Notes on Intuitionistic Fuzzy Sets
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On the most extended interval-valued intuitionistic fuzzy modal operators from both types

Krassimir T. Atanassov[1,2]

[1] Department of Bioinformatics and Mathematical Modelling ,
Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 105, Sofia 1113, Bulgaria

[2] Intelligent Systems Laboratory
Prof. Asen Zlatarov University, Burgas 8010, Bulgaria
e-mail: krat@bas.bg

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Abstract: The most extended (by the moment) interval-valued intuitionistic fuzzy modal operators from both types are introduced. A theorem for equivalence of two of them is proved.
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1 Introduction

The Interval Valued Intuitionistic Fuzzy Sets (IVIFSs) are extensions of the Intuitionistic Fuzzy Sets (IFSs) (see [1, 2, 5] and their theory was enriched with a lot of operators that do not have analogues in the standard fuzzy sets theory and in the rest of the fuzzy sets extensions. In the present paper, we introduce two groups of operators that are extensions of the existing by the moment interval-valued intuitionistic fuzzy modal operators from the two types.

2 Extensions of the interval-valued intuitionistic fuzzy modal operators from the first type

The first of the extensions, introduced in the last years, was given in [3]. We use it as a basis of our research, correcting some misprints in it. Here, we introduce new operators, one of which is the operator from [3].

First, all forms of the operators are given and the conditions for the validity of each one of them is discussed and after this, we will reduce the more detailed research to its simplest case.

Let $\text{ext}_1, \text{ext}_2, \text{ext}_3, \text{ext}_4, \text{ext}_5, \text{ext}_6, \text{ext}_7, \text{ext}_8 \in \{\inf, \sup\}$. Let

$$X^{\begin{pmatrix} \text{ext}_1 & \text{ext}_2 & \text{ext}_3 & \text{ext}_4 \\ \text{ext}_5 & \text{ext}_6 & \text{ext}_7 & \text{ext}_8 \end{pmatrix}}_{\begin{pmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{pmatrix}}(A)$$

$$\equiv \{\langle x, [\inf M_X(x), \sup M_X(x)], [\inf N_X(x), \sup N_X(x)]\rangle | x \in E\}$$
$$= \{\langle x, [a_1 \inf M_A(x) + b_1(1 - \text{ext}_1 M_A(x) - c_1 \text{ext}_2 N_A(x)),$$
$$a_2 \sup M_A(x) + b_2(1 - \text{ext}_3 M_A(x) - c_2 \text{ext}_4 N_A(x))],$$
$$[d_1 \inf N_A(x) + e_1(1 - f_1 \text{ext}_5 M_A(x) - \text{ext}_6 N_A(x)),$$
$$d_2 \sup N_A(x) + e_2(1 - f_2 \text{ext}_7 M_A(x) - \text{ext}_8 N_A(x))]\rangle | x \in E\},$$

where $a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2 \in [0, 1]$.

The definition will be correct, if

$$0 \leq \inf M_X(x) \leq \sup M_X(x) \leq 1, \tag{1}$$

$$0 \leq \inf N_X(x) \leq \sup N_X(x) \leq 1, \tag{2}$$

$$\sup M_X(x) + \sup N_X(x) \leq 1. \tag{3}$$

Not the most complex (while not the simplest either) form of the operator is:

$$X^{\begin{pmatrix} \text{ext}_1 & \text{ext}_2 & \text{ext}_3 & \text{ext}_4 \\ \text{ext}_1 & \text{ext}_2 & \text{ext}_3 & \text{ext}_4 \end{pmatrix}}_{\begin{pmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{pmatrix}}(A)$$

$$= \{\langle x, [a_1 \inf M_A(x) + b_1(1 - \text{ext}_1 M_A(x) - c_1 \text{ext}_2 N_A(x)),$$
$$a_2 \sup M_A(x) + b_2(1 - \text{ext}_3 M_A(x) - c_2 \text{ext}_4 N_A(x))],$$
$$[d_1 \inf N_A(x) + e_1(1 - f_1 \text{ext}_1 M_A(x) - \text{ext}_2 N_A(x)),$$
$$d_2 \sup N_A(x) + e_2(1 - f_2 \text{ext}_3 M_A(x) - \text{ext}_4 N_A(x))]\rangle | x \in E\}$$
$$\equiv X^{\begin{pmatrix} \text{ext}_1 & \text{ext}_2 & \text{ext}_3 & \text{ext}_4 \end{pmatrix}}_{\begin{pmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{pmatrix}}(A).$$

For brevity, in the upper index of X we will write i and s instead of $\inf$ and $\sup$, respectively.

From the above records of the X-operator it is clear that in the first case it will have 256 different forms and in the second case – 16 different forms.

