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ON THE MODULUS OF CONTINUITY OF HARMONIC QUASIREGULAR MAPPINGS ON THE UNIT BALL IN $\mathbb{R}^n$

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Abstract

We show that, for a class of moduli functions $\omega(\delta),\ 0 \le \delta \le 2$, the property $|\varphi(\xi) - \varphi(\eta)| \le \omega(|\xi - \eta|),\ \xi,\ \eta \in \mathbb{S}^{n-1}$ implies the corresponding property $|u(x)-u(y)| \le C\omega(|x-y|)\ x, y \in \mathbb{B}^n$, for $u = P[\varphi]$, provided u is a quasiregular mapping. Our class of moduli functions includes $\omega(\delta) = \delta^\alpha\ (0 < \alpha \le 1)$, so our result generalizes earlier results on Hölder continuity (see [1]) and Lipschitz continuity (see [2]).

1 Introduction and notations

We set, for any $n \ge 2$

$$P[\varphi](x) = \int_{\mathbb{S}^{n-1}} P(x, \xi)\varphi(\xi)d\sigma(\xi)$$

where $P(x, \xi) = \dfrac{1 - |x|^2}{|x - \xi|^n}$ is the Poisson kernel for $\mathbb{B}^n = \{x \in \mathbb{R} : |x| < 1\},\ d\sigma$ is the normalized surface measure on $\mathbb{S}^{n-1}$ and $\varphi : \mathbb{S}^{n-1} \to \mathbb{R}^n$ is a continuous mapping.

We are going to work with moduli functions $\omega(\delta),\ 0 \le \delta \le 2$, satisfying the following conditions:

1° $\omega(\delta)$ is continuous, increasing and $\omega(0) = 0$,

2° $\omega(\delta)/\delta$ is a decreasing function,

3° $\displaystyle\int_0^\delta \frac{\omega(\rho)}{\rho} d\rho \le C\omega(\delta)$,

We say that f is $\omega-$continuous if $|f(x) - f(y)| \le \omega(|x - y|)$ for all x and y in the domain of f.

We note that the following properties of ω follow from the conditions 1° and 2°:

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4° $\displaystyle\int_\delta^2 \frac{\omega(\rho)}{\rho^3} d\rho \le C\frac{\omega(\delta)}{\delta^2}$,

5° $\displaystyle\int_0^\delta \omega(\rho)\rho^{n-1} d\rho \le C\delta^n\omega(\delta)$.

2 The Main Result

Theorem 2.1 *Assume $\varphi : \mathbb{S}^{n-1} \to \mathbb{R}^n$ is $\omega-$continuous mapping, where ω is a modulus function satisfying properties $1° - 3°$. If the harmonic extension $u = P[\varphi]$ of φ to $\mathbb{B}^n$ is $K-$quasiregular, then u is $C\omega-$continuous, where C depends on n, K and ω only.*

In the case of Lipschitz continuity, i.e. $\omega(\delta) = L\delta$, this was proved in [2]. If $\omega(\delta) = L\delta^\alpha$, $0 < \alpha < 1$, then the conclusion holds without the assumption of quasiregularity of $u = P[\varphi]$, see [3].

We use the same method of proof as in [2], adapted to deal with our class of moduli functions.

Let us choose $x_0 = r\xi_0 \in \mathbb{B}$, $r = |x_0|$, $\xi_0 \in \mathbb{S}^{n-1}$; let $T = T_{x_0}r\mathbb{S}^{n-1}$ be the $(n-1)$-dimensional tangent plane to the sphere $r\mathbb{S}^{n-1}$ at point x_0. The proof is based on the following estimate

$$||D(u|_T)(x_0)|| \le C(\omega, n) \cdot \frac{\omega(\delta)}{\delta},\ \delta = 1 - |x_0|, \qquad (*)$$

which is of independent interest.

Without loss of generality $x_0 = re_n$, where $e_n = (0, 0, \ldots, 0, 1) \in \mathbb{S}^{n-1}$. We have, by a simple calculation,

$$\frac{\partial}{\partial x_j} P(x, \xi) = -\frac{2x_j}{|x - \xi|^n} - n(1 - |x|^2)\frac{x_j - \xi_j}{|x - \xi|^{n+2}}.$$

Hence, for $1 \le j < n$ and for $x_0 = re_n$ we have

$$\frac{\partial}{\partial x_j} P(x_0, \xi) = n(1 - |x_0|^2)\frac{\xi_j}{|x_0 - \xi|^{n+2}}.$$

Note that this integral kernel is odd in $\xi \in \mathbb{S}^{n-1}$. We have, using this property of the kernel,

$$\begin{aligned} \frac{\partial u}{\partial x_j}(x_0) &= n(1 - |x_0|^2)\int_{\mathbb{S}^{n-1}} \varphi(\xi) \cdot \frac{\xi_j}{|x_0 - \xi|^{n+2}} d\sigma(\xi) \\ &= n(1 - |x_0|^2)\int_{\mathbb{S}^{n-1}} [\varphi(\xi) - \varphi(\xi_0)]\frac{\xi_j}{|x_0 - \xi|^{n+2}} d\sigma(\xi). \end{aligned}$$

