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# On the moduli space $M(0, 4)$ of vector bundles

By

Isao NARUKI

**Introduction.**

In this note we fix an algebraically closed field $k$ with $\mathrm{ch}(k)\neq 2$ as the ground field and we denote the moduli space of vector bundles over $\boldsymbol{P}_2(k)$ of rank 2 with chern classes $c_1, c_2$ by $M(c_1, c_2)$ according to the general usage. Barth [1] gave a beautiful description to $M(0, n)$ ($n\geqq 2$) and claimed that it is always rational. We note however that some gap in his proof was recently pointed out by Maruyama [3]. Fortunately Ellingsrud-Strømme [2] proved by using a different method that $M(0, n)$ is rational for $n$ ($\geqq 3$) odd. But, as far as we know, the rationality-question has not yet been answered for any even $n$ ($\geqq 4$). One might even ask if the answer could eventually be negative, since the situation is so complicated already for the simplest case $n=4$. Thus it is worthwhile establishing the rationality of $M(0, 4)$ as an affirmative example, which is the purpose of this note. We note that our result is a little stronger than is required. In fact we will prove the rationality for the quotient of the quadric cone of decomposable elements in $\Lambda^2 V$ by $W=W(B_4)$ where $V$ is a Cartan subalgebra of the complex simple group $B_4$ and $W$ is the Weyl group associated with $V$. (According to [1] and [3], $M(0, 4)$ is birational to the direct product of this quotient and the affine space $k^8$. The corresponding stronger statement for $n=3$ is proved in the appendix of [3]) Throughout this note we let $M$ stand for this quotient variety.

Our method of proof is essentially based on the fact that the group $W$ has a substantial chain of normal subgroups which enables us to divide the quotient formation into several steps. To explain our idea clearly and also to prepare some necessary notation for later use, we now outline our stepwise quotient formation briefly: Let $V$ and $W$ be as above and $Z$ the center $\{\pm\}$ of $W$. We set

$$L:=\Lambda^2 V$$

$$G:=W/Z\,.$$

$G$ acts faithfully on $L$; it acts also on the cone of decomposable elements:

$$Q:=\{x\in L\,;\,x\wedge x=0\}.$$

(Note that $g(x\wedge x)=\det(g|V)\cdot(x\wedge x)$ for $g\in W$.) Thus we have the precise

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definition for $M$:

$$M=Q/G\,.$$

Now we define the character sgn of $G$ by requiring the commutativity for the diagram

$$\begin{tikzcd} W \arrow[d] \arrow[dr, "\det"] & \\ G \arrow[r, "\mathrm{sgn}"'] & \{\pm 1\} \end{tikzcd}$$

and we put

$$\dot{G}=\{g\in G\,;\,\mathrm{sgn}(g)=1\} \qquad (=\mathrm{Ker}\,(\mathrm{sgn}))\,.$$

The representation of $\dot{G}$ on $L$ is not irreducible but it splits into the direct sum of two 3-dimensional (irreducible) representations denoted by $\rho_i: \dot{G}\to GL(L_i)$ $i=1, 2$ $(L=L_1\oplus L_2)$. We set

$$H_i:=\mathrm{Ker}\,(\rho_i) \qquad i=1, 2\,.$$

$H_1$ and $H_2$ are both isomorphic to Klein's 4-group $(\boldsymbol{Z}/2\boldsymbol{Z})^2$. They commute each other and $H_1\cap H_2=\{1\}$ since the representation is faithful. The direct product

$$H:=H_1\cdot H_2$$

is thus a normal subgroup of $G$ isomorphic to $(\boldsymbol{Z}/2\boldsymbol{Z})^4$. We put

$$G^*:=G/H$$

$$L^*:=L/H \qquad Q^*:=Q/H\,.$$

(The fact that $H$ is abelian makes the description of $L^*$, $Q^*$ particularly simple.) $G^*$ has again the non-trivial center $Z^*$ which is of order 2, and we further set

$$\tilde{G}:=G^*/Z^*$$

$$\tilde{L}:=L^*/Z^* \qquad \tilde{Q}:=Q^*/Z^*\,.$$

$\tilde{G}$ is isomorphic to the symmetric group $S_3$ and the action on $\tilde{L}$ is very near to the usual diagonal action of $S_3$ over $(k^3)\times(k^3)$. Now we get the final description of $M$

$$\hat{L}:=\tilde{L}/\tilde{G} \qquad M=\tilde{Q}/\tilde{G}\,.$$

We will prove the rationality for $\hat{L}$ and $M$ simultaneously, by choosing good coordinate systems for (suitable Zariski open subsets of) the algebraic varieties $L^*, Q^*, \cdots, M$ step by step.

