On The Monitoring of Linear Profiles

Keunpyo Kim, Mahmoud A. Mahmoud, and William H. Woodall

Virginia Tech, Blacksburg, VA 24061-0439

We propose control chart methods for process monitoring when the quality of a process or product is characterized by a linear function. In the historical analysis of Phase I data, we recommend methods including the use of a bivariate T^2 chart to check for stability of the regression coefficients in conjunction with a univariate Shewhart chart to check for stability of the variation about the regression line. We recommend the use of three univariate control charts in Phase II. These three charts are used to monitor the *Y*-intercept, the slope, and the variance of the deviations about the regression line, respectively. A simulation study shows that this type of Phase II method can detect sustained shifts in the parameters better than competing methods in terms of average run length performance. We also relate the monitoring of linear profiles to the control charting of regression-adjusted variables and other methods.

Introduction

In many practical situations, the quality of a process or product is characterized by a relationship (or profile) between two or more variables instead of by the distribution of a single quality characteristic. Lawless et al. (1999) gave examples in automotive engineering. Kang and Albin (2000) presented two examples of situations for which product profiles are of interest. The first example involved aspartame (an artificial sweetener), which is characterized by the amount of dissolved aspartame per liter of water at different levels of temperature. In this case, there is a desired functional relationship between the amount of dissolved aspartame and the temperature. The other example is a semiconductor manufacturing application involving calibration of a mass

Mr. Kim is a Ph.D. student in the Department of Statistics. His email address is kekim@vt.edu

Mr. Mahmoud is a Ph.D. student in the Department of Statistics. His email address is mamahmou@vt.edu

Dr. Woodall is a Professor in the Department of Statistics. He is a Fellow of ASQ. His email address is bwoodall@vt.edu

flow controller in which the performance of the process is characterized by a linear function. Mestek et al. (1994) and Stover and Brill (1998) gave similar calibration applications.

Kang and Albin (2000) proposed Phase I and Phase II control chart methods for monitoring a process for which the quality of a product is characterized by a linear relationship. Stover and Brill (1998) also considered the Phase I problem, while Brill (2001) considered possible extensions to more general relationships than a linear one.

In this paper we propose statistical process control charts for monitoring in Phase II a process or product that is characterized by a linear profile. Kang and Albin (2000) proposed two control chart strategies to monitor such a process. One approach involves a multivariate T^2 chart. The other uses statistics based on the successive samples of deviations from the in-control line in a combination of an exponentially weighted moving average (EWMA) chart to monitor the average deviation and a range (R-) chart to monitor the variation of the deviations. Both approaches are described in the next section. Our method is more similar to their second approach. Instead of using the deviations from the in-control line, however, we code the independent variable so that the average value is zero and use the estimated regression coefficients from each sample, i.e., the estimates of the *Y*-intercept and slope, to construct two univariate EWMA charts. Also, we study two different one-sided EWMA charts for monitoring a process standard deviation as replacements for their R-chart. One of these charts was developed by Crowder and Hamilton (1992). In practice a two-sided chart might be more appropriate in many applications to detect decreases in variability about the regression line as well as increases.

The statistical performance of the combined use of the three EWMA charts is compared to that of the methods of Kang and Albin (2000) later in our paper. We used simulation to show that our proposed method has better overall performance than the competing methods. Later in the paper we make recommendations for the Phase I analysis.

The monitoring of linear profiles is very closely related to the control charting of regression-adjusted variables as proposed by Mandel (1969), Zhang (1992), Hawkins (1991, 1993), Wade and Woodall (1993), and Hauck et al. (1999). In these approaches a regression model is often used to account for the effect of an input quality variable *X* on the output quality variable *Y* when monitoring a particular stage of a manufacturing process. One can adjust the output variable, however, based on any number of upstream process or quality variables in what Hawkins (1993) referred to as a "cascade process." In a cascade process, variables have a natural ordering and if any variable undergoes a parameter shift, it may affect some or all of the variables following it but none preceding it. The use of regression adjustment of a single quality variable based on a simple linear regression model is very similar to the linear profile situation except that the Phase I data usually consist of a single set of bivariate data points. In Phase II, one observes a sequence of deviations from the predicted values of *Y* based on the fitted Phase I regression model. In the regression-adjusted applications, however, the *X*-variable is usually considered to be a random variable, not taking fixed values as typically assumed in the linear profile monitoring application.

The problem of modeling and monitoring process or product quality using a function has been approached with other methods. Walker and Wright (2002) used additive models to represent the curves of interest in the monitoring of density profiles of particleboard. Jin and Shi (2001) used wavelets to monitor "waveform signals" for diagnosis of process faults. The use of linear functions as responses in designed experiments has also been studied recently. See, for example, Miller (2002) and Nair et al. (2002).

Phase II Methods

We assume that for the j^{th} random sample collected over time we have the observations (x_i, y_{ij}), $i = 1, 2, \ldots, n$. It is assumed that when the process is in statistical control, the underlying model is

$$Y_{ij} = A_0 + A_1 X_i + e_{ij}\ , i = 1, 2, \ldots, n, \tag{1}$$

where the e_{ij}'s are independent, identically distributed (i.i.d.) normal random variables with mean zero and variance σ^2. For simplicity, we first consider the case for which the X-values are fixed and take the same set of values for each sample. In this section we consider the Phase II case in which the in-control values of the parameters A_0, A_1, and σ^2 in Equation (1) are assumed to be known.

