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On the moment problem

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The purpose of this paper is to provide some additional insight into the moment problem by connecting a condition by Lin, Bondesson's class of hyperbolically completely monotone densities, and the theory of regularly varying functions. In particular, two questions addressed in a recent paper by Stoyanov concerning powers of random variables and functions that (do not) preserve uniqueness will be investigated.

Keywords: generalized gamma distribution; hyperbolically completely monotone; Krein integral; Lin condition; moment problem; moments; regular variation; slow variation; uniqueness

1. Introduction

The moment problem concerns the question whether or not a probability distribution or random variable is uniquely determined by the sequence of moments, all of which are supposed to exist. If the answer is positive we call the distribution or random variable *M-determinate*; if not, we call it *M-indeterminate*. So far no conveniently applicable necessary *and* sufficient condition has been found.

Inspired by a recent paper by Stoyanov (2000) and its reference to Lin (1997), and by connections to the theory of regularly varying functions and, apparently, to the Bondesson (1992) class $\mathcal{C}$ of probability densities that are hyperbolically completely monotone, the aim of this paper is to shed some additional light on the moment problem. However, since these densities have their support on the positive half-axis, we shall restrict ourselves to that case – the so-called *Stieltjes problem*.

Sections 2 and 3 contain some background material and preliminaries. An important theorem is given in Section 4, and is applied to some examples concerning powers of random variables in Section 5. Section 6 is devoted to a class of exponential distributions, and to a connection to the domain of attraction to the Gumbel distribution.

A typical behaviour of powers of random variables is that M-determinacy is preserved for 'low' powers, but lost for 'higher' powers. Motivated by a remark in Stoyanov (2000), we find criteria for when and how this happens in Section 7. Finally, inspired by another remark in Stoyanov (2000), we treat the problem of which functions (do not) preserve uniqueness.

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2. Background

We are thus interested in conditions for determining whether or not a given sequence of moments, $\{m_n,\ n \geqslant 1\}$, uniquely determines the associated probability distribution. A trivial sufficient condition is the existence of the moment generating function (mgf). A more subtle one is the Carleman condition, which, for non-negative random variables, states that the distribution is M-determinate if

$$\sum_{n=1}^{\infty} m_n^{-1/2n} = \infty. \tag{2.1}$$

We refer to Stoyanov (1997, Section 11; 2000) for surveys of the moment problem, some examples and further references. We also mention two recent papers by Pakes *et al.* (2001) and Pakes (2001).

In order to prove non-uniqueness in the absolutely continuous case there exists an integral test due to Krein (1944), the one-sided analogue of which is given by Slud (1993, Corollary 1), where it is shown that if the density f is positive on the whole positive half-axis, and

$$K = \int_0^{\infty} \frac{-\log f(x^2)}{1+x^2}\,\mathrm{d}x < \infty, \tag{2.2}$$

then the distribution is M-indeterminate (cf. also Lin 1997, Theorem 3). The integral in (2.2) is called the *logarithmic normalized integral*.

We close this section by setting the scene. Henceforth we assume that all random variables are positive and absolutely continuous, and that all moments are finite.

Definition 2.1. *The absolutely continuous random variable X satisfies the condition $X \in \mathcal{L}$ if, for some $x_0 \geqslant 0$, it has a positive and differentiable density f, and $f(x) \searrow 0$ and*

$$L(x) = -\frac{xf'(x)}{f(x)} = -x\frac{\mathrm{d}}{\mathrm{d}x}(\log f(x)) \nearrow \infty \qquad \text{as } x_0 < x \to \infty.$$

Remark 2.1. Whenever convenient, we shall write L_X for the L-function associated with X, and L_r for L_{X^r}. We shall also use L_g for the L-function of a function g, density or not.

Remark 2.2. Equivalently, $X \in \mathcal{L}$ if

$$\frac{\mathrm{d}}{\mathrm{d}x}(-\log f(\mathrm{e}^x)) \nearrow \infty \qquad \text{as } x_0 < x \to \infty.$$

The condition also implies (Bondesson 1992, p. 28) that the density of $\log X$ is log-concave, that is, strongly unimodal.

Our conclusions will be based on the following modified Krein integral:

$$K(a) = \int_a^\infty \frac{-\log f(x^2)}{1+x^2}\,\mathrm{d}x.$$

Definition 2.2. *The absolutely continuous random variable X satisfies the condition $X \in \mathcal{K}_c$ ($X \in \mathcal{K}_d$) if the density f is positive on (a, ∞) for some $a > 0$ and $K(a) < \infty$ ($K(a) = \infty$).*

Proposition 2.1. *Suppose that X possesses finite moments of all orders.*

(i) *If $X \in \mathcal{K}_c$, then X is* M-*indeterminate.*
(ii) *If $X \in \mathcal{L} \cap \mathcal{K}_d$, then X is* M-*determinate.*

Proof. Part (i) is Pakes (2000, Proposition 1), which extends Slud (1993, Corollary 1). Part (ii) is due to Lin (1997). □

The Lin function is intimately related to the theory of regularly varying functions. We therefore provide a definition and some properties. For an introduction to regular variation, see Feller (1971), de Haan (1970), Resnick (1987) and/or Bingham *et al.* (1987).