The simplest form of the X-operator is

$$X^{\left(\begin{smallmatrix} \text{ext}_1 & \text{ext}_1 & \text{ext}_2 & \text{ext}_2 \\ \text{ext}_1 & \text{ext}_1 & \text{ext}_2 & \text{ext}_2 \end{smallmatrix}\right)}_{\left(\begin{smallmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{smallmatrix}\right)}(A)$$

$$= \{\langle x, [a_1 \inf M_A(x) + b_1(1 - \text{ext}_1 M_A(x) - c_1 \text{ext}_1 N_A(x)),$$

$$a_2 \sup M_A(x) + b_2(1 - \text{ext}_2 M_A(x) - c_2 \text{ext}_2 N_A(x))],$$

$$[d_1 \inf N_A(x) + e_1(1 - f_1 \text{ext}_1 M_A(x) - \text{ext}_1 N_A(x)),$$

$$d_2 \sup N_A(x) + e_2(1 - f_2 \text{ext}_2 M_A(x) - \text{ext}_2 N_A(x))]\rangle | x \in E\}$$

$$\equiv X^{\left(\begin{smallmatrix} \text{ext}_1 & \text{ext}_2 \end{smallmatrix}\right)}_{\left(\begin{smallmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{smallmatrix}\right)}(A).$$

So, here will take into consideration the simpler situation, that, obviously, has only 4 cases that we will study sequentially.

Let everywhere for the 4 cases, $a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2 \in [0, 1]$. Then

$$X^{\left(\begin{smallmatrix} i & i \end{smallmatrix}\right)}_{\left(\begin{smallmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{smallmatrix}\right)}(A)$$

$$= \{\langle x, [a_1 \inf M_A(x) + b_1(1 - \inf M_A(x) - c_1 \inf N_A(x)),$$

$$a_2 \sup M_A(x) + b_2(1 - \inf M_A(x) - c_2 \inf N_A(x))],$$

$$[d_1 \inf N_A(x) + e_1(1 - f_1 \inf M_A(x) - \inf N_A(x)),$$

$$d_2 \sup N_A(x) + e_2(1 - f_2 \inf M_A(x) - \inf N_A(x))]\rangle | x \in E\}$$

and to see that the operator is correct, we must find the conditions under which the inequalities (1) – (3) are valid. Below, we shall study the mentioned above four cases, each of which with four sub-cases.

1.1. We see that

$$a_1 \inf M_A(x) + b_1(1 - \inf M_A(x) - c_1 \inf N_A(x))$$

$$\geq a_1 \inf M_A(x) + b_1(1 - \inf M_A(x) - \inf N_A(x))$$

$$\geq a_1 \inf M_A(x) \geq 0.$$

1.2. We check

$$a_2 \sup M_A(x) + b_2(1 - \inf M_A(x) - c_2 \inf N_A(x))$$

$$= a_2 \sup M_A(x) - b_2 \inf M_A(x) - b_2 c_2 \inf N_A(x) + b_2$$

$$\leq a_2 \sup M_A(x) + b_2 \leq a_2 + b_2.$$

Therefore, the condition is $a_2 + b_2 \leq 1$. Analogously, if $d_2 + e_2 \leq 1$, then

$$d_2 \sup N_A(x) + e_2(1 - f_2 \inf M_A(x) - \inf N_A(x)) \leq 1.$$

1.3. We obtain

$$a_2 \sup M_A(x) + b_2(1 - \inf M_A(x) - c_2 \inf N_A(x))$$

$$-a_1 \inf M_A(x) - b_1(1 - \inf M_A(x) - c_1 \inf N_A(x))$$

$$= a_2 \sup M_A(x) - a_1 \inf M_A(x) + (b_2 - b_1)(1 - \inf M_A(x)) + (b_1c_1 - b_2c_2) \inf N_A(x)) \geq 0$$

for $a_2 \geq a_1$, $b_2 \geq b_1$ and $b_1c_1 \geq b_2c_2$; and

$$d_2 \sup N_A(x) + e_2(1 - f_2 \inf M_A(x) - \inf N_A(x))$$

$$-d_1 \inf N_A(x) - e_1(1 - f_1 \inf M_A(x) - \inf N_A(x)) \geq 0,$$

for $d_2 \geq d_1, e_2 \geq e_1$ and $e_1f_1 \geq e_2f_2$.

1.4. We have

$$a_2 \sup M_A(x) + b_2(1 - \inf M_A(x) - c_2 \inf N_A(x))$$

$$+d_2 \sup N_A(x) + e_2(1 - f_2 \inf M_A(x) - \inf N_A(x))$$

$$\leq a_2 \sup M_A(x) + d_2 \sup N_A(x) + b_2 + e_2$$

$$\leq \max(a_2, d_2) + b_2 + e_2.$$

Hence, the condition is $\max(a_2, d_2) + b_2 + e_2 \leq 1$.