Kang and Albin (2000) proposed two strategies to monitor a process when the regression parameters are all known. Their first strategy is to use a bivariate T^2 chart to monitor the regression coefficients. This chart is based on the fact that the least squares estimators of A_0 and A_1 have a bivariate normal distribution. The least squares estimators of A_0 and A_1 for sample j are given by the following formulas:

$$a_{0j} = \bar{y}_j - a_{1j}\bar{x} \quad \text{and} \quad a_{1j} = \frac{S_{xy(j)}}{S_{xx}}, \tag{2}$$

where $\bar{y}_j = \frac{1}{n}\sum_{i=1}^{n} y_{ij}$, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $S_{xy(j)} = \sum_{i=1}^{n}(x_i - \bar{x})y_{ij}$, and $S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2$.

The least squares estimators a_{0j} and a_{1j} have a bivariate normal distribution with the mean vector

$$\boldsymbol{\mu} = (A_0, A_1)^T \tag{3}$$

and the variance-covariance matrix

$$\boldsymbol{S} = \begin{pmatrix} s_0^2 & s_{01}^2 \\ s_{01}^2 & s_1^2 \end{pmatrix}, \tag{4}$$

where $s_0^2 = s^2\left(\frac{1}{n}+\frac{\bar{x}^2}{S_{xx}}\right)$, $s_1^2 = s^2\frac{1}{S_{xx}}$ and $s_{01}^2 = -s^2\frac{\bar{x}}{S_{xx}}$ are the variance of a_{0j}, the variance of a_{1j} and the covariance between a_{0j} and a_{1j}, respectively.

In their first monitoring strategy one computes the vector of sample estimators $\boldsymbol{Z}_j = (a_{0j}, a_{1j})^T$ for sample j, where a_{0j} and a_{1j} are the sample intercept and the sample slope as defined in Equation (2). Then one calculates the T^2 statistic given by

$$T_j^2 = (\boldsymbol{Z}_j - \boldsymbol{\mu})^T \boldsymbol{S}^{-1}(\boldsymbol{Z}_j - \boldsymbol{\mu}), \quad (5)$$

where $\boldsymbol{\mu}$ and $\boldsymbol{S}$ are defined as in Equations (3) and (4), respectively. When the process is in-control, T_j^2 follows a central chi-square distribution with 2 degrees of freedom. Therefore, the recommended upper control limit for the chart is UCL= $c_{2,a}^2$, where $c_{2,a}^2$ is the $100(1-a)$ percentile of the chi-square distribution with 2 degrees of freedom. When there are shifts from the in-control parameter values, Kang and Albin (2000) pointed out that T_j^2 in Equation (5) follows a non-central chi-square distribution with a non-centrality parameter

$$t = n(l + b\ \bar{x})^2 + b^2 S_{xx},$$

where l is the shift in the intercept A_0 and b is the shift in the slope A_1. Both l and b are measured in units of s by Kang and Albin (2000), so the actual shifts in this case are $l s$ and $b s$ from A_0 and A_1, respectively.

Kang and Albin (2000) studied the average run length (ARL) performance of their T^2 control chart using a simulation study. However, since T_j^2 has a non-central chi-square distribution, one can evaluate the exact ARL of the T^2 chart using the following formula:

$$\text{ARL}=\frac{1}{\Pr(T_j^2 > c^2_{2,a})}.$$

Their second strategy is to apply some standard control chart schemes to the regression "residuals" obtained at sample j using

$$e_{ij} = y_{ij} - A_0 - A_1 x_i \ , i = 1, 2, \ldots, n. \tag{6}$$

They applied an EWMA chart to monitor the average value of these deviations. They suggested an R-chart be used in combination with the EWMA chart.

The average of the residuals for the j^{th} sample is denoted by $\bar{e}_j$, calculated using

$$\bar{e}_j = \frac{\sum_{i=1}^{n} e_{ij}}{n}.$$

The EWMA control chart statistics, denoted by z_j, $j = 1, 2, \ldots$, are given by

$$z_j = q\ \bar{e}_j + (1-q) z_{j-1}, \tag{7}$$

with q $(0 < q \le 1)$ a smoothing constant and $z_0 = 0$. An out-of-control signal is given as soon as z_j is less than the lower control limit (LCL) or z_j is greater than the upper control limit (UCL), where

$$LCL = -L s \sqrt{\frac{q}{(2-q)n}}\ , \quad UCL = L s \sqrt{\frac{q}{(2-q)n}} \tag{8}$$

and $L(> 0)$ is a constant selected to give a specified in-control ARL. Kang and Albin (2000) proposed an R-chart to be used in conjunction with this EWMA chart for two reasons. The first reason is to detect shifts in the process variance s^2 since the EWMA chart is not sensitive to shifts in the process variation. The other reason is that the EWMA chart based on the average residual is not sensitive to some shifts in A_0 and A_1 for which the magnitudes of the residuals tend to be large, but the average residual tends to be very small. This can occur, for example, when the slope of the line changes, but the average value of Y does not.

For the R-chart, Kang and Albin (2000) recommended that one calculate and plot the sample ranges $R_j = \max_i(e_{ij}) - \min_i(e_{ij})$, j = 1, 2, …. The lower and upper control limits for the R-chart are

$$LCL = s\,(d_2 - Ld_3) \text{ and } UCL = s\,(d_2 + Ld_3), \tag{9}$$

respectively, where $L(> 0)$ is a constant chosen to give a specified in-control ARL. The values of d_2 and d_3 are commonly-used constants that depend on the sample size n. These values can be found, for example, in tables in Montgomery (2001) or Ryan (2000). One disadvantage of this approach, however, is that if n < 7 there is no lower control limit and one cannot detect decreases in variation about the regression line.

