$. \ {}_q\nabla_{t_0}^\alpha V(t,x) \leq -c_3 \| x \|^2 \tag{28}$$

for all $(t,x) \in T_q \times S_\rho$, $t \geq t_0$*, then the trivial solution of (20) is Mittag-Leffler stable.*

Proof. From (27), we have $V(t_0, x_0) \geq 0$. Consequently, from Lemma 1 we have ${}_q^C\nabla_{t_0}^\alpha V(t,x) \leq {}_q\nabla_{t_0}^\alpha V(t,x)$ for $t \geq t_0$. Thus we have from (28), ${}_q^C\nabla_{t_0}^\alpha V(t,x) \leq -c_3 \| x \|^2$. Therefore the hypotheses of Theorem 1 are satisfied and thus the trivial solution of the q-fractional dynamical system (20) is Mittag-Leffler stable.

THEOREM 4. *If there exist a scalar function* $V(t,x) \in C[T_q \times \mathbb{R}^n, \mathbb{R}_+]$ *and positive constants* c_1, c_2*, and* c_3 *such that*

$$c_1 \| x \|^2 \leq V(t,x) \leq c_2 \| x \|^2 \tag{29}$$

and

$$. \ {}_q\nabla_{t_0}^\alpha V(t,x) \leq -c_3 \| x \|^2, \tag{30}$$

for all $(t,x) \in T_q \times \mathbb{R}^n$, $t \geq t_0$*, then the trivial solution of the the q-fractional dynamical system (20) is globally Mittag-Leffler stable.*

Proof. The proof is analogous to the proof of Theorem 3.

4. CONCLUSIONS

The exponential stability of systems of differential equations is used in many areas of science and engineering. The Mittag-Leffler stability of fractional systems is a new concept introduced recently and analogously defined. In this paper, we defined the Mittag-Leffler stability for q-fractional dynamical systems and stated the sufficient conditions for such kind of stability. We believe that this kind of stability will have applications in control theory and other branches of physics, mathematics and engineering when q-fractional nonlinear dynamical systems are considered.

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