To close this introduction, the author heartily thanks Masaki Maruyama for introducing him to the subject and for being always ready for helpful discussions.

## 1. Explicit formulation.

Since $V, W$ are not necessary anymore, we begin by defining $G$ and $L$ directly. Let $E$ be the abelian group with the generators $e_i$ $(i=1,2,3,4)$ and the fundamental relations:

$$e_i^2=1\,, \qquad e_ie_j=e_je_i$$

$$e_1e_2e_3e_4=1\,.$$

The permutation group $S_4$ of the indices $\{1,2,3,4\}$ acts on $E$ as automorphisms by

$$\sigma(e_i)=e_{\sigma(i)} \qquad \sigma\in S_4\,.$$

$G$ is then represented by the semi-direct product:

$$G=E\times S_4\,.$$

Now we regard the symbols $[i, j]$ $1\leqq i<j\leqq 4$ as the coordinates of $L\simeq k^6$ and we let $[j, i]$ denote $-[i, j]$. (These are exactly the Plücker coordinates of $L$.) By the law

$$e_i([i,j])=-[i,j]$$

$$e_i([j,k])=[j,k] \qquad (j,k\neq i)$$

$$\sigma([i,j])=[\sigma(i),\sigma(j)] \qquad \sigma\in S_4$$

the group certainly acts on $L$. The character sgn coincides with the usual sign on $S_4$ and we have

$$\operatorname{sgn}(e_i)=-1\,.$$

The decomposable elements are defined by the equation:

$$[1,2]\cdot[3,4]+[1,3]\cdot[4,2]+[1,4]\cdot[2,3]=0$$

To observe the splitting $L=L_1\oplus L_2$ mentioned in the introduction, we introduce the following coordinates:

$$\begin{cases} u_1=[1,2]+[3,4] \\ u_2=[1,3]+[4,2] \\ u_3=[1,4]+[2,3] \end{cases} \qquad \begin{cases} v_1=[1,2]-[3,4] \\ v_2=[1,3]-[4,2] \\ v_3=[1,4]-[2,3] \end{cases}$$

Then we obtain a list of important operations:

(1.1) $\varepsilon_1:=e_1e_2=e_3e_4$ operates as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \longrightarrow \begin{pmatrix} u_1 \\ -u_2 \\ -u_3 \end{pmatrix} \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \longrightarrow \begin{pmatrix} v_1 \\ -v_2 \\ -v_3 \end{pmatrix}.$$

(1.2) $\varepsilon_2 := e_1e_3 = e_2e_4$ operates as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \longrightarrow \begin{pmatrix} -u_1 \\ u_2 \\ -u_3 \end{pmatrix} \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \longrightarrow \begin{pmatrix} -v_1 \\ v_2 \\ -v_3 \end{pmatrix}.$$

(1.3) $\varepsilon_3 := e_1e_4 = e_2e_3$ operates as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \longrightarrow \begin{pmatrix} -u_1 \\ -u_2 \\ u_3 \end{pmatrix} \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \longrightarrow \begin{pmatrix} -v_1 \\ -v_2 \\ v_3 \end{pmatrix}.$$

(1.4) $A := e_3(1, 2)$ operates as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \longrightarrow \begin{pmatrix} -u_1 \\ -u_3 \\ -u_2 \end{pmatrix} \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \longrightarrow \begin{pmatrix} -v_1 \\ v_3 \\ v_2 \end{pmatrix}.$$

(1.5) $\sigma_1 := (1, 2)(3, 4)$ operates as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \longrightarrow \begin{pmatrix} -u_1 \\ -u_2 \\ u_3 \end{pmatrix} \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \longrightarrow \begin{pmatrix} -v_1 \\ v_2 \\ -v_3 \end{pmatrix}.$$

(1.6) $\sigma_2 := (1, 3)(4, 2)$ operates as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \longrightarrow \begin{pmatrix} u_1 \\ -u_2 \\ -u_3 \end{pmatrix} \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \longrightarrow \begin{pmatrix} -v_1 \\ -v_2 \\ v_3 \end{pmatrix}.$$

(1.7) $\sigma_3 := (1, 4)\cdot(2, 3)$ operates as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \longrightarrow \begin{pmatrix} -u_1 \\ u_2 \\ -u_3 \end{pmatrix} \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \longrightarrow \begin{pmatrix} v_1 \\ -v_2 \\ -v_3 \end{pmatrix}.$$

(1.8) $e_1$ operates as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \longrightarrow \begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \end{pmatrix} \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \longrightarrow \begin{pmatrix} -u_1 \\ -u_2 \\ -u_3 \end{pmatrix}.$$

The other elements $e_i(j, k) \in G$ operates like $A$ above, and we have thus described the action of $G$ over $L$ completely.