In our proposed alternative approach, we first code the X-values so that the average coded value is zero. This simplifies the resulting analysis and removes much of the need for a T^2 approach since, in this situation, the least squares estimators of the Y-intercept and slope for each sample are independent random variables. See, for example, Myers (1990, pp. 11-15) and Ryan (1997, pp. 38-39).

After transforming the X-values, we obtain an alternative form of the underlying model in Equation (1) as

$$Y_{ij} = B_0 + B_1 X_i' + e_{ij},\ i = 1, 2, \ldots, n, \tag{10}$$

where $B_0 = A_0 + A_1\overline{X}$, $B_1 = A_1$, and $X_i' = (X_i - \overline{X})$.

For the j^{th} sample, the least squares estimator of B_0 is $b_{0j} = \overline{y}_j$ while the least squares estimator of B_1 is the same as that for A_1 in Equation (2). Both b_{0j} and b_{1j} are normally distributed with means B_0 and B_1 and variances $\frac{s^2}{n}$ and $\frac{s^2}{S_{xx}}$, respectively. Also the covariance between b_{0j} and b_{1j} is zero in Equation (4). Thus, we can apply a separate control chart for each sequence of random variables without the problems that would result if the estimators were highly correlated.

Our proposed method in Phase II to detect sustained shifts in the underlying parameters is to use three separate univariate control charts. We use EWMA charts to monitor the *Y*-intercept, the slope and the error variance, respectively, although one could use other types of charts, such as the cumulative sum and Shewhart charts, if desired. The three EWMA charts are used jointly, with the combination of charts signaling with the first chart to signal. The basic motivation for our proposed method comes from the fact that when assignable causes are present in a process for which the output of a product is characterized by a linear profile, it seems reasonable that at least one of three parameters, the *Y*-intercept, the slope and the error variance, would be directly affected. Thus, use of the three EWMA charts should result in at least one signal when there has been a sustained change in the process. Also, having a control chart corresponding to each parameter leads to an easier diagnosis of any process change than the omnibus methods of Kang and Albin (2000).

As an example, once the parameters are established in a baseline calibration study, one would want to detect any change in the *Y*-intercept or slope and any increase in the variation about the regression line since such shifts correspond to greater inaccuracies in the measurement process. A change in the *Y*-intercept corresponds to the introduction of bias and a change in the slope could lead to a dilation or contraction of the measurement scale. A decrease in the variation about the line would correspond to an improvement in the measurement process, as long as the other parameters do not change. The effect of assignable causes, however, will vary from application to application. Sometimes, for example, one may wish to detect isolated outliers.

In our proposed method, we first incorporate the technique proposed by Crowder and Hamilton (1992), where an EWMA chart based on the logarithmic transformation of the sample variances was used to monitor for increases in process variability. Crowder and Hamilton (1992) showed, as expected, that their EWMA chart is superior to the usual R-chart in detecting small and moderate-sized increases in a process variance. If one wishes to detect decreases in variation, as will frequently be the case, then appropriate methods are discussed by Acosta-Mejia et al. (1999) and Lowry et al. (1995).

For the EWMA chart for monitoring the *Y*-intercept (B_0), we use the estimates of the *Y*-intercept, b_{0j}, to compute the EWMA statistics

$$\text{EWMA}_\text{I}(j) = ?\, b_{0j} + (1 - ?)\, \text{EWMA}_\text{I}(j-1), \qquad (11)$$

j = 1, 2, …, with q $(0 < q \leq 1)$ a smoothing constant and $\text{EWMA}_\text{I}(0) = B_0$. An out-of-control signal is given as soon as $\text{EWMA}_\text{I}(j)$ < LCL or $\text{EWMA}_\text{I}(j)$ > UCL, where

$$\text{LCL} = B_0 - L_I s \sqrt{\frac{q}{(2-q)n}} \text{ and UCL} = B_0 + L_I s \sqrt{\frac{q}{(2-q)n}}. \qquad (12)$$

In Equation (12), $L_I (> 0)$ is chosen to give a specified in-control ARL.

The estimates of the slope B_1, b_{1j}, are used in the EWMA chart for monitoring the slope. The EWMA statistics are given by

$$\text{EWMA}_\text{S}(j) = ?\, b_{1j} + (1 - ?)\, \text{EWMA}_\text{S}(j-1), \qquad (13)$$

j = 1, 2, …, with q $(0 < q \le 1)$ a smoothing constant and EWMA$_S$(0) = B_1. The lower and upper control limits for the chart are given by

$$\text{LCL} = B_1 - L_S s \sqrt{\frac{q}{(2-q)S_{xx}}} \text{ and } \text{UCL} = B_1 + L_S s \sqrt{\frac{q}{(2-q)S_{xx}}}, \qquad (14)$$

respectively, where $L_S (> 0)$ is chosen to give a specified in-control ARL.