Definition 2.3. *A positive measurable function u on $[a, \infty)$, for some $a \geqslant 0$, varies regularly at infinity with exponent ρ, $-\infty < \rho < \infty$ – which we write as $u \in \mathcal{RV}(\rho)$ – if and only if*

$$\frac{u(tx)}{u(t)} \to x^\rho \quad \text{as } t \to \infty \qquad \text{for all } x > 0.$$

If $\rho = 0$ the function is said to be slowly varying at infinity – written $u \in \mathcal{SV}$.

The following lemma contains some properties, proofs of which can be found in the sources cited.

Lemma 2.1. *Let $u \in \mathcal{RV}(\rho)$ be positive on the positive half-axis.*

(i) *If $-\infty < \rho < \infty$, then*

$$u(x) = x^\rho \ell(x) \qquad \text{as } x \to \infty,$$

where $\ell \in \mathcal{SV}$. If, in addition, u has a monotone derivative u', then

$$L_u(x) = -\frac{xu'(x)}{u(x)} \to -\rho \qquad \text{as } x \to \infty.$$

If, moreover, $\rho \neq 0$, then $\operatorname{sgn}(u) \cdot u' \in \mathcal{RV}(\rho - 1)$.
(ii) *Let $\rho > 0$, and set $u^{-1}(y) = \inf\{x : u(x) \geqslant y\}$, $y \geqslant 0$. Then $u^{-1} \in \mathcal{RV}(1/\rho)$.*
(iii) $\log u \in \mathcal{SV}$.
(iv) *Suppose that $u_i \in \mathcal{RV}(\rho_i)$, $i = 1, 2$. Then $u_1 + u_2 \in \mathcal{RV}(\max\{\rho_1, \rho_2\})$.*
(v) *Suppose that $u_i \in \mathcal{RV}(\rho_i)$, $i = 1, 2$, that $u_2(x) \to \infty$ as $x \to \infty$, and set $u(x) = u_1(u_2(x))$. Then $u \in \mathcal{RV}(\rho_1 \cdot \rho_2)$.*

Remark 2.3. Since slow and regular variation are asymptotic properties, *ultimate* monotonicity of the derivatives is enough when necessary.

3. Preliminaries

In his Chapter 4, Bondesson (1992) introduces the *hyperbolically completely monotone* functions: a positive function g on $(0, \infty)$ is hyperbolically completely monotone if, for every $u > 0$, the function $g(uv)g(u/v)$ is a completely monotone function of $v + 1/v$. Some examples and properties are given in his Chapter 5. Next, he defines the class $\mathcal{C}$ of probability density functions that are hyperbolically completely monotone, and provides several examples and properties – it turns out that the class contains a wide collection of densities. Although it is the *density* that belongs to the class $\mathcal{C}$, we shall, for convenience, write $X \in \mathcal{C}$.

The important feature is that it follows from the arguments in Bondesson (1992, p. 72) that, in the present context, we must have $L(x) \nearrow \infty$ as $x \to \infty$, since all moments are finite, in particular $X \in \mathcal{L}$. This establishes the following fact.

Lemma 3.1. *Suppose that X has moments of all orders. If $X \in \mathcal{C}$, then $X \in \mathcal{L}$.*

The following two lemmas deal with the Lin condition and the Krein integral for powers of random variables.

Lemma 3.2. *If $X \in \mathcal{L}$, then $X^r \in \mathcal{L}$ for all $r > 0$.*

Proof. The conclusion follows easily from the fact that

$$L_r(x) = -x\left(\frac{1/r - 1}{x} + \frac{f'(x^{1/r})}{f(x^{1/r})} \cdot \frac{1}{r} x^{(1/r)-1}\right) = 1 - \frac{1}{r} + \frac{1}{r} L(x^{1/r}). \qquad \square$$

Remark 3.1. In order to check the Lin condition it thus suffices to check it for a suitable, possibly more tractable, power.

Remark 3.2. In particular, if $X \in \mathcal{C}$ then $X^r \in \mathcal{L}$ for all $r > 0$. However, $X^r \in \mathcal{C}$ *provided* $r > 1$ *only* (Bondesson 1992, p. 69). For more on this, see Remark 5.1.