We finally remark that the quadric-cone $Q$ of decomposable elements is defined by the equation

$$u_1^2+u_2^2+u_3^2=v_1^2+v_2^2+v_3^2$$

in the new coordinates $u_i$, $v_i$.

## 2. Function field of $L^*$.

The group $\dot{G}=\mathrm{Ker}\,(\mathrm{sgn})$ is generated by $\varepsilon_i$, $\sigma_i$ $(i=1,2,3)$ and $e_i(j,k)$ $(i,j,k$ : different); so, by the table of Section 1, we see that it has two invariant subspaces i.e. the subspace $L_1$ given by $v_1=v_2=v_3=0$ and the subspace $L_2$ given by $u_1=u_2=u_3=0$. Then $(u_1,u_2,u_3)$ (resp. $(v_1,v_2,v_3)$) is regarded as a coordinate system of $L_1$ (resp. $L_2$). We immediately check

$$H_1=\{1,\sigma_1\varepsilon_3,\sigma_2\varepsilon_1,\sigma_3\varepsilon_2\}$$

$$H_2=\{1,\sigma_1\varepsilon_2,\sigma_2\varepsilon_3,\sigma_3\varepsilon_1\}.$$

$H_2$ (resp. $H_1$), which by definition fixes each $v_i$ (resp. $u_i$), operates as the sign changes of even number of $u_i$'s (resp. $v_i$'s). This implies that the function field of $L^*=L/H$, denoted by $k(L^*)$, is generated by the following $H$-invariants:

$$\begin{cases} a_i:=u_i^2 & b_i:=v_i^2 \qquad (i=1,2,3) \\ s:=u_1u_2u_3 & t:=v_1v_2v_3\,. \end{cases}$$

$$k(L^*)=k(a_1,a_2,a_3,b_1,b_2,b_3,s,t)$$

where we have the relations among the generators

$$s^2=a_1a_2a_3 \tag{2.1}$$

$$t^2=b_1b_2b_3\,. \tag{2.2}$$

The hypersurface $Q^*=Q/H$ of $L^*$ is given by the following linear equation:

$$a_1+a_2+a_3=b_1+b_2+b_3\,. \tag{2.3}$$

## 3. Function field of $\widetilde{L}$.

The center $Z^*$ of $G^*$ is of order 2 and generated by the image of element $e_1\in G$. By (1.8) we see that this transposes $a_i$ and $b_i$ for $i=1,2,3$ and transforms $s,t$ to $-t,-s$. We thus obtain $Z^*$-invariants

$$\alpha_i:=a_i+b_i \qquad (i=1,2,3)$$

$$\beta_i:=(s+t)(a_i-b_i)$$

$$\gamma\ :=(s+t)^2$$

$$\delta\ :=s-t\,.$$

To see that these generate the function field $k(\widetilde{L})$ we need only write down the inversion formula:

$$a_i=\frac{1}{2}\left(\alpha_i+\frac{\beta_i}{\sqrt{\gamma}}\right) \qquad (i=1, 2, 3\,;\ \sqrt{\gamma}=s+t)$$

$$b_i=\frac{1}{2}\left(\alpha_i-\frac{\beta_i}{\sqrt{\gamma}}\right)$$

$$s=\frac{1}{2}(\sqrt{\gamma}+\delta)$$

$$t=\frac{1}{2}(\sqrt{\gamma}-\delta)$$

which shows that $k(L^*)$ is of degree 2 over the field generated by the invariants. By calculating $\gamma+\delta^2=2(s^2+t^2)=2(a_1a_2a_3+b_1b_2b_3)$, $\sqrt{\gamma}\delta=s^2-t^2=a_1a_2a_3-b_1b_2b_3$ explicitly, we obtain the following two relations which are corresponding to (2.1) and (2.2)

(3.1) $$2\gamma\{\gamma+\delta^2\}=\gamma\alpha_1\alpha_2\alpha_3+\alpha_1\beta_2\beta_3+\alpha_2\beta_1\beta_3+\alpha_3\beta_1\beta_2$$