Finally, we consider the EWMA chart for monitoring the error variance (s^2) based on the approach of Crowder and Hamilton (1992). In this proposed method, we apply the values of MSE_j, i.e., the usual estimator of s^2 based on the residuals corresponding to the fitted line at sample j, to obtain the EWMA statistics instead of their use of sample variances. In contrast to the two EWMA charts described above, this chart is a one-sided EWMA scheme to detect only increases in process variability. We have the EWMA statistics

$$\text{EWMA}_E(j) = \max\{q \; ln(MSE_j) + (1-q)\,\text{EWMA}_E(j\text{-}1),\; ln(s_0^2)\} \qquad (15)$$

for j = 1, 2, …, with q $(0 < q \le 1)$ again a smoothing constant and EWMA$_E$(0) = $ln(s_0^2)$. In our proposed method, the assumption that s_0^2, the in-control value of s^2, is 1 is made without loss of generality, so we have EWMA$_E$(0) = 0. Lawless (1980, pp 21-23) provided an exact expression for Var[$ln(MSE_j)$] using the log-gamma distribution, but for convenience we use the following approximation that is very similar to a result derived by Crowder and Hamilton (1992):

$$\text{Var}[ln(MSE_j)] \,\tilde{}\, \frac{2}{n-2} + \frac{2}{(n-2)^2} + \frac{4}{3(n-2)^3} - \frac{16}{15(n-2)^5}.$$

The procedure signals when $\text{EWMA}_E(j)$ is greater than an upper control limit given by

$$\text{UCL} = L_E \sqrt{\frac{q}{(2-q)} \text{Var}[\ln(MSE_j)]} \qquad (16)$$

and the multiplier $L_E (> 0)$ is again chosen to give a specified in-control ARL. One could, of course, use a two-sided procedure if detecting decreases in the error variance was also considered to be important, as will frequently be the case in applications.

ARL Comparisons

In this section we compare the ARL performance of our EWMA-based method in Phase II to the performance of the methods proposed by Kang and Albin (2000). We consider the same example used in their simulation study. All chart combinations studied are designed to have the same overall in-control ARL of 200. The smoothing constants q in Equations (11), (13), and (15) are set equal to 0.2 as in the EWMA chart used by Kang and Albin (2000). One could, of course, use different smoothing constants for each chart. In general, smaller smoothing constants lead to quicker detection of smaller shifts, as shown by Lucas and Saccucci (1990). A total of 10,000 replications were used in our simulation study to estimate each ARL value. Exact ARL values, however, are given for the T^2 chart method.

The underlying in-control linear profile model used by Kang and Albin (2000) is $Y_{ij} = 3 + 2X_i + e_{ij}$, where the e_{ij}'s are i.i.d. normal random variables with mean zero and variance one. The fixed X_i-values of 2, 4, 6, and 8 (with $\bar{x} = 5$) were used in their simulation study (Albin (2002)). In our proposed method, we transform these X_i-values using the alternate form of the model in Equation (10) and obtain the new linear function $Y_{ij} = 13 + 2X_i' + e_{ij}$, where the e_{ij}'s are i.i.d. N(0,1) random variables. The X_i'-values are -3, -1, 1, and 3 with $\bar{x}' = 0$.

We denote our combination of the three EWMA charts by EWMA_3. In EWMA_3, we design each of the three EWMA charts to have the same in-control ARL when considered individually. For the EWMA chart for monitoring the *Y*-intercept, L_I in Equation (12) is chosen as 3.0156 to give the in-control ARL of 584.7. For the EWMA chart for monitoring the slope, L_S in Equation (14) is set equal to 3.0109 to have an in-control ARL of 584.6. Finally, L_E in Equation (16) is chosen as 1.3723 to achieve the in-control ARL of 584.4 for the EWMA chart for monitoring the error variance. The combination of all three EWMA charts has an overall in-control ARL of roughly 200.

The EWMA and R charts used by Kang and Albin (2000) in their simulation study have the same multiplier L in Equations (8) and (9). We found that the value L =3.1151 yields an in-control ARL of approximately 802 for the EWMA chart for monitoring the average deviation and an in-control ARL of approximately 261 for the R-chart, which has no lower control limit with $n = 4$. These ARL estimates were each obtained using 10,000 simulations. The in-control ARL of their combined procedure is close to 200, as they claim.

Four different types of shifts are considered in our simulation study. These are intercept shifts and slope shifts under the model in Equation (1), error variance increases, and slope shifts under the model in Equation (10) for which the average value of the independent variable is coded to be zero. The plots in each of our figures are based on ARL values for 200 equally spaced shifts within each range of shifts considered.

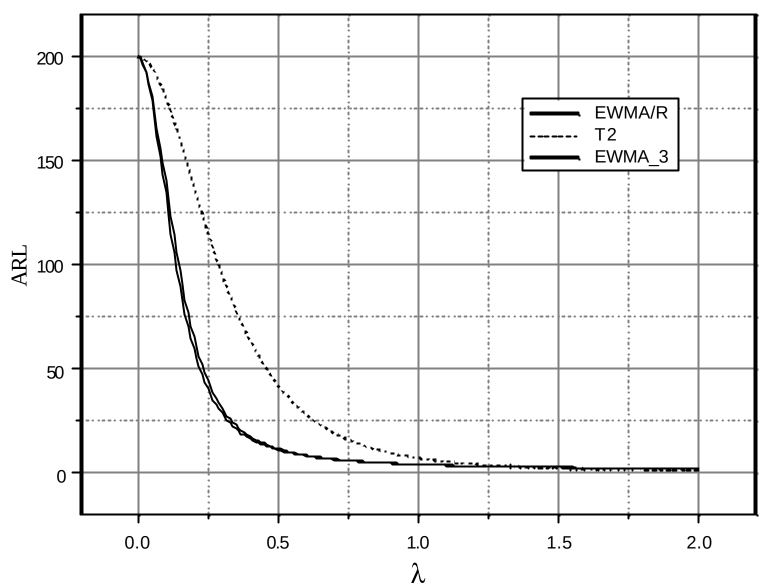
Table 1 and Figure 1 give the ARL values for shifts in A_0 (or equivalently in B_0) in units of s. (Kang and Albin (2000) measured the sizes of the shifts in these units, although use of units of $s/n^{1/2}$ would lead to greater generality.) Our proposed method, EWMA_3, performs a little better than the EWMA/R chart and much better than the T^2 chart in detecting small-to-moderate shifts in the intercept. The T^2 chart works better than our proposed method for large shifts. For improved performance for the EWMA_3 method in detecting large shifts, one could incorporate Shewhart limits into each of the three EWMA charts in a manner discussed by Lucas and Saccucci (1990). Our use of the constant asymptotic control limits instead of the “exact” limits discussed by Steiner (1999) also results in a slight delay in detecting shifts occurring with the first sample as is assumed in our simulations.