Lemma 3.3. *Let $r > 0$. We have*

$$X^r \in \mathcal{K}_d \Leftrightarrow K_r(a) = \int_a^\infty \frac{y^{r-1}(-\log f(y^2))}{1 + y^{2r}}\, dy = \infty \quad \text{for some } a < 0.$$

Proof. Insert the density of X^r into the Krein integrals and put $y = x^{1/r}$. $\square$

We shall also exploit the following facts:

$$\int_0^\infty \frac{\log x}{1+x^2}\,\mathrm{d}x = 0, \tag{3.1}$$

$$\int_0^\infty \frac{x^\alpha |\log x|^\beta}{1+x^\gamma}\,\mathrm{d}x \begin{cases} < \infty & \text{if } \gamma - \alpha > 1,\ -\infty < \beta < \infty, \\ = \infty & \text{if } \gamma - \alpha \leqslant 1. \end{cases} \tag{3.2}$$

4. A theorem

Given the above prerequisites, the proof of the following theorem is fairly immediate.

Theorem 4.1. *Suppose that $X \in \mathcal{C}$, and that X has moments of all orders. If $X \in \mathcal{K}_d$, then X is* M-*determinate*.

Proof. Combine Lemma 3.1 and Theorem 2.1. □

Remark 4.1. Theorem 4.1 is obviously weaker than Lin's theorem. On the other hand – and this is the point – the class $\mathcal{C}$ provides us with a large class of densities for which the uniqueness problem thus reduces to checking the relevant Krein integral.

5. Powers of random variables

Berg (1988) proves *inter alia* that all even powers of order greater than 4 of a centred normal variable are M-indeterminate. Thus, even though a random variable may well be M-determinate, this need not necessarily be the case for a power. The following is another consequence of the preliminaries above. After a quick proof, we provide some illustrative examples. Although most of them have been treated before, we include them in order to show that the discussion becomes a lot simpler and swifter.

Theorem 5.1. *Let $r > 0$, and suppose that X possesses finite moments of all orders*.

(i) *If $K_r(a) < \infty$ for some $a > 0$, then X^r is* M-*indeterminate*.
(ii) *If $K_r(a) = \infty$ for some $a > 0$ and $X \in \mathcal{L}$, then X^r is* M-*determinate*.

Proof. Combine Lemmas 3.3 and 3.2, and Theorem 2.1. □

5.1. The lognormal distribution

This distribution has finite moments of all orders, but no mgf. Heyde (1963) proved that X is M-indeterminate by exhibiting a family of distributions having the same moment sequence; see also Shohat and Tamarkin (1943). Stoyanov (2000, Proposition 1), proves that X^r is M-indeterminate for every $r > 0$ by showing that $K_r(0) < \infty$ for all $r > 0$. Although this settles the problem, let us add that, since $X \in \mathcal{C}$ (Bondesson 1992, pp. 58–59), it follows from Lemma 3.1 that $X^r \in \mathcal{L}$ for every $r > 0$, that is, there exist random variables which are M-indeterminate for all $r > 0$ and yet satisfy the Lin condition for every $r > 0$.

5.2. The generalized gamma distribution

Let $\alpha, \beta, \gamma > 0$, and set

$$f(x) = C \cdot x^{\beta-1} \exp\{-x^\alpha/\gamma\}, \qquad x > 0.$$

This class is denoted by $GG(\alpha, \beta, \gamma)$ and consists of powers of gamma-distributed variables. For $\alpha = 1$ we have the gamma distribution and for $\alpha = 2$, $\beta = 1$ the absolute value of a centred normal variable. The class also contains the Weibull and the Rayleigh distributions. The moment problem has been dealt with in Pakes and Khattree (1992) and in Pakes *et al.* (2001) with different methods.

When $\alpha \geqslant 1$ the mgf exists, so that the distributions are M-determinate. As for the Lin condition, it is shown in Bondesson (1992, Chapter 5), that $X \in \mathcal{C}$ for $0 < \alpha < 1$. Thus $X \in \mathcal{L}$ (Lemma 3.1). Since $X^r \in GG(\alpha/r, \beta/r, \gamma)$, it follows that $X^r \in \mathcal{C}$ and, hence, that $X^r \in \mathcal{L}$ whenever $r > \alpha$. An application of Lemma 3.2 shows that $X^r \in \mathcal{L}$ for all α and r; in particular, all GG-distributed variables belong to $\mathcal{L}$.

It remains to check the Krein integral. Using (3.1), a simple computation shows that

$$K(a) < \infty \Leftrightarrow \int_a^\infty \frac{x^{2\alpha}}{1+x^2}\,\mathrm{d}x < \infty \Leftrightarrow \int_0^\infty \frac{x^{2\alpha}}{1+x^2}\,\mathrm{d}x < \infty.$$

Summarizing, we conclude that X is M-indeterminate for $0 < \alpha < \frac{1}{2}$, and, by Theorem 2.1, that X is M-determinate for $\alpha \geqslant \frac{1}{2}$.

For X^r we similarly obtain

$$K_r(a) < \infty \Leftrightarrow \int_a^\infty \frac{y^{r-1} y^{2\alpha}}{1+y^{2r}}\,\mathrm{d}y < \infty \Leftrightarrow \int_0^\infty \frac{y^{r-1} y^{2\alpha}}{1+y^{2r}}\,\mathrm{d}y < \infty,$$

that is, X^r is M-indeterminate when $r > 2\alpha$, and M-determinate when $r \leqslant 2\alpha$.