Therefore, for this first case, the inequalities (1) – (3) have the concrete forms

$$a_2 \geq a_1, b_2 \geq b_1, d_2 \geq d_1, e_2 \geq e_1, \tag{4}$$

$$a_2 + b_2 \leq 1, \tag{5}$$

$$d_2 + e_2 \leq 1, \tag{6}$$

$$b_1c_1 \geq b_2c_2, \tag{7}$$

$$e_1f_1 \geq e_2f_2, \tag{8}$$

$$\max(a_2, d_2) + b_2 + e_2 \leq 1. \tag{9}$$

We must mention immediately, that the validity of conditions (5) and (6) follows directly from (9), i.e., these two conditions can be omitted.

The second X-operator is

$$X_{\left(\begin{smallmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{smallmatrix}\right)}^{\left(\begin{smallmatrix} i & s \end{smallmatrix}\right)}(A)$$

$$= \{\langle x, [a_1 \inf M_A(x) + b_1(1 - \inf M_A(x) - c_1 \inf N_A(x)),$$

$$a_2 \sup M_A(x) + b_2(1 - \sup M_A(x) - c_2 \sup N_A(x))],$$

$$[d_1 \inf N_A(x) + e_1(1 - f_1 \inf M_A(x) - \inf N_A(x)),$$

$$d_2 \sup N_A(x) + e_2(1 - f_2 \sup M_A(x) - \sup N_A(x))]\rangle | x \in E\}.$$

2.1 We see that this case coincides with case **1.1**.

2.2 This case follows directly from

$$a_2 \sup M_A(x) + b_2(1 - \sup M_A(x) - c_2 \sup N_A(x)) \leq a_2 \sup M_A(x) + b_2 \leq 1.$$

Therefore, the conditions are $a_2 + b_2 \leq 1$ and, respectively, for the second inequality $d_2 + e_2 \leq 1$.

2.3. We obtain

$$Z \equiv a_2 \sup M_A(x) + b_2(1 - \sup M_A(x) - c_2 \sup N_A(x))$$

$$-a_1 \inf M_A(x) - b_1(1 - \inf M_A(x) - c_1 \inf N_A(x))$$

$$= (a_2 - b_2) \sup M_A(x) - (a_1 - b_1) \inf M_A(x) + b_2 - b_1$$

$$-b_2 c_2 \sup N_A(x) + b_1 c_1 \inf N_A(x)$$

$$\geq (a_2 - b_2 - a_1 + b_1) \inf M_A(x) + b_2 - b_1 - b_2 c_2 \sup N_A(x)$$

(from $\sup N_A(x) \leq 1 - \sup M_A(x) \leq 1 - \inf M_A(x)$)

$$\geq (a_2 - b_2 - a_1 + b_1) \inf M_A(x) + b_2 - b_1 - b_2 c_2 + b_2 c_2 \inf M_A(x)$$

$$= (a_2 - b_2 - a_1 + b_1 + b_2 c_2) \inf M_A(x) + b_2 - b_1 - b_2 c_2.$$

If $a_2 - a_1 \geq b_2 - b_1 - b_2 c_2$, then

$$Z \geq b_2 - b_1 - b_2 c_2.$$

Therefore, $Z \geq 0$ if $a_2 - a_1 \geq b_2 - b_1 - b_2 c_2 \geq 0$.

If $a_2 - a_1 \leq b_2 - b_1 - b_2 c_2$, then

$$Z_1 \geq a_2 - a_1.$$

Therefore, the condition is $\min(a_2 - a_1, b_2 - b_1 - b_2 c_2) \geq 0$.

2.4. We check

$$a_2 \sup M_A(x) + b_2(1 - \sup M_A(x) - c_2 \sup N_A(x))]$$

$$+d_2 \sup N_A(x) + e_2(1 - f_2 \sup M_A(x) - \sup N_A(x))$$

$$= (a_2 - b_2 - e_2 f_2) \sup M_A(x) + (d_2 - e_2 - b_2 c_2) \sup N_A(x) + b_2 + e_2$$

$$\leq \max(a_2 - b_2 - e_2 f_2, d_2 - e_2 - b_2 c_2) + b_2 + e_2,$$

i.e., now the condition is $\max(a_2 - b_2 - e_2 f_2, d_2 - e_2 - b_2 c_2) + b_2 + e_2 \leq 1$.

Therefore, for the second case, the inequalities (1) – (3) have the concrete forms

$$\min(a_2 - a_1, b_2 - b_1 - b_2 c_2) \geq 0, \tag{10}$$

$$\min(d_2 - d_1, e_2 - e_1 - e_2 f_2) \geq 0, \tag{11}$$

$$\max(a_2 - b_2 - e_2 f_2, d_2 - e_2 - b_2 c_2) + b_2 + e_2 \leq 1. \tag{12}$$

The third X-operator is

$$X^{\left(\begin{array}{cc} s & i \end{array} \right)}_{\left(\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{array} \right)}(A)$$

$$= \{\langle x, [a_1 \inf M_A(x) + b_1(1 - \sup M_A(x) - c_1 \sup N_A(x)),$$

$$a_2 \sup M_A(x) + b_2(1 - \inf M_A(x) - c_2 \inf N_A(x))],$$
$$[d_1 \inf N_A(x) + e_1(1 - f_1 \sup M_A(x) - \sup N_A(x)),$$
$$d_2 \sup N_A(x) + e_2(1 - f_2 \inf M_A(x) - \inf N_A(x))]\rangle | x \in E\}.$$

