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ON THE MORÁVEK AND VLACH CONDITIONS FOR THE EXISTENCE OF A SOLUTION TO THE MULTI-INDEX PROBLEM

GRAHAM SMITH

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For the multi-index problem:

maximize

$$\sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n c_{ijk} x_{ijk},$$

subject to:

$$\sum_{k=1}^n x_{ijk} = A_{ij} \quad (i = 1, 2, \dots, l; j = 1, 2, \dots, m),$$

$$\sum_{j=1}^m x_{ijk} = B_{ik} \quad (i = 1, 2, \dots, l; k = 1, 2, \dots, n),$$

$$\sum_{i=1}^l x_{ijk} = C_{jk} \quad (j = 1, 2, \dots, m; k = 1, 2, \dots, n),$$

where:

$$x_{ijk} \geq 0 \quad (i = 1, 2, \dots, l; j = 1, 2, \dots, m; k = 1, 2, \dots, n),$$

$$\sum_{i=1}^l A_{ij} = \sum_{k=1}^n C_{jk} \quad (j = 1, 2, \dots, m),$$

$$\sum_{j=1}^m C_{jk} = \sum_{i=1}^l B_{ik} \quad (k = 1, 2, \dots, n),$$

$$\sum_{k=1}^n B_{ik} = \sum_{j=1}^m A_{ij} \quad (i = 1, 2, \dots, l)$$

necessary conditions for the existence of a solution have been given by Morávek and Vlach [2] which were restated by Haley [1] in the form:

$$(1) \quad \sum_{i \in I} \sum_{k \in K} B_{ik} + \sum_{j \in J} \sum_{k \in K} C_{jk} - \sum_{i \in I} \sum_{j \in J} A_{ij} \geq 0$$

where

$$\begin{aligned} I &\subseteq \{1, 2, \dots, l\}, \\ J &\subseteq \{1, 2, \dots, m\}, \\ K &\subseteq \{1, 2, \dots, n\}, \\ \bar{K} &\subseteq \{1, 2, \dots, n\}, \\ K \cap \bar{K} &= \emptyset, \\ K \cup \bar{K} &= \{1, 2, \dots, n\}. \end{aligned}$$

It is known [1] that the conditions (1) are sufficient for the existence of a solution if at least one of  $l$ ,  $m$  or  $n$  is less than or equal to 2.

The sufficiency of these conditions for other classes of problems has been questioned by Morávek and Vlach [3]. This note will demonstrate that the conditions (1) are not sufficient for the existence of a solution for the smallest possible problem with  $l, m, n > 2$ , that is:  $l = m = n = 3$ .

A computationally intractable procedure for determining necessary and sufficient conditions for the existence of a solution to the multiindex problem has been given by SMITH [5]. In the incomplete application of this procedure to a problem with  $l = m = n = 3$ , all of the conditions of the set (1) were generated, together with some conditions not belonging to (1). One such condition not belonging to (1) was:

$$(2) \quad -A_{21} - A_{31} - A_{32} + B_{23} + B_{32} + B_{33} + C_{11} + C_{12} + C_{21} \geq 0.$$

To test whether or not the condition (2) was implied by the conditions (1) the linear programming problem:

Minimize:

$$-A_{21} - A_{31} - A_{32} + B_{23} + B_{32} + B_{33} + C_{11} + C_{12} + C_{21}.$$

Subject to

$$\sum_{i=1}^l A_{ij} - \sum_{k=1}^n C_{jk} = 0 \quad (j = 1, 2, \dots, m),$$

$$\sum_{j=1}^m C_{jk} - \sum_{i=1}^l B_{ik} = 0 \quad (k = 1, 2, \dots, n),$$

$$\sum_{k=1}^n B_{ik} - \sum_{j=1}^m A_{ij} = 0 \quad (i = 1, 2, \dots, l),$$

$$\sum_{i \in I} \sum_{k \in K} B_{ik} + \sum_{j \in J} \sum_{k \in \bar{K}} C_{jk} - \sum_{i \in I} \sum_{j \in J} A_{ij} \geq 0 \quad \text{all } I, J, K,$$

$$0 \leq A_{ij} \leq 1 \quad (i = 1, 2, \dots, l; j = 1, 2, \dots, m),$$

$$0 \leq B_{ik} \leq 1 \quad (i = 1, 2, \dots, l; k = 1, 2, \dots, n),$$

$$0 \leq C_{jk} \leq 1 \quad (j = 1, 2, \dots, m; k = 1, 2, \dots, n)$$

was solved. The bounds on  $A_{ij}B_{ik}$  and  $C_{jk}$  are required in order that the objective function be bounded.