Remark 5.1. For $r < \alpha$, X^r provides an example of an M-determinate random variable that, in spite of possessing an mgf, belongs to $\mathcal{K}_d$ and to $\mathcal{L}$, but does *not* belong to the class $\mathcal{C}$; recall Remark 3.2.

5.3. A boundary case

If $X \in GG(\alpha, \beta, \gamma)$, the mgf exists for $\alpha \geqslant 1$, but not for $0 < \alpha < 1$. The density

$$f(x) = C \cdot x^{\beta-1} \exp\{-x/\ell(x)\}, \qquad x > 0,$$

where $\ell \in \mathcal{SV}$ and $\ell(x) \nearrow \infty$ as $x \to \infty$, is a boundary case in that the mgf 'barely' does not exist. For the case $\ell(x) = \log x$, see Stoyanov (1997).

Recalling (3.1),

$$K(a) = C + 0 + \int_a^\infty \frac{x^2}{(1+x^2)\ell(x^2)}\,\mathrm{d}x = \infty \qquad \text{for all } a > 0,$$

and, for $\ell(x)$ smooth enough,

$$L(x) = 1 - \beta + \frac{x}{\ell(x)} - \frac{x^2\ell'(x)}{(\ell(x))^2} = 1 - \beta + \frac{x}{\ell(x)} - \frac{x\ell'(x)}{\ell(x)} \cdot \frac{x}{\ell(x)} \to \infty \qquad \text{as } x \to \infty,$$

so that $X \in \mathcal{L}$, which proves that X is M-determinate.

As for powers, $X^r \in \mathcal{L}$ is automatic, $X^r \in \mathcal{K}_c$ for $r > 2$ and $X^r \in \mathcal{K}_d$ for $r < 2$, so that X^r is M-determinate for $r < 2$ and M-indeterminate for $r > 2$. For $r = 2$ the conclusion depends on the asymptotics of $\ell(x)$; cf. Remark 6.1.

5.4. The generalized inverse Gaussian distribution

Let $-\infty < \beta < \infty$ and $b_1, b_2 > 0$, and set

$$f(x) = C \cdot x^{\beta-1} \exp\{-(b_1 x + b_2/x)\}, \qquad x > 0.$$

This is another case where $X \in \mathcal{C}$ (Bondesson 1992, p. 59). The distribution is M-determinate, since the mgf exists. The case $\beta = -\frac{1}{2}$ corresponds to the inverse Gaussian distribution, which has been studied in Stoyanov (1997; 1999).

Now $K_r(a) < \infty$ (Lemma 3.3) if and only if

$$K_r'(a) = \int_a^\infty \frac{y^{r-1}((1-\beta)\log y + b_1 y^2 + b_2/y^2)}{1 + y^{2r}}\,\mathrm{d}y < \infty,$$

that is, if and only if $r > 2$. Furthermore, $X^r \in \mathcal{L}$ for all $r > 0$, since $X \in \mathcal{C}$. It follows that X^r is M-indeterminate for $r > 2$ and M-determinate for $r \leqslant 2$.

6. M-determinacy and regular variation

In his Comment 2, Stoyanov (2000, p. 947) mentions that densities of the form

$$f(x) = C \cdot u(x) \exp\{-v(x)\}, \qquad x > 0, \tag{6.1}$$

where u and v are positive functions on the positive half-axis, can be given a unified treatment with conditions expressed in terms of u and v. Although the problem can be studied more generally, we confine ourselves to the case $u \in \mathcal{RV}(\rho_u)$ and $v \in \mathcal{RV}(\rho_v)$. Once again we refer to the related paper by Pakes *et al.* (2001).

A first observation is that we must have $\rho_v > 0$ and $\rho_u > -1$ in order for all moments to exist (and conversely). The case $\rho_v = 0$, that is $v \in \mathcal{SV}$, is in fact also possible, depending on the slowly varying factor, but we refrain from going into details.

Theorem 6.1. *Let the density of X be given as in* (6.1) *with $u \in \mathcal{RV}(\rho_u)$, $\rho_u > -1$, and $v \in \mathcal{RV}(\rho_v)$, $\rho_v > 0$.*

(i) *If* $\rho_v < \frac{1}{2}$, *then* X *is* M-*indeterminate.*
(ii) *Furthermore, suppose that* u *and* v *have ultimately monotone derivatives. If* $\rho_v > \frac{1}{2}$, *then* X *is* M-*determinate.*

Canonical examples are the generalized gamma distributions. Note again that if $X \in \mathcal{C}$, then uniqueness is just a matter of determining ρ_v.

The proof follows via the following two lemmas.