(3.2) $$4\gamma^2\delta=\gamma(\beta_1\alpha_2\alpha_3+\beta_2\alpha_1\alpha_3+\beta_3\alpha_1\alpha_2)+\beta_1\beta_2\beta_3\,.$$

The hypersurface $\tilde{Q}=Q^*/Z^*$ is defined by the linear equation:

(3.3) $$\beta_1+\beta_2+\beta_3=0\,.$$

## 4. Function field of $\hat{L}$,

The group $\tilde{G}=G^*/Z^*$ is now isomorphic to $S_3$ and it operates on $\tilde{L}=L^*/Z^*$ whose function field is $k(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma, \delta)$. As is seen from (1.4), $\tilde{G}$ permutes $\alpha_i$'s as the index changes and $\gamma$ is $\tilde{G}$-invariant. But $\delta$ is a relative invariant belonging to the charactor sgn i.e. we have

$$\sigma(\delta)=\operatorname{sgn}(\sigma)\delta \qquad \sigma\in\tilde{G}\simeq S_3\,.$$

Also by (1.4) we have the rather complicated operation of $\tilde{G}$ over $\beta_i$'s i.e. $\sigma(\beta_i)=\operatorname{sgn}(\sigma)\beta_{\sigma(i)}$. This disadvantage is however immediately removed if we introduce the following variables as the substitute for $\beta_i$'s:

$$\bar{\beta}_i:=\beta_i/\delta \qquad i=1, 2, 3\,.$$

Now $\tilde{G}$ permutes $\bar{\beta}_i$'s as the index changes. We further introduce the following substitute of $\alpha_i$'s

$$\bar{\alpha}_i:=\alpha_i/\bar{\beta}_i \qquad i=1, 2, 3\,.$$

These variables are more convenient since the equations (3.1) and (3.2) are written in the much simpler form:

(4.1) $$2\gamma\{\gamma+\delta^2\}=p_3(\bar{\beta})\{\gamma p_3(\bar{\alpha}))+\delta^2p_1(\bar{\alpha})\}$$

(4.2) $$4\gamma^2=p_3(\bar{\beta})\{\gamma p_2(\bar{\alpha})+\delta^2\},$$

where $p_i(\bar{\alpha})$ (resp. $p_i(\bar{\beta})$) denotes the $i$-th elementary symmetric polynomial of $\bar{\alpha}_i$'s (resp. $\bar{\beta}_i$'s). By rewriting (4.2) in the form

$$\delta^2=\frac{4\gamma^2}{p_3(\bar{\beta})}-\gamma p_2(\bar{\alpha})=\gamma\left\{\frac{4\gamma}{p_3(\bar{\beta})}-p_2(\bar{\alpha})\right\} \tag{4.2$'$}$$

and eliminating $\delta^2$ from (4.1), we obtain

$$2\bar{\gamma}^2+(1-p_2(\bar{\alpha})-2p_1(\bar{\alpha}))\bar{\gamma}+p_1(\bar{\alpha})p_2(\bar{\alpha})-p_3(\bar{\alpha})=0 \tag{4.3}$$

where we have introduced the new variable

$$\bar{\gamma}:=2\gamma/p_3(\bar{\beta}).$$

Now we recall that $\delta$ is not yet quite $\tilde{G}$-invariant and for this reason we put

$$\bar{\delta}:=\Delta(\bar{\alpha})\delta \qquad \Delta(\bar{\alpha}):=(\bar{\alpha}_1-\bar{\alpha}_2)(\bar{\alpha}_2-\bar{\alpha}_3)(\bar{\alpha}_3-\bar{\alpha}_1).$$

We then obtain from (4.2)$'$

$$\bar{\delta}^2=\Delta(\bar{\alpha})^2p_3(\bar{\beta})\bar{\gamma}(2\bar{\gamma}-p_2(\bar{\alpha}))/2. \tag{4.4}$$