Feasible solutions to this linear programming problem define the constants of multi-index problems satisfying the conditions (1).

There are in fact several optimal solutions, one of which is given in figure 1.

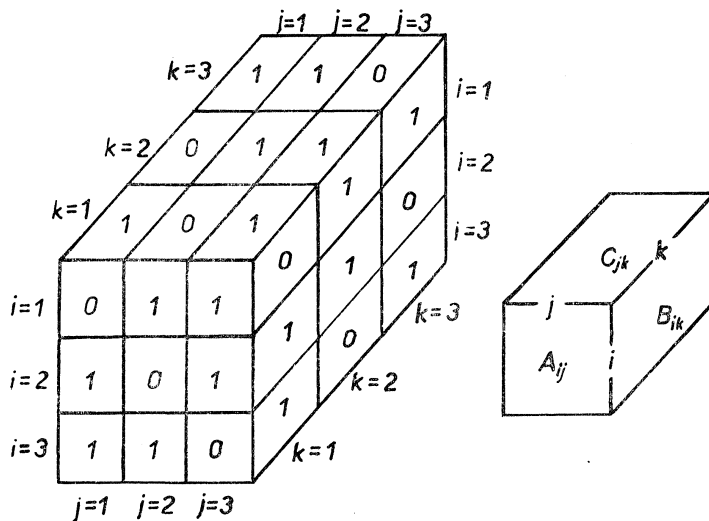


Fig. 1.

For the problem specified by figure 1 condition (2) is not satisfied since:

$$-A_{21} - A_{31} - A_{32} + B_{23} + B_{32} + B_{33} + C_{11} + C_{12} + C_{21} = -1$$

and therefore no solution exists, even though the conditions (1) are satisfied.

If

$$L = \{1, 2, \dots, l\},$$

$$M = \{1, 2, \dots, m\},$$

$$N = \{1, 2, \dots, n\},$$

$$\alpha \subseteq L \times M,$$

$$\beta \subseteq L \times N,$$

$$\gamma \subseteq M \times N$$

such that if  $(i, j) \in \alpha$ , then for each  $k$ , either  $(i, k) \in \beta$  or  $(j, k) \in \gamma$ .

Then for a solution to exist, Smith [4] has shown that it is necessary that

$$(3) \quad - \sum_{(i,j) \in \alpha} A_{ij} + \sum_{(i,k) \in \beta} B_{ik} + \sum_{(j,k) \in \gamma} C_{jk} \geq 0 \quad \text{all } \alpha, \beta.$$

The conditions (1) are the subset of the conditions (3) given by:  $\alpha = I \times J$ ,  $\beta = I \times K$  and  $\gamma = J \times \bar{K}$ .

All of the necessary conditions generated for the problem with  $l = m = n = 3$  (for example the condition (2)) are included in the conditions (3). It is tempting therefore to conjecture that the conditions (3) are necessary and sufficient for the existence of a solution to the multi-index problem having  $l = m = n = 3$ .

#### References

- [1] *K. B. Haley*: Note on the Letter by Morávek and Vlach. *Opns. Res.* 15 (1967), 545—546.
- [2] *J. Morávek & M. Vlach*: On the Necessary Conditions for the Existence of a Solution to the Multi-index Problem. *Opns. Res.* 15 (1967), 542—545.
- [3] *J. Morávek & M. Vlach*: On Necessary Conditions for a Class of Systems of Linear Inequalities. *Aplikace Matematiky* 13 (1968), 299—303.
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#### Souhrn

### O MORÁVKOVÝCH A VLACHOVÝCH PODMÍNKÁCH PRO EXISTENCI ŘEŠENÍ VÍCEINDEXOVÉHO PROBLÉMU

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V článku je dán nejmenší možný příklad trojindexového problému, který vyhovuje podmínkám Morávka a Vlacha z r. 1967 a který nemá přípustné řešení.

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