Table 1. ARL Comparisons Under Intercept Shifts From A_0 To $A_0 + l\,s$

(In-control ARL = 200)

	l									
Chart	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
EWMA/R	66.5	17.7	8.4	5.4	3.9	3.2	2.7	2.3	2.1	1.9
T^2	137.7	63.5	28.0	13.2	6.9	4.0	2.6	1.8	1.5	1.2
EWMA_3	59.1	16.2	7.9	5.1	3.8	3.1	2.6	2.3	2.1	1.9

Figure 1. ARL Comparisons Under Intercept Shifts λ



In Table 2 and Figure 2, we consider the case when there are shifts in the slope parameter A_1 in model (1) in units of s . Our proposed method performs uniformly slightly better than the EWMA/R chart over the entire range of shifts considered. It has much better performance than the T^2 chart except for the largest shift considered. The difference here, however, is relatively small. The use of a multivariate EWMA control chart, as proposed by Lowry et al. (1992), would result in better statistical performance than that of the T^2 chart, but the interpretation of an out-of-control signal would still not be straightforward.

Table 2. ARL Comparisons Under Slope Shifts in Model (1) From A_1 To $A_1 + b\,s$ (In-control ARL = 200)

	b									
Chart	0.025	0.050	0.075	0.100	0.125	0.150	0.175	0.200	0.225	0.250
EWMA/R	119.0	43.9	19.8	11.3	7.7	5.8	4.7	3.9	3.4	3.0
T^2	166.0	105.6	60.7	34.5	20.1	12.2	7.8	5.2	3.7	2.7
EWMA_3	101.6	36.5	17.0	10.3	7.2	5.5	4.5	3.8	3.3	2.9

Figure 2. ARL Comparisons Under Slope Shifts β

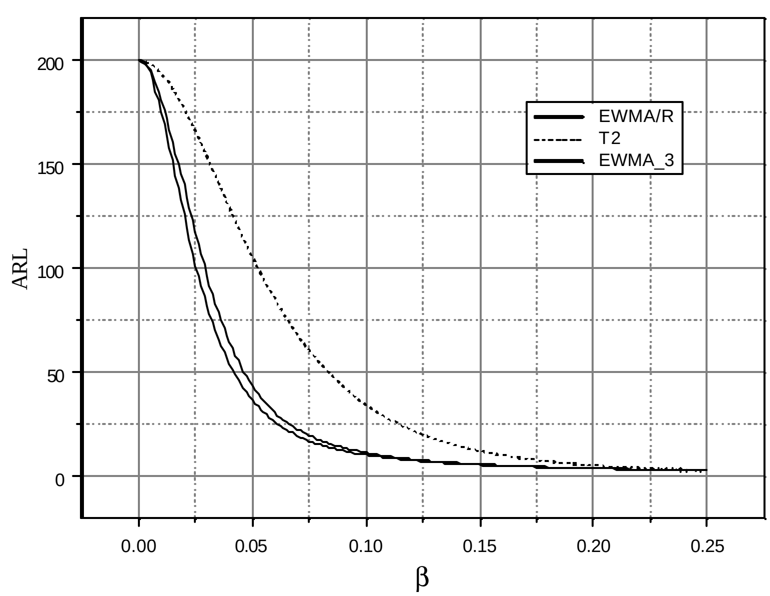


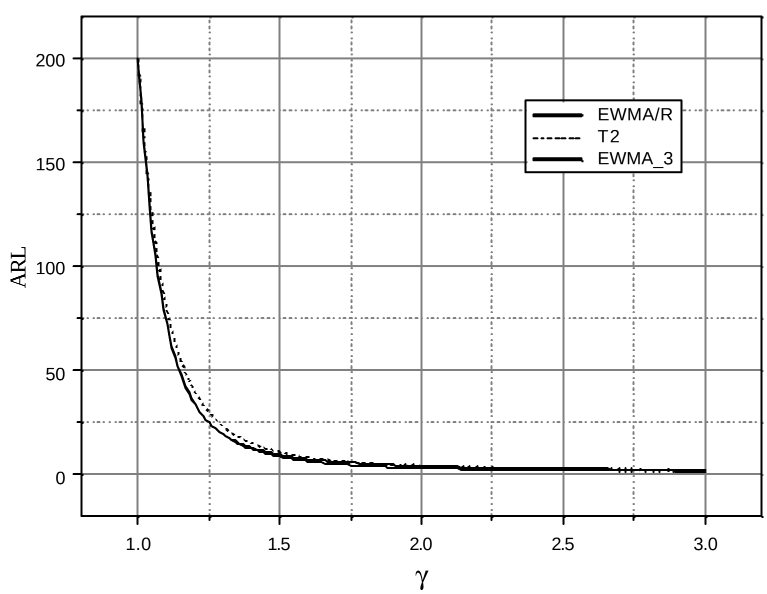
Table 3 and Figure 3 show that the ARL performance of our proposed method is comparable to that of the T^2 chart for detecting increases in s. Our method does not work quite as well as the EWMA/R method because the standard deviation shifts are mainly detected by the R-chart. The in-control ARL value of 261 for the R-chart is much lower than the in-control ARL value of 584.4 for our EWMA chart designed to detect shifts in the standard deviation. It was surprising to us that the T^2 chart worked so well given that no information is accumulated over time as with the EWMA charts. Since an increase in the standard deviation increases the variance of the estimators of the *Y*-intercept and slope, smoothing these estimators over time with EWMAs does not appear to be helpful.