3.1. We obtain directly that

$$a_1 \inf M_A(x) + b_1(1 - \sup M_A(x) - c_1 \sup N_A(x)) \geq a_1 \inf M_A(x) \geq 0$$

and

$$d_1 \inf N_A(x) + e_1(1 - f_1 \sup M_A(x) - \sup N_A(x)) \geq d_1 \inf N_A(x) \geq 0.$$

3.2. This case coincides with **1.2**.
3.3. We have:

$$a_2 \sup M_A(x) + b_2(1 - \inf M_A(x) - c_2 \inf N_A(x))$$
$$-a_1 \inf M_A(x) - b_1(1 - \sup M_A(x) - c_1 \sup N_A(x))$$
$$= b_2 - b_1 + (a_2 + b_1) \sup M_A(x) - (a_1 + b_2) \inf M_A(x)$$
$$+b_1 c_1 \sup N_A(x) - b_2 c_2 \inf N_A(x) \geq 0$$

for $a_2 \geq a_1, b_2 \geq b_1, c_1 \geq c_2$.
3.4. We check

$$a_2 \sup M_A(x) + b_2(1 - \inf M_A(x) - c_2 \inf N_A(x))$$
$$+d_2 \sup N_A(x) + e_2(1 - f_2 \inf M_A(x) - \inf N_A(x))$$
$$= b_2 + e_2 + a_2 \sup M_A(x) + d_2 \sup N_A(x)$$
$$-(b_2 + e_2 f_2) \inf M_A(x) - (b_2 c_2 + e_2) \inf N_A(x))$$
$$\leq b_2 + e_2 + a_2 \sup M_A(x) + d_2 \sup N_A(x)$$
$$\leq \max(a_2, d_2) + b_2 + e_2.$$

Therefore, for the third case, the inequalities (1) – (3) have the concrete forms

$$a_2 \geq a_1, b_2 \geq b_1, c_1 \geq c_2, \tag{13}$$
$$a_2 + b_2 \leq 1, \tag{14}$$
$$d_2 + e_2 \leq 1, \tag{15}$$
$$\max(a_2, d_2) + b_2 + e_2 \leq 1. \tag{16}$$

The fourth X-operator is

$$X_{\begin{pmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{pmatrix}}^{\begin{pmatrix} s & s \end{pmatrix}}(A)$$
$$= \{\langle x, [a_1 \inf M_A(x) + b_1(1 - \sup M_A(x) - c_1 \sup N_A(x)),$$
$$a_2 \sup M_A(x) + b_2(1 - \sup M_A(x) - c_2 \sup N_A(x))],$$

$$[d_1 \inf N_A(x) + e_1(1 - f_1 \sup M_A(x) - \sup N_A(x)),$$

$$d_2 \sup N_A(x) + e_2(1 - f_2 \sup M_A(x) - \sup N_A(x))]\rangle | x \in E\}.$$

4.1. This case coincides with **3.1**.
4.2. This case coincides with **2.2**.
4.3. We obtain directly that

$$a_2 \sup M_A(x) + b_2(1 - \sup M_A(x) - c_2 \sup N_A(x))$$

$$-a_1 \inf M_A(x) + b_1(1 - \sup M_A(x) - c_1 \sup N_A(x)) \geq 0$$

for $a_2 \geq a_1, b_2 \geq b_1, c_1 \geq c_2$.
4.4. This case coincides with **2.4**.

Therefore, for the fourth case, the inequalities (1) – (3) have the forms of (10) – (12), as for the third case.

For all cases, one basic condition must be valid

$$b_2 + e_2 \leq 1. \tag{17}$$

From all these checks it follow the validity of the following theorem.

Theorem 1. For each IVIFS A and for each one of the four X-operators,

$$X^{\left(\begin{smallmatrix} ext_1 & ext_2 \end{smallmatrix} \right)}_{\left(\begin{smallmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{smallmatrix} \right)}(A)$$

is an IVIFS.

3 Extensions of the interval-valued intuitionistic fuzzy modal operators from the second type

Now, having in mind the four forms of the X-operator, here, for a first time we will introduce an extension of the operator $\boxed{\circ}_{\left(\begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \zeta_2 \end{smallmatrix} \right)}$.