Since $\bar{\gamma}$, $\bar{\delta}$ are $\tilde{G}$-invariant, we arrive at the following important assertion: *The function field $k(\hat{L})$ of $\hat{L}=\tilde{L}/\tilde{G}$ is obtained by adioining the algebraic quantities $\bar{\gamma}$, $\bar{\delta}$ defined by* (4.3), (4.4) *to the fixed field $k(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3)^{\tilde{G}}$. The quotient $M=\tilde{Q}/\tilde{G}$ is defined by the equation*

$$\bar{\beta}_1+\bar{\beta}_2+\bar{\beta}_3=0. \tag{4.5}$$

## 5. The rationality of $\hat{L}$ and $M$.

Now we have almost everything to prove the desired rationality. We begin by finding suitable generators for the fixed field $k(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)^{\tilde{G}}$: We set

$$\begin{aligned} p&:=\bar{\beta}_1+\bar{\beta}_2+\bar{\beta}_3\\ q&:=\bar{\alpha}_1\bar{\beta}_1+\bar{\alpha}_2\bar{\beta}_2+\bar{\alpha}_3\bar{\beta}_3\\ r&:=\bar{\alpha}_1^2\bar{\beta}_1+\bar{\alpha}_2^2\bar{\beta}_2+\bar{\alpha}_3^2\bar{\beta}_3\end{aligned}$$

These obviously belong to the fixed field and we can solve the above identities with respect to $\bar{\beta}_1$, $\bar{\beta}_2$, $\bar{\beta}_3$:

$$\begin{cases} \bar{\beta}_1=-\dfrac{\bar{\alpha}_2\bar{\alpha}_3p-(\bar{\alpha}_2+\bar{\alpha}_3)q+r}{(\bar{\alpha}_1-\bar{\alpha}_3)(\bar{\alpha}_2-\bar{\alpha}_1)}\\[2ex] \bar{\beta}_2=-\dfrac{\bar{\alpha}_1\bar{\alpha}_3p-(\bar{\alpha}_1+\bar{\alpha}_3)q+r}{(\bar{\alpha}_2-\bar{\alpha}_1)(\bar{\alpha}_3-\bar{\alpha}_2)}\\[2ex] \bar{\beta}_3=-\dfrac{\bar{\alpha}_1\bar{\alpha}_2p-(\bar{\alpha}_1+\bar{\alpha}_2)q+r}{(\bar{\alpha}_3-\bar{\alpha}_2)(\bar{\alpha}_1-\bar{\alpha}_3)}. \end{cases} \tag{5.1}$$

By using this we see that any element of the fixed field can be expressed to be a rational function of $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\bar{\alpha}_3$, $p$, $q$, $r$. We then see by the $\tilde{G}$-invariance of the element and $p$, $q$, $r$ that it is also rationally expressible by $p_1(\bar{\alpha})$, $p_2(\bar{\alpha})$, $p_3(\bar{\alpha})$, $p$, $q$, $r$. Thus we have proved

$$k(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3)^{\widetilde{G}}=k(p_1(\bar{\alpha}), p_2(\bar{\alpha}), p_3(\bar{\alpha}), p, q, r).$$

Now, if we express $p_3(\bar{\beta})$ by (5.1) as an element of the right hand side, we then obtain

$$p_3(\bar{\beta})=\frac{P(p_1(\bar{\alpha}), p_2(\bar{\alpha}), p_3(\bar{\alpha}), p, q, r)}{\Delta(\alpha)^2}.$$

Here $P$ is a polynomial of the arguments and it is homogeneous of degree 3 with respect to $p, q, r$; so we further introduce the substitutes for $p, q$;

$$\bar{p}:=p/r \qquad \bar{q}:=q/r.$$

The equation (4.4) is then written in the form

$$\bar{\delta}^2=r^3\bar{\gamma}(2\bar{\gamma}-p_3(\bar{\alpha}))\times P(p_1(\bar{\alpha}), p_2(\bar{\alpha}), p_3(\bar{\alpha}), \bar{p}, \bar{q}, 1)/2. \tag{5.2}$$

The final variable change $\bar{\bar{\delta}}:=\bar{\delta}/r$ gives now

$$(\bar{\bar{\delta}})^2=rR(\bar{\gamma}, p_1(\bar{\alpha}), p_2(\bar{\alpha}), p_3(\bar{\alpha}), \bar{p}, \bar{q}) \tag{5.3}$$

where $R$ is a polynomial of its arguments. Now we can obviously solve the relations (4.3), (5.3) with respect to $p_3(\bar{\alpha})$ and $r$, which, together with the last assertion of Section 4, implies the identity

$$k(\hat{L})=k(p_1(\bar{\alpha}), p_2(\bar{\alpha}), \bar{p}, \bar{q}, \bar{\gamma}, \bar{\bar{\delta}}).$$

Since the subvariety $M$ of $\hat{L}$ is defined by $\bar{p}=0$, we have thus proved the desired rationality both for $\hat{L}$ and $M$.

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES
KYOTO UNIVERSITY

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