Table 3. ARL Comparisons Under Standard Deviation Shifts From s To gs

(In-control ARL = 200)

	g									
Chart	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
EWMA/R	34.3	12.0	6.1	3.9	2.9	2.3	1.9	1.7	1.5	1.4
T^2	39.6	14.9	7.9	5.1	3.8	3.0	2.5	2.2	2.0	1.8
EWMA_3	33.5	12.7	7.2	5.1	3.9	3.2	2.8	2.5	2.3	2.1

Figure 3. ARL Comparisons Under Standard Deviation Shifts γ



One of the reviewers of our manuscript stated that an EWMA chart based on the natural units, instead of the logarithmic scale, would provide improved performance in detecting increases in the variation about the line. To investigate this issue, we compared the ARL performance of EWMA_3 with the EWMA chart of Crowder and Hamilton (1992) replaced by an EWMA chart with EWMA statistics

$$\text{EWMA}_{\text{N}}(j) = \max\{q\ (MSE_j - 1) + (1-q)\ \text{EWMA}_{\text{N}}(j\text{-}1),\ 0\}$$

for j = 1, 2, …, with q $(0 < q \le 1)$ a smoothing constant and $\text{EWMA}_{\text{N}}(0) = 0$. The procedure signals when $\text{EWMA}_{\text{N}}(j)$ is greater than an upper control limit given by

$$\text{UCL} = L_N \sqrt{\frac{q}{(2-q)} \text{Var}[MSE_j]}\ .$$

The multiplier $L_N\ (> 0)$ is again chosen to give a specified in-control ARL. We used $L_N = 4.0734$ to achieve the in-control ARL of 589.3 for this chart. The combination of all three EWMA charts has an overall in-control ARL of roughly 200. The ARL comparisons in Table 4 show, indeed, that use of the natural units does result in slightly better performance. The substitution of this EWMA chart in place of that of Crowder and Hamilton (1992) did not affect any of the other reported ARL values of our proposed method for shifts in any of the other parameters.

Table 4. ARL Comparisons Between the Natural-scale and Log-scale EWMA_3 Under Standard Deviation Shifts (In-control ARL = 200)

	g									
Chart	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
Log-Scale	33.5	12.7	7.2	5.1	3.9	3.2	2.8	2.5	2.3	2.1
Natural-Scale	31.6	11.7	6.5	4.5	3.4	2.7	2.3	2.1	1.8	1.7

Kang and Albin (2000) considered the case for which the relationship between the intercept shift, l (in units of s) and the slope shift, b (in units of s) is given as $l + b\ \bar{x} = 0$ with $\bar{x} = 5$. However, after transforming the X_i-values (2, 4, 6, and 8) into X_i'-values (-3, -1, 1, and 3), this case corresponds to simply a change in slope for the model in Equation (10). In Table 5 and Figure 4, we denote by d the shift in the slope (in units of s) in Equation (10). (Note that use of units of $s/(S_{xx})^{1/2}$ would lead to some greater generality.) Since it can be shown that the ARL values for shifts d are symmetric around $d = 0$, we use positive shifts in Figure 4. When the slope B_1 in Equation (10) shifts a small-to-moderate amount, one can observe that our EWMA_3 method has much better ARL performance. As the shift sizes increase, we see slightly higher out-of-control ARL values compared with the T^2 chart, again a situation that could be remedied to some extent by the use of Shewhart limits with the EWMA charts or control limits based on the exact variances of the EWMA statistics.

Table 5. ARL Comparisons Under Slope Shifts in Model (10) From B_1 To $B_1 + ds$

(In-control ARL = 200)

	d								
Chart	-0.2	-0.3	-0.4	-0.5	-0.6	-0.7	-0.8	-0.9	-1.0
EWMA/R	76.7	33.7	15.3	7.5	4.2	2.6	1.8	1.4	1.2
T^2	52.2	21.2	9.6	4.9	2.9	1.9	1.5	1.2	1.1
EWMA_3	13.1	6.6	4.4	3.3	2.7	2.3	2.1	1.9	1.7