Lemma 6.1. *Suppose that* $\rho_u > -1$ *and* $\rho_v > 0$. *Then* $X \in \mathcal{L}$.

Proof. We have

$$L_X(x) = -\frac{xu'(x)}{u(x)} + xv'(x) = L_u(x) - v(x)L_v(x) \sim -\rho_u + v(x)\rho_v \qquad \text{as } x \to \infty. \qquad \square$$

Lemma 6.2. *Suppose that* $\rho_u > -1$ *and* $\rho_v > 0$, *and set*

$$K_v(a) = \int_a^\infty \frac{x^{2\rho_v}\ell_v(x^2)}{1+x^2}\,\mathrm{d}x.$$

(i) $K(a) < \infty \Leftrightarrow K_v(a) < \infty \Leftrightarrow K_v = K_v(0) < \infty$.
(ii) *If* $\rho_v < \frac{1}{2}$, *then* $X \in \mathcal{K}_c$, *and if* $\rho_v > \frac{1}{2}$, *then* $X \in \mathcal{K}_d$.

Proof. Since $\log u \in \mathcal{SV}$, we have

$$\int_a^\infty \frac{|\log u(x^2)|}{1+x^2}\,\mathrm{d}x < \infty,$$

so that

$$K(a) = \int_a^\infty \frac{C - \log u(x^2) + v(x^2)}{1+x^2}\,\mathrm{d}x = C + \int_a^\infty \frac{v(x^2)}{1+x^2}\,\mathrm{d}x = C + \int_a^\infty \frac{x^{2\rho_v}\ell_v(x^2)}{1+x^2}\,\mathrm{d}x,$$

from which (i) and (ii) are immediate. $\square$

Remark 6.1. When $\rho_v = \frac{1}{2}$ the conclusion depends on the slowly varying contribution. Typical examples are $\ell(x) = 1/(\log x)^2$ and $\ell(x) = 1$.

Remark 6.2. With reference to the lognormal distribution, we mention that the proofs also cover the case $v(x) = (\log x)^\beta$ (with $\beta > 1$ since all moments exist).

Similar results can be given for the case when only one of u and v is regularly varying by compensating with some other assumption. This makes the result more general. However, the compensation for regular variation is 'what remains' to make the result come true.

We finally consider the case when the v-function is the sum of two regularly varying functions, one with a positive exponent and one with a negative one.

Theorem 6.2. *Let f be given as in* (6.1) *with* $u \in \mathcal{RV}(\rho_u)$, $-\infty < \rho_u < \infty$, *and* $v = v_1 + v_2$, *where* $v_1 \in \mathcal{RV}(\rho_1)$, $\rho_1 > 0$, *and* $v_2 \in \mathcal{RV}(-\rho_2)$, $\rho_2 > 0$.

(i) *If* $\rho_1 < \frac{1}{2}$, *then* X *is* M-*indeterminate.*
(ii) *Furthermore, suppose that* u, v_1 *and* v_2 *all have ultimately monotone derivatives. If* $\rho_1 > \frac{1}{2}$, *then* X *is* M-*determinate.*

Proof. The relevant Krein integral is

$$\int_a^\infty \frac{x^{2\rho_1}\ell_1(x^2) + x^{-2\rho_2}\ell_2(x^2)}{1+x^2}\,\mathrm{d}x, \tag{6.2}$$

which converges for $\rho_1 < \frac{1}{2}$ and diverges when $\rho_1 > \frac{1}{2}$ for every $a > 0$.

As for the Lin condition, it follows from Lemma 2.1 that

$$L_X(x) \sim -\rho_u + v(x)\rho_1 \sim -\rho_u + v_1(x)\rho_1 \qquad \text{as } x \to \infty. \qquad \square$$

Remark 6.3. Note that the conclusion is independent of ρ_2 in this case.

Remark 6.4. When ρ_1 or ρ_2 *equals* $\frac{1}{2}$ the convergence or divergence of the Krein integrals depends on the asymptotics of the slowly varying components.

The canonical example is the (generalized) inverse Gaussian distribution; recall Section 5. Using the notational convention $\log^+ x = \max\{1, \log x\}$, a more general example is

$$f(x) = C \cdot x^5 \frac{(\log^+ x)^4}{(\log^+ \log^+ x)^7} \exp\left\{-\left(\mathrm{x}^{2/3}\sqrt{\log^+ x} + \frac{x^{-1/3}}{\log^+ \log^+ \log^+ x}\right)\right\}, \qquad x > 0.$$

Analogous results can be formulated for the case when one v-function is slowly varying under some additional assumptions on the slowly varying factors, although for restricted ρ_u-ranges.