Let again $\text{ext}_1, \text{ext}_2 \in \{\inf, \sup\}$. We define

$$\boxed{\circ}^{\left(\begin{smallmatrix} \text{ext}_1 & \text{ext}_2 \end{smallmatrix} \right)}_{\left(\begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \zeta_2 \end{smallmatrix} \right)} A$$

$$= \{\langle x, [\alpha_1 \inf M_A(x) - \varepsilon_1 \text{ext}_1 N_A(x) + \gamma_1, \alpha_2 \sup M_A(x) - \varepsilon_2 \text{ext}_2 N_A(x) + \gamma_2],$$

$$[\beta_1 \inf N_A(x) - \zeta_1 \text{ext}_1 M_A(x) + \delta_1, \beta_2 \sup N_A(x) - \zeta_2 \text{ext}_2 M_A(x) + \delta_2]\rangle | x \in E\}. \tag{18}$$

The components of this operator must satisfy the following conditions in a general form:

$$0 \leq \alpha_1 \inf M_A(x) - \varepsilon_1 \text{ext}_1 N_A(x) + \gamma_1$$

$$\leq \alpha_2 \sup M_A(x) - \varepsilon_2 \text{ext}_2 N_A(x) + \gamma_2 \leq 1, \tag{19}$$

$$\begin{aligned}&0 \leq \beta_1 \inf N_A(x) - \zeta_1 \text{ext}_1 M_A(x) + \delta_1 \\ &\leq \beta_2 \sup N_A(x) - \zeta_2 \text{ext}_2 M_A(x) + \delta_2 \leq 1,\end{aligned} \tag{20}$$

$$\begin{aligned}&\alpha_2 \sup M_A(x) - \varepsilon_2 \text{ext}_2 N_A(x) + \gamma_2 \\ &+\beta_2 \sup N_A(x) - \zeta_2 \text{ext}_2 M_A(x) + \delta_2 \leq 1.\end{aligned} \tag{21}$$

For example, for the case with operator $\boxed{\circ}^{\left(\begin{smallmatrix}\inf & \sup\end{smallmatrix}\right)}_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \zeta_2\end{smallmatrix}\right)}$ the conditions are

$$\max(\alpha_i - \zeta_i, \beta_i - \varepsilon_i) + \gamma_i + \delta_i \leq 1, \tag{22}$$

$$\alpha_2 + \gamma_2 \leq 1, \quad \beta_2 + \delta_2 \leq 1, \tag{23}$$

$$\gamma_i \geq \varepsilon_i, \quad \delta_i \geq \zeta_i, \tag{24}$$

$$\gamma_2 - \gamma_1 - \varepsilon_2 \geq 0, \tag{25}$$

$$\gamma_2 + \delta_2 \geq 1, \tag{26}$$

$$\alpha_1 \leq \alpha_2, \ \ \beta_1 \leq \beta_2, \ \ \gamma_1 \leq \gamma_2, \ \ \delta_1 \leq \delta_2, \ \ \varepsilon_1 \geq \varepsilon_2, \ \ \zeta_1 \leq \zeta_2. \tag{27}$$

In this section, following [4] and correcting some misprints in it, we introduce and prove the following theorem.

Theorem 2. The two most extended modal operators

$$X^{\left(\begin{smallmatrix} i & s \end{smallmatrix}\right)}_{\left(\begin{smallmatrix}a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2\end{smallmatrix}\right)} \text{ and } \boxed{\circ}^{\left(\begin{smallmatrix} i & s \end{smallmatrix}\right)}_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_1 & \delta_2 & \varepsilon_1 & \zeta_1\end{smallmatrix}\right)}$$

defined over IVIFSs are equivalent.

Proof. Let $a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2 \in [0, 1]$ and satisfy (10) – (12). Let for $i = 1, 2$:

$$\alpha_i = a_i - b_i, \ \ \beta_i = d_i - e_i, \ \ \gamma_i = b_i, \ \ \delta_i = e_i, \ \ \varepsilon_i = b_i c_i, \ \ \zeta_i = e_i f_i.$$

Also, let

$$X_1 \equiv \alpha_1 \inf M_A(x) - \varepsilon_1 \inf N_A(x) + \gamma_1,$$

$$Y_1 \equiv \beta_1 \inf N_A(x) - \zeta_1 \inf M_A(x) + \delta_1,$$

$$X_2 \equiv \alpha_2 \sup M_A(x) - \varepsilon_2 \sup N_A(x) + \gamma_2,$$

$$Y_2 \equiv \beta_2 \sup N_A(x) - \zeta_2 \sup M_A(x) + \delta_2.$$

Then

$$X_1 = (a_1 - b_1) \inf M_A(x) - b_1 c_1 \inf N_A(x) + b_1,$$

$$Y_1 = (d_1 - e_1) \inf N_A(x) - e_1 f_1 \inf M_A(x) + e_1,$$

$$X_2 = (a_2 - b_2) \sup M_A(x) - b_2 c_2 \sup N_A(x) + b_2,$$

$$Y_2 = (d_2 - e_2) \sup N_A(x) - e_2 f_2 \sup M_A(x) + e_2.$$

We obtain sequentially the following inequalities.