Of course, since $x_0 = re_n$, we have $\xi_0 = e_n$. Now, using

$$|\xi_j| \leq |\xi - \xi_0|, \qquad (1 \leq j < n,\ \xi \in \mathbb{S}^{n-1})$$

and ω continuity of φ we get, for $1 \leq j < n$,

$$\Big|\frac{\partial u}{\partial x_j}(x_0)\Big| \leq n(1-|x_0|^2) \cdot \int_{\mathbb{S}^{n-1}} \frac{|\xi - \xi_0|\omega(|\xi - \xi_0|)}{|x_0 - \xi|^{n+2}} d\sigma(\xi).$$

In order to estimate the last integral, we split $\mathbb{S}^{n-1}$ into two disjoint subsets $E = \{\xi \in \mathbb{S}^{n-1} : \ |\xi - \xi_0| \leq 1 - |x_0|\}$ and $F = \{\xi \in \mathbb{S}^{n-1} : \ |\xi - \xi_0| > 1 - |x_0|\}$.

Since $|\xi - x_o| \geq 1 - |x_0|$ for $\xi \in \mathbb{S}^{n-1}$ we have

$$\begin{aligned}
\int_E \frac{|\xi - \xi_0|\omega(|\xi - \xi_0|)}{|x_0 - \xi|^{n+2}} d\sigma(\xi) &\leq (1-|x_0|)^{-n-2} \cdot \int_E |\xi - \xi_0|\omega(|\xi - \xi_0|) d\sigma(\xi) \\
&\leq (1-|x_0|)^{-n-2} \int_0^\delta \rho\omega(\rho)\rho^{n-2} d\rho \\
&\leq C \cdot \frac{\omega(\delta)}{\delta^2},
\end{aligned}$$

where $\delta = 1 - |x_0|$. Here we used property 5° of the modulus function ω.

On the other hand, there is a constant C_n such that

$$\frac{|\xi - \xi_0|}{\xi - x_0|} \leq C_n \quad \text{for} \quad \xi \in F$$

and therefore, using property 4° of ω, we have

$$\begin{aligned}
\int_F \frac{|\xi - \xi_0|\omega(|\xi - \xi_0|)}{|x_0 - \xi|^{n+2}} d\sigma(\xi) &\leq C_n^{n+2} \int_E \frac{\omega(|\xi - \xi_0|) d\sigma(\xi)}{|\xi - \xi_0|^{n+1}} \\
&\leq C \cdot \int_\delta^2 \rho^{-n-1}\omega(\rho)\rho^{n-2} d\rho \\
&\leq C \cdot \frac{\omega(\delta)}{\delta^2}, \qquad \delta = 1 - |x_0|.
\end{aligned}$$

Combining the above estimates for integrals over E and F we obtain, for $1 \leq j < n$,

$$\Big|\frac{\partial u}{\partial x_j}(x_0)\Big| \leq C(n, \omega) \cdot \frac{\omega(\delta)}{\delta}, \quad \delta = 1 - |x_0|$$

and this is precisely estimate $(*)$ in direction of coordinate axis x_j $(1 \leq j < n)$. However, the same estimate is true for any tangential direction, by rotational symmetry and $(*)$ is proved.

Now, $K-$quasiregularity gives the estimate of the derivative of u:

$$||u'(x)|| \leq KC(n, \omega)\frac{\omega(\delta)}{\delta}, \qquad \delta = 1 - |x|.$$

Using property 3° of ω and a simple argument involving integration one concludes the proof.

References

[1] Craig A. Nodler and Daniel M. Oberlin, *Moduli of continuity and a Hardy-Littlewood theorem.* In Complex analysis, Joensuu 1987, volume 1351 of *Lecture Notes in Math.*, pages 265-272. Springer, Berlin 1988.

[2] Arsenović M., Kojić V. and Mateljević M., *On Lipschitz continuity of harmonic quasiregular maps on the unit ball in* $\mathbb{R}^n$, Ann. Acad. Sci. Fenn., Volumen 33, 2008, 315-318.

[3] Antoni Zygmund, *Trigonometrical Series*, 2^{nd} ed. Vol I, II, Cambridge university Press, New York, 1959.

[4] Sheldon Axler, Paul Bourdon and Wade Ramey, *Harmonic Function Theory.*, Springer Verlag New York 1992.

[5] M. Vuorinen, *Conformal Geometry and quasiregular mappings*, Lecture Notes in Math., 1319, Springer, Berlin, 1988

[6] Pavlović M., *Remarks on* L^p*-oscillation of the modulus of a holomorphic function*, J. Math. Anal. Appl. 326 (2007), 1-11.

[7] Pavlović M., *Lipschitz conditions on the modulus of a harmonic function*, Revista Mat. Iberoamericana, 23 (2007), no. 3, 831-845.

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