We conclude this section with a brief discussion of the connection between regular variation and sample extremes (de Haan 1970). Let F be a distribution function with density f. The function $R = f/(1-F)$ is called the *hazard rate* of F or the *intensity function*, depending on the context. A result, due to von Mises, states that if F has infinite upper end-point and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{R(x)}\right) \to 0 \qquad \text{as } x \to \infty,$$

then F belongs to the domain of attraction of the Gumbel distribution, the distribution function of which equals $\Lambda(x) = \exp\{-\mathrm{e}^{-x}\}$, $-\infty < x < \infty$; see, for example, Bingham *et al.* (1987, Theorem 8.13.7).

Let f be given by (6.1) with $u = v'$. The observation that

$$R(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\log(1 - F(x))$$

allows us to interpret v' as the hazard rate of F. Assuming that $v \in \mathcal{RV}(\rho_v)$ for some $\rho_v > 0$, that $v''(= u')$ exists, and that both derivatives are ultimately monotone, we obtain, after repeated use of Lemma 2.1,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{R(x)}\right) = -\frac{1}{(v'(x))^2}\cdot v''(x) = -\frac{L_{v'}(x)}{L_v(x)}\cdot\frac{1}{v(x)} \to -\frac{\rho_v - 1}{\rho_v}\cdot 0 = 0 \qquad \text{as } x \to \infty,$$

that is, these distributions belong to the domain of attraction of the Gumbel distribution.

An analogous argument (which we omit) can be made if $v = v_1 + v_2$, where $v_i \in \mathcal{RV}(\rho_i)$, $i = 1, 2$, with $\rho_1 > 0$ and $\rho_2 < 0$, both have ultimately monotone derivatives.

7. From what power on is X^r M-indeterminate?

In Section 5 we found that the typical situation seems to be that the powers of a random variable are M-determinate up to some level, after which they are M-indeterminate; cf. also Stoyanov (2000, p. 946). Is there a general pattern?

An examination of our findings concerning the generalized gamma distributions with $\alpha \geqslant 1$ suggests that, more generally, if X possesses an mgf, and therefore is M-determinate, then the same is true for X^r up to some power $r_m \geqslant 0$, after which X^r remains M-determinate up to another power $r_k \geqslant r_m$, because the Lin condition is satisfied and $X^r \in \mathcal{K}_d$, after which $X^r \in \mathcal{K}_c$, and hence is M-indeterminate. The case $0 < \alpha < 1$ suggests that when X has no mgf then there is just the boundary point r_k. The lognormal distribution, finally, calls for cases where all powers are M-indeterminate.

We now proceed to show that this is indeed the case. Thus, let the density of X be given by (6.1) with $u \in \mathcal{RV}(\rho_u)$, $\rho_u > -1$, and $v \in \mathcal{RV}(\rho_v)$, $\rho_v > 0$, and suppose that the mgf exists. Let $r > 1$. Upon observing that

$$E\exp\{tX^r\} \leqslant 1 \qquad \text{for } t \leqslant 0,$$

and that

$$E\exp\{tX^r\} \leqslant \mathrm{e}^t + \int_1^\infty \exp\{tx^r\} f(x)\,\mathrm{d}x \qquad \text{for } t > 0,$$

we define

$$r_m = \sup\{r : E\exp\{tX^r\} < \infty\}, \qquad \text{for some } t > 0.$$

Next, the Krein integral equals

$$K_r(a) = \int_a^\infty \frac{y^{r-1}(C - \log u(y^2) + v(y^2))}{1 + y^{2r}}\,\mathrm{d}y.$$

Since $\log u \in \mathcal{SV}$ (Lemma 2.1) it first follows that

$$\int_a^\infty \frac{y^{r-1}|\log u(y^2)|}{1 + y^{2r}}\,\mathrm{d}y < \infty \qquad \text{for all } a \geqslant 0.$$

An elementary investigation then shows that, for any $r > 1$,

$$0 < z(y) = \frac{y^{r-1}(1+y^2)}{1+y^{2r}} \leqslant 2 \qquad \text{for all } y > 0, \tag{7.1}$$

which implies that

$$K_r(a) < \infty \ (=\infty) \Leftrightarrow \int_a^\infty \frac{y^{r-1}v(y)}{1+y^{2r}}\,\mathrm{d}y = \int_a^\infty \frac{y^{r-1+2\rho_v}\ell_v(y^2)}{1+y^{2r}}\,\mathrm{d}y < \infty \ (=\infty).$$

Since convergence (divergence) of the integral holds simultaneously for all $a \geqslant 0$ (cf. Lemma 6.2), it follows that

$$X \in \mathcal{K}_c \ (X \in \mathcal{K}_d) \Leftrightarrow K_r^* = \int_0^\infty \frac{y^{r-1+2\rho_v}\ell_v(y^2)}{1+y^{2r}}\,\mathrm{d}y < \infty \ (=\infty).$$

Moreover, since $y^{r-1}/(1+y^{2r})$, *viewed as a function of* $r > 1$, is decreasing for any fixed $y > 0$, it follows in addition that K_r^*, *viewed as a function of* r, is decreasing for $r > 1$. We therefore define

$$r_k = \sup\{r \geqslant 1 : K_r^* = \infty\}.$$

Theorem 7.1. *Let f be given as in* (6.1) *with* $u \in \mathcal{RV}(\rho_u)$, $\rho_u > -1$, *and* $v \in \mathcal{RV}(\rho_v)$, $\rho_v > 0$, *both functions being positive*.