Figure 4. ARL Comparisons Under Slope Shifts δ

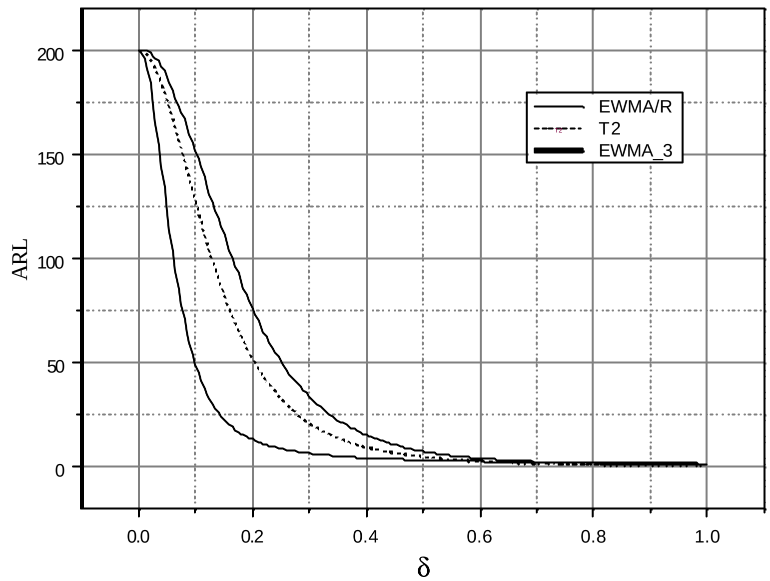


Table 6 gives the ARL values when simultaneous intercept shifts and slope shifts under the model in Equation (10) are considered. Our proposed method performs uniformly better than the EWMA/R chart combination.

Table 6. ARL Comparisons of EWMA/R Charts and EWMA_3

Under Combinations of Intercept and Slope Shifts in Model (10)

(In-control ARL = 200)

EWMA/R ARL EWMA_3 ARL		*d*									
		0.025	0.050	0.075	0.100	0.125	0.150	0.175	0.200	0.225	0.250
l	0.05	179.1 157.6	169.9 114.7	156.6 74.8	140.8 48.3	123.2 32.2	105.1 22.5	88.8 16.9	73.4 13.2	60.2 10.7	49.6 8.9
	0.10	139.5 122.1	133.6 94.6	125.4 66.4	115.5 44.9	103.5 30.7	90.4 21.9	78.3 16.6	65.7 13.1	55.6 10.6	46.3 8.9
	0.15	96.8 84.6	94.2 70.8	90.3 54.5	85.1 39.6	78.5 28.5	70.9 20.9	63.0 16.1	55.3 12.8	47.7 10.4	40.9 8.8
	0.20	64.8 57.1	63.8 51.1	62.1 42.4	59.7 33.3	56.6 25.4	52.9 19.5	48.5 15.4	44.0 12.4	39.2 10.2	34.6 8.7
	0.25	44.3 39.5	43.8 36.5	42.9 32.3	41.8 27.1	40.3 22.0	38.4 17.8	36.1 14.4	33.6 11.9	30.8 10.0	28.1 8.5
	0.30	31.0 28.2	30.8 26.9	30.5 24.7	29.9 22.0	29.2 18.8	28.3 15.7	27.1 13.2	25.7 11.2	24.2 9.6	22.5 8.3

	0.35	22.9 20.9	22.8 20.2	22.6 19.1	22.2 17.6	21.9 15.8	21.4 13.9	20.7 12.1	19.9 10.5	19.1 9.1	18.0 8.0
	0.40	17.4 16.2	17.3 15.9	17.2 15.3	17.1 14.5	16.8 13.5	16.6 12.1	16.2 10.9	15.8 9.7	15.3 8.6	14.7 7.6
	0.45	13.9 13.1	13.9 12.9	13.9 12.6	13.8 12.1	13.6 11.4	13.5 10.6	13.3 9.8	13.0 8.9	12.6 8.0	12.2 7.3
	0.50	11.5 10.8	11.5 10.8	11.4 10.6	11.3 10.3	11.3 9.9	11.1 9.3	11.0 8.7	10.8 8.1	10.6 7.5	10.3 6.9

In Table 7, the fixed X_i-values of 1, 2, 3, and 4 are considered to study the ARL performance for a different value of S_{xx}. In this case, we have $S_{xx} = 5$ instead of $S_{xx} = 20$ used in Tables 1-6. We include only EWMA/R and EWMA_3 charts in our comparisons since the T^2 chart did not demonstrate good overall ARL performance. As the slope shifts d increase, one can observe that our EWMA_3 method has much better ARL performance than the EWMA/R charts. We also observed that the ARL performance for the situations considered in Tables 1-3 and Table 5 has almost the same pattern as shown there. The ARL performance for larger values of S_{xx} was also considered, but the overall conclusions are the same.

Table 7. ARL Comparisons Under Slope Shifts in Model (10) From B_1 To $B_1 + d\,s$

(X_i -values are 1, 2, 3, and 4 and In-control ARL = 200)

	d								
Chart	-0.2	-0.3	-0.4	-0.5	-0.6	-0.7	-0.8	-0.9	-1.0
EWMA/R	149.1	110.1	75.5	50.7	33.3	22.3	15.0	10.5	7.5
EWMA_3	49.1	22.8	13.1	8.9	6.6	5.3	4.4	3.8	3.3

Kang and Albin (2000) considered shifts in the parameters in model (1) corresponding to shifts in B_1 in model (10) as “unusual”, although their view is unsupported. In general, we find the model in Equation (10) to be more useful in some applications than the one in Equation (1) since it can be used to focus attention more clearly on changes in the regression line within presumably the most important region of values of the independent variable. The types of parameter shifts of interest, however, will depend on the particular application.

Phase I Methods

In practical applications, the in-control values of parameters are not known and must be estimated using historical data collected from the process. We assume here that k samples of simple linear regression data are available in a set of historical data.