If $a_1 \geq b_1$, then

$$X_1 \geq -b_1c_1 + b_1 = b_1(1 - c_1) \geq 0.$$

If $a_1 \leq b_1$, then

$$X_1 \geq (a_1 - b_1)\inf M_A(x) - b_1c_1 + b_1c_1 \inf M_A(x) + b_1$$

$$= (a_1 - b_1 + b_1c_1)\inf M_A(x) - b_1c_1 + b_1.$$

If $a_1 - b_1 + b_1c_1 \geq 0$, then

$$X_1 \geq -b_1c_1 + b_1 \geq 0.$$

If $a_1 - b_1 + b_1c_1 \leq 0$, then

$$X_1 \geq a_1 - b_1 + b_1c_1 - b_1c_1 + b_1 = a_1 \geq 0,$$

i.e., in all cases $X_1 \geq 0$.

$$X_2 \leq (a_2 - b_2)\sup M_A(x) + b_2.$$

If $a_2 - b_2 \geq 0$, then

$$X_2 \leq a_2 - b_2 + b_2 = a_2 \leq 1.$$

If $a_2 - b_2 \leq 0$, then

$$X_2 \leq b_2 \leq 1,$$

i.e. $X_2 \leq 1$ and analogously, $Y_2 \leq 1$.

$$X_2 - X_1 = (a_2 - b_2)\sup M_A(x) - b_2c_2 \sup N_A(x) + b_2$$

$$-(a_1 - b_1)\inf M_A(x) + b_1c_1 \inf N_A(x) - b_1$$

$$\geq (a_2 - b_2 - a_1 + b_1)\sup M_A(x) - b_2c_2 \sup N_A(x) + b_2 - b_1$$

$$= (a_2 - b_2 - a_1 + b_1 + b_2c_2)\sup M_A(x) - b_2c_2 + b_2 - b_1.$$

If $a_2 - a_1 \geq b_2 - b_1 - b_2c_2$, then from (10) it follows

$$X_2 \geq b_2 - b_1 - b_2c_2 \geq 0.$$

If $a_2 - a_1 \leq b_2 - b_1 - b_2c_2$, then again from (10) it follows

$$X_2 \geq a_2 - b_2 - a_1 + b_1 + b_2c_2 - b_2c_2 + b_2 - b_1 \geq a_2 - a_1 \geq 0,$$

i.e., always $X_2 \geq X_1$. By the same manner we check that $Y_2 \geq Y_1$.

$$X_2 + Y_2 = (a_2 - b_2)\sup M_A(x) - b_2c_2 \sup N_A(x) + b_2$$

$$+(d_2 - e_2)\sup N_A(x) - e_2f_2 \sup M_A(x) + e_2$$

$$= (a_2 - b_2 - e_2f_2)\sup M_A(x) + (d_2 - e_2 - b_2c_2)\sup N_A(x) + b_2 + e_2.$$

Now, there are four cases that we must study sequentially.

If $a_2 - b_2 - e_2 f_2 \geq 0$ and $d_2 - e_2 - b_2 c_2 \geq 0$, then

$$X_2 + Y_2 \leq (a_2 - b_2 - e_2 f_2 - d_2 + e_2 + b_2 c_2) \sup M_A(x)$$

$$+ d_2 - e_2 - b_2 c_2 + b_2 + e_2$$

$$\leq a_2 - b_2 - e_2 f_2 - d_2 + e_2 + b_2 c_2 + d_2 - e_2 - b_2 c_2 + b_2 + e_2$$

(from (12))

$$= a_2 - e_2 f_2 + e_2 \leq 1.$$

If $a_2 - b_2 - e_2 f_2 \geq 0$ and $d_2 - e_2 - b_2 c_2 \leq 0$, then as above

$$X_2 + Y_2 \leq (a_2 - b_2 - e_2 f_2) \sup M_A(x) + b_2 + e_2$$

$$\leq a_2 - b_2 - e_2 f_2 + b_2 + e_2 = a_2 - e_2 f_2 + e_2 \leq \max(a_2, d_2) + b_2 + e_2 \leq 1.$$

If $a_2 - b_2 - e_2 f_2 \leq 0$ and $d_2 - e_2 - b_2 c_2 \geq 0$, then

$$X_2 + Y_2 \leq (d_2 - e_2 - b_2 c_2) \sup N_A(x) + b_2 + e_2$$

$$\leq d_2 - e_2 - b_2 c_2 + b_2 + e_2$$

$$\leq d_2 - b_2 c_2 + b_2 \leq \max(a_2, d_2) + b_2 + e_2 \leq 1 \leq 1.$$

If $a_2 - b_2 - e_2 f_2 \leq 0$ and $d_2 - e_2 - b_2 c_2 \leq 0$, then from (12)

$$X_2 + Y_2 \leq b_2 + e_2 \leq 1,$$

i.e., always $X_2 + Y_2 \leq 1$.

Also, from (12) we obtain that for $i = 1, 2$:

$$\alpha_i + \gamma_i - \zeta_i + \delta_i = (a_i - b_i) + b_i - e_i f_i + e_i$$

$$= a_i - e_i f_i + e_i \leq a_2 - e_2 f_2 + e_2 \leq \max(a_2 - b_2 - e_2 f_2, d_2 - e_2 - b_2 c_2) + b_2 + e_2 \leq 1$$

and analogously

$$\beta_i + \gamma_i - \varepsilon_i + \delta_i = (d_i - e_i) + b_i - b_i c_i + e_i$$

$$= b_i + d_i - b_i c_i \leq 1.$$

Hence,

$$\max(\alpha_1 - \zeta_1, \beta_1 - \varepsilon_1) + \gamma_1 - \varepsilon_1 + \delta_1 \leq 1,$$

i.e. (22) is valid.

On the other hand,

$$\alpha_2 + \gamma_2 = a_2 - b_2 + b_2 = a_2 \leq 1$$

and

$$\beta_2 + \delta_2 = d_2 - e_2 + e_2 = d_2 \leq 1,$$

i.e. (23) is valid.

$$\gamma_i - \varepsilon_i = b_i - b_i c_i \geq 0$$

and

$$\delta_i - \zeta_i = e_i - e_i f_i \geq 0,$$

i.e. (24) is valid.

From (10)

$$\gamma_2 - \gamma_1 - \varepsilon_2 = b_2 - b_1 - b_1 c_1 \geq 0,$$

i.e. (25) is valid.

Inequality (17) follows directly from (27).

The rest conditions for the operator $\boxed{\circ}^{\left(\begin{smallmatrix} i & s \end{smallmatrix} \right)}_{\left(\begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_1 & \delta_2 & \varepsilon_1 & \zeta_1 \end{smallmatrix} \right)}$ are checked directly.

Thus, we obtain

$$\boxed{\circ}^{\left(\begin{smallmatrix} i & s \end{smallmatrix} \right)}_{\left(\begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_1 & \delta_2 & \varepsilon_1 & \zeta_1 \end{smallmatrix} \right)}(A)$$

$$= \{\langle x, [\alpha_1 \inf M_A(x) - \varepsilon_1 \inf N_A(x) + \gamma_1, \alpha_2 \sup M_A(x) - \varepsilon_2 \sup N_A(x) + \gamma_2],$$

$$[\beta_1 \inf N_A(x) - \zeta_1 \inf M_A(x) + \delta_1, \beta_2 \sup N_A(x) - \zeta_2 \sup M_A(x) + \delta_2]\rangle | x \in E\}$$

$$= \{\langle x, [(a_1 - b_1) \inf M_A(x) - b_1 c_1 \inf N_A(x) + b_1,$$

$$(a_2 - b_2) \sup M_A(x) - b_2 c_2 \sup N_A(x) + b_2],$$

$$[(d_1 - e_1) \inf N_A(x) - e_1 f_1 \inf M_A(x) + e_2, b$$

$$(d_2 - e_2) \sup N_A(x) - e_2 f_2 \sup M_A(x) + e_2]\rangle | x \in E\}$$

$$= \{\langle x, [a_1 \inf M_A(x) + b_1(1 - \inf M_A(x) - c_1 \inf N_A(x)),$$

$$a_2 \sup M_A(x) + b_2(1 - \sup M_A(x) - c_2 \sup N_A(x))],$$

$$[d_1 \inf N_A(x) + e_1(1 - f_1 \inf M_A(x) - \inf N_A(x)),$$

$$d_2 \sup N_A(x) + e_2(1 - f_2 \sup M_A(x) - \sup N_A(x))]\rangle | x \in E\},$$

$$X^{\left(\begin{smallmatrix} i & s \end{smallmatrix} \right)}_{\left(\begin{smallmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{smallmatrix} \right)}(A).$$

Conversely, let $\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1, \zeta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \varepsilon_2, \zeta_2 \in [0, 1]$, and let them satisfy (22) – (27).

Then, let $\gamma_i, \delta_i > 0$ and

$$a_i = \alpha_i + \gamma_i \ (\leq 1),$$

$$b_i = \gamma_i,$$

$$c_i = \frac{\varepsilon_i}{\gamma_i} \ (\leq 1),$$

$$d_i = \beta_i + \delta_i \ (\leq 1),$$

$$e_i = \delta_i,$$

$$f_i = \frac{\zeta_i}{\delta_i} \ (\leq 1).$$

Let

$$X_1 \equiv a_1 \inf M_A(x) + b_1(1 - \inf M_A(x) - c_1 \inf N_A(x)),$$

$$Y_1 \equiv d_1 \inf N_A(x) + e_1(1 - f_1 \inf M_A(x) - \inf N_A(x)),$$

$$X_2 \equiv a_2 \sup M_A(x) + b_2(1 - \sup M_A(x) - c_2 \sup N_A(x)),$$

$$Y_2 \equiv d_2 \sup N_A(x) + e_2(1 - f_2 \sup M_A(x) - \sup N_A(x)).$$

Then

$$X_1 = (\alpha_1 + \gamma_1) \inf M_A(x) + \gamma_1(1 - \inf M_A(x) - \frac{\varepsilon_1}{\gamma_1} \inf N_A(x)),$$