(i) *Then* X^r *is* M-*indeterminate for* $r > 2\rho_v$.

(ii) *Suppose that* $\rho_v > 0$ *and, in addition, that* u *and* v *have ultimately monotone derivatives. Then* X^r *is* M-*determinate for* $r < 2\rho_v$.

Proof. (i) follows from the fact that $r_k = 2\rho_v$, and (ii) follows since $X \in \mathcal{L}$ by Lemma 6.1, so that $X^r \in \mathcal{L}$ for all $r > 0$ by Lemma 3.2. □

Example 7.1. For the $GG(\alpha, \beta, \gamma)$ distribution we have $r_m = \alpha$, $r_k = 2\alpha$ when $\alpha \geqslant 1$, and $r_m = 0$, $r_k = 2\alpha$ when $\alpha < 1$, and the lognormal distribution (Remark 6.2 applies here too) has $r_m = r_k = 0$ ($\sup\{\varnothing\} = 0$).

Once again,

- if $X \in \mathcal{C}$ the problem reduces to an inspection of ρ_v;
- the boundary cases $r = r_k$ and $r = r_m$ require special attention;
- one can formulate results when v equals the sum of two regularly varying functions.

8. Functions preserving M-(in)determinacy

A topic 'currently under study' (Stoyanov 2000, p. 947) is the determination of functions that preserve or destroy M-determinacy and M-indeterminacy, respectively. In order to extend the results for powers, let X be positive and absolutely continuous with density f, suppose that g is positive and strictly increasing, let h denote the inverse, and set $Y = g(X)$. A trivial observation is that, if g is bounded, then Y is bounded, hence trivially M-determinate no matter what the case happens to be with X. In the remainder of this section we therefore suppose that g, and thus also h, increase to infinity.

Now suppose, in addition, that g is twice continuously differentiable, and note that the assumptions on g carry over to h.

Since the density of Y at the point y^2 equals $f_Y(y^2) = f(h(y^2))h'(y^2)2y$, for $y > 0$, the Krein integral for Y becomes, except for constants (recall (3.1)),

$$K_h(a) = \int_a^\infty \frac{-\log f(h(y^2)) - \log h'(y^2)}{1+y^2}\,\mathrm{d}y, \tag{8.1}$$

and the Lin function of Y becomes

$$L_Y(y) = -y\frac{\mathrm{d}}{\mathrm{d}y}(\log f(h(y)) + \log h'(y)) = -L_h(y)\cdot L_X(h(y)) = L_{h'}(y) \tag{8.2}$$

Next let f, as before, be as in (6.1) with $u \in \mathcal{RV}(\rho_u)$ and $v \in \mathcal{RV}(\rho_v)$, and suppose that $g \in \mathcal{RV}(\rho)$. Since g increases to $+\infty$, we must have $\rho \geqslant 0$. For simplicity we omit the somewhat special slowly varying case from the discussion.

Theorem 8.1. *Let X have density*

$$f(x) = C\cdot u(x)\exp\{-v(x)\}, \qquad x > 0,$$

where $u \in \mathcal{RV}(\rho_u)$, $\rho_u > -1$, and $v \in \mathcal{RV}(\rho_v)$, $\rho_v > 0$, both functions being positive. Furthermore, let $g \in \mathcal{RV}(\rho)$, $\rho > 0$, be positive with inverse h, and set $Y = g(X)$.

(i) *If $X \in \mathcal{K}_c$, then* M-*indeterminacy is preserved for $\rho > 2\rho_v$.*
(ii) *If X possesses an mgf, then* M-*determinacy is preserved for $\rho < \rho_v$.*
(iii) *Suppose, in addition, that u', v', g, g' and g'' are ultimately monotone. If $X \in \mathcal{L}\cap\mathcal{K}_d$, then* M-*determinacy is preserved for $\rho < 2\rho_v$.*

Proof. The proof amounts to determining r_m and r_k as defined in Section 7, and to checking the Lin condition.

The mgf of Y equals

$$C\int_0^\infty \exp\{tx^\rho \ell(x)\}x^{\rho_u}\ell_u(x)\exp\{-x^{\rho_v}\ell_v(x)\}\,\mathrm{d}x,$$

which converges for $\rho < \rho_v$ and diverges for $\rho > \rho_v$, that is, $r_m = \rho_v$.