Kang and Albin (2000) recommended substituting estimates of the parameters in their control charts in Phase I. First, they estimated the regression parameters of the underlying model in Equation (1) based on the k samples available from historical data. They obtained the reference line, $\hat{Y} = a_0 + a_1 X$, and $\hat{s}^2 = MSE$, where

$$a_0 = \frac{\sum_{j=1}^{k} a_{0j}}{k}, \quad a_1 = \frac{\sum_{j=1}^{k} a_{1j}}{k}, \text{ and } MSE = \frac{\sum_{j=1}^{k} MSE_j}{k}. \tag{17}$$

Here MSE_j is the usual unbiased estimator of s^2 from sample j.

After determining the initial estimates of the regression parameters as given in Equation (17), they advised using these estimates to calculate control limits and then applying their control algorithms. If any values fall outside the control limits for which assignable causes can be determined, then those samples are to be removed from the data and the control limits recalculated. Again, one checks to see if the points are within the control limits and, if not, the process of deleting samples and recalculating limits is repeated. The control limits used in Phase I for their multivariate approach and their residual approach are given below.

In the multivariate approach, under the assumption that $\boldsymbol{\mu}$ and $\boldsymbol{S}$ in Equations (3) and (4) are unknown, Kang and Albin (2000) evaluated the modified T^2 chart statistic for the j^{th} sample from Equation (5) as

$$T_{0j}^2 = \frac{k}{k-1}(\boldsymbol{Z}_j - \boldsymbol{Z})^T \boldsymbol{S}^{-1}(\boldsymbol{Z}_j - \boldsymbol{Z}), \tag{18}$$

where $\boldsymbol{Z} = (a_0, a_1)^T$ and $\boldsymbol{S}$ are unbiased estimators of $\boldsymbol{\mu}$ and $\boldsymbol{S}$, respectively. They obtained the sample variance-covariance matrix $\boldsymbol{S}$ by modifying the elements of Equation (4), with σ^2 estimated by MSE from Equation (17); that is,

$$\boldsymbol{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix},$$

where $S_{11} = MSE\left(\frac{1}{n} + \frac{\overline{x}^2}{S_{xx}}\right)$, $S_{22} = MSE\frac{1}{S_{xx}}$ and $S_{12} = -MSE\frac{\overline{x}}{S_{xx}}$. Kang and Albin (2000) showed that the distribution of the modified T^2 chart statistic, T_{0j}^2, in Equation (18) is related to the F distribution with (2, $(n-2)k$) degrees of freedom. Thus, they recommend the upper control limit for the chart as UCL= $2F_{2,(n-2)k,a}$, where $F_{2,(n-2)k,a}$ is the $100(1-a)$ percentile of the F distribution with (2, $(n-2)k$) degrees of freedom.

For the EWMA and R charts in Phase I, Kang and Albin (2000) modified the lower and upper control limits for the EWMA chart in Equation (8) and the R-chart in Equation (9) by replacing s by $\sqrt{MSE}$; that is, for the EWMA chart, we have

$$LCL = -L\sqrt{MSE}\sqrt{\frac{q}{(2-q)n}}, \quad UCL = L\sqrt{MSE}\sqrt{\frac{q}{(2-q)n}}$$

and for the R-chart we have

$$LCL = \sqrt{MSE}(d_2 - Ld_3) \ , \quad UCL = \sqrt{MSE}(d_2 + Ld_3) \ .$$

One difficulty, however, in applying an EWMA chart in Phase I is that several samples could be contributing to any out-of-control signal. Thus, the rule for deleting samples and recalculating limits is not clearly defined. In general, use of an EWMA chart is not appropriate in Phase I.

Stover and Brill (1998) propose two methods in Phase I. One is a Hotelling's T^2 approach based on vectors containing estimates of the Y-intercept and slope. This method is similar to the T^2 method of Kang and Albin (2000), although the estimator of the variance-covariance matrix is different. Neither pair of authors acknowledge, however, the fact that successive values of their Phase I T^2 statistics are dependent. Thus, the marginal distribution of the control chart statistic cannot be used to determine the overall probability of a Phase I signal.

The other Phase I method proposed by Stover and Brill (1998) is a univariate chart based on the first principal component corresponding to the vectors containing the Y-intercept and slope estimators. We support the use of the Shewhart-type T^2 chart for the Y-intercept and slope, but also recommend a Shewhart chart to monitor the error variance. If one codes the X-values, as we recommend, to yield Equation (10), then it also seems reasonable to use separate Shewhart-type charts for monitoring the Y-intercept and slope since the estimators of the Y-intercept and slope are independent for each sample. We advise against using the principal component chart proposed by Stover and Brill (1998), however, since it will not be able to detect combinations of shifts in the Y-intercept and the slope in the direction perpendicular to the major axis corresponding to the first principal component.

A principal component analysis as described by Jones and Rice (1992) can be very useful in summarizing and interpreting Phase I profile data with equally spaced X-values. Change-point methods can be very useful in Phase I if one suspects that instability can be modeled adequately by step shifts in the underlying parameter(s). Andrews et al. (1996) discussed change-point methods for linear regression models.

Conclusions

Our ARL comparisons show that our methods are generally more effective than the methods of Kang and Albin (2000) in Phase II for detecting sustained shifts in either the *Y*-intercept or slope or increases in the error variance. The proposed methods are considerably more effective in detecting shifts in the slope of the line when the average *Y*-value does not change. Our proposed methods also seem much more interpretable.

We also make recommendations for an effective Phase I analysis, although much more work is needed on this type of application. In general, an overall strategy needs to be developed for monitoring process profiles. Our methods need to be extended to more complicated relationships between the independent and dependent variables. The simple linear regression model considered in our paper, for example, can be extended to polynomial and multiple regression models. In our view the study of profile monitoring is a very rich and promising area of research.

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