$$Y_1 = (\beta_1 + \delta_1) \inf N_A(x) + \delta_1(1 - \frac{\zeta_1}{\delta_1} \inf M_A(x) - \inf N_A(x)),$$

$$X_2 = (\alpha_2 + \gamma_2) \sup M_A(x) + \gamma_2(1 - \sup M_A(x) - \frac{\varepsilon_2}{\gamma_2} \sup N_A(x)),$$

$$Y_2 = (\beta_2 + \delta_2) \sup N_A(x) + \delta_2(1 - \frac{\zeta_2}{\delta_2} \sup M_A(x) - \sup N_A(x)).$$

Using (23) and (24), we obtain sequentially:

$$0 \leq \gamma_1 - \varepsilon_1 \leq X_1 = \alpha_1 \inf M_A(x) + \gamma_1 - \varepsilon_1 \inf N_A(x) \leq \alpha_1 + \gamma_1 \leq 1,$$

$$0 \leq \delta_1 - \zeta_1 \leq Y_1 = \beta_1 \inf N_A(x) + \delta_1 - \zeta_1 \inf M_A(x) \leq \beta_1 + \delta_1 \leq 1,$$

$$0 \leq \gamma_2 - \varepsilon_2 \leq X_2 \leq \alpha_2 \sup M_A(x) + \gamma_2 \leq \alpha_2 + \gamma_2 \leq 1,$$

$$0 \leq \delta_2 - \zeta_2 \leq Y_2 \leq \beta_2 \sup N_A(x) + \delta_2 \leq \beta_2 + \delta_2 \leq 1.$$

$$X_2 - X_1 = (\alpha_2 + \gamma_2) \sup M_A(x) + \gamma_2(1 - \sup M_A(x) - \frac{\varepsilon_2}{\gamma_2} \sup N_A(x))$$

$$-(\alpha_1 + \gamma_1) \inf M_A(x) - \gamma_1(1 - \inf M_A(x) - \frac{\varepsilon_1}{\gamma_1} \inf N_A(x))$$

$$= \alpha_2 \sup M_A(x) - \alpha_1 \inf M_A(x) + \gamma_2 - \gamma_1 - \varepsilon_2 \sup N_A(x) + \varepsilon_1 \inf N_A(x)$$

$$\geq \alpha_2 \sup M_A(x) - \alpha_1 \sup M_A(x) + \gamma_2 - \gamma_1 - \varepsilon_2 + \varepsilon_2 \sup M_A(x)$$

$$= (\alpha_2 - \alpha_1 + \varepsilon_2) \sup M_A(x) + \gamma_2 - \gamma_1 - \varepsilon_2$$

(from (25))

$$\geq \gamma_2 - \gamma_1 - \varepsilon_2 \geq 0.$$

$$X_2 + Y_2 = (\alpha_2 + \gamma_2) \sup M_A(x) + \gamma_2(1 - \sup M_A(x) - \frac{\varepsilon_2}{\gamma_2} \sup N_A(x))$$

$$+(\beta_2 + \delta_2) \sup N_A(x) + \delta_2(1 - \frac{\zeta_2}{\delta_2} \sup M_A(x) - \sup N_A(x))$$

$$= (\alpha_2 - \zeta_2) \sup M_A(x) + (\beta_2 - \varepsilon_2) \sup N_A(x) + \gamma_2 + \delta_2$$

(from (22))

$$\leq \max(\alpha_2 - \zeta_2, \beta_2 - \varepsilon_2) + \gamma_2 + \delta_2 \leq 1.$$

Also, (10) that is valid because from (27) it follows that

$$a_2 - a_1 = \alpha_2 + \gamma_2 - \alpha_1 - \gamma_1 \geq 0$$

and from (25)

$$b_2 - b_1 - b_2c_2 = \gamma_2 - \gamma_1 - \gamma_2.\frac{\varepsilon_2}{\gamma_2} = \gamma_2 - \gamma_1 - \varepsilon_2 \geq 0,$$

i.e.

$$\min(a_2 - a_1, b_2 - b_1 - b_2c_2) \geq 0.$$

By analogy we check the validity of (11).

For (12) we obtain

$$\max(a_2 - b_2 - e_2f_2, d_2 - e_2 - b_2c_2) + b_2 + e_2$$

$$= \max(\alpha_2 + \gamma_2 - \gamma_2 - \zeta_2, \beta_2 + \delta_2 - \delta_2 - \varepsilon_2) + \gamma_2 + \delta_2$$
$$= \max(\alpha_2 - \zeta_2, \beta_2 + \varepsilon_2) + \gamma_2 + \delta_2 \leq 1$$

from (22).

Inequality (27) follows directly from (17).

Then, we obtain

$