The Krein integral in (8.1) equals

$$\int_a^\infty \frac{C - \log u(h(y^2)) + v(h(y^2)) - \log h'(y^2)}{1+y^2}\,\mathrm{d}y.$$

Repeated applications of Lemma 2.1 show that $\log u(h(\cdot)) \in \mathcal{SV}$, $h \in \mathcal{RV}(1/\rho)$, $v(h(\cdot)) \in \mathcal{RV}(\rho_v/\rho)$, and $\log h' \in \mathcal{SV}$, so that (cf. Lemma 6.2 and the proof of Theorem 7.1)

$$Y \in \mathcal{K}_c\ (Y \in \mathcal{K}_d) \Leftrightarrow K_h^* = \int_0^\infty \frac{v(h(y^2))}{1+y^2}\,\mathrm{d}y = \int_0^\infty \frac{y^{2\rho_v/\rho}\ell(y^2)}{1+y^2}\,\mathrm{d}y < \infty\ (=\infty),$$

from which we conclude that $r_k = 2\rho_v$.

Combining (8.2), the fact that $h \in \mathcal{R}(\rho^{-1})$, and Lemma 2.1, finally shows that

$$L_Y(y) \sim \rho^{-1} L_X(h(y)) + 1 - \rho^{-1} \to \infty \qquad \text{as } y \to \infty,$$

which concludes the proof. □

Remarks 8.1.

- The proof also works if X is lognormal.
- As for the case $g \in \mathcal{SV}$, if, for example, $g(x) = \log(1+x)$, $x > 0$, then $\mathrm{E}\exp\{tY\} = \mathrm{E}(1+X)^t < \infty$ for all t, that is, Y is always M-determinate, irrespective of X.
- Once again, if X and Y both belong to the class $\mathcal{C}$, then preservation or not is only a matter of checking Krein integrals.
- As before, results of this kind can be given for the case when v is a sum of two regularly varying functions.

Example 8.1. As mentioned above, powers (more generally, regularly varying functions) of random variables in $\mathcal{L}$ also belong to $\mathcal{L}$. However, this is not necessarily the case for functions growing more rapidly than regularly varying functions. An inspection of (8.2) shows that one complication might arise when $\lim_{y\to\infty} L_Y(y)$ is of the form $\infty - \infty$, although it follows from the proof above that this is not possible if g is regularly varying.

The other possible complication is when the first term on the right-hand side of (8.2) is of the form $0 \cdot \infty$. Let $\lambda > 1$, and set

$$g(x) = \exp\{(\log x)^\lambda\} \quad \left(= x^{(\log x)^{\lambda-1}}\right), \qquad x > 0.$$

The inverse is $h(y) = \exp\{(\log y)^{1/\lambda}\}$, $y > 0$. As far as regular variation is concerned, it is (well) known that $h \in \mathcal{SV}$. Moreover, g is *not* regularly varying. In fact,

$$\frac{g(tx)}{g(x)} = \exp\{(\log tx)^\lambda - (\log x)^\lambda\} = \exp\left\{(\log x)^\lambda\left(1 + \frac{\log t}{\log x}\right)^\lambda - 1\right\}$$

$$\sim \exp\left\{(\log x)^\lambda \frac{\lambda \log t}{\log x}\right\} \to \begin{cases} 0 & \text{for } 0 < t < 1, \\ \infty & \text{for } t > 1, \end{cases} \qquad \text{as } x \to \infty.$$

This means that g is what is called *rapidly varying at infinity*; cf. de Haan (1970), Bingham *et al.* (1987) and Resnick (1987) ($g(y)$ increases faster than any power of y).

Straightforward computations now show (recall (8.2)) that

$$L_Y(y) = \frac{1}{\lambda(\log y)^{1-1/\lambda}}\{L_X(\mathrm{e}^{(\log y)^{1/\lambda}}) - 1\} + 1 + \frac{1 - 1/\lambda}{\log y} \sim 0 \cdot \infty + 1 + 0 \qquad \text{as } x \to \infty.$$

If, for example, $X \in GG(\alpha, \beta, 1)$, then $Y \in \mathcal{L}$, since

$$L_Y(y) = \frac{\alpha e^{\alpha(\log y)^{1/\lambda}} - \beta}{\lambda(\log y)^{1-1/\lambda}} + 1 + \frac{1 - 1/\lambda}{\log y} \to \infty \qquad \text{as } y \to \infty.$$

If, on the other hand, X is lognormal with parameters μ and $\sigma^2 = 1$, then $L_X(x) = \log x - \mu + 1$, so that

$$L_Y(y) = \frac{(\log y)^{2/\lambda - 1}}{\lambda} - \frac{\mu}{\lambda(\log y)^{1-1/\lambda}} + 1 + \frac{1 - 1/\lambda}{\log y} \to \begin{cases} \infty, & \text{if } 1 < \lambda < 2, \\ 1.5, & \text{if } \lambda = 2, \\ 0, & \text{if } \lambda > 2; \end{cases}$$

in particular, $Y \in \mathcal{L}$ when $1 < \lambda < 2$ and $Y \notin \mathcal{L}$ when $\lambda \geqslant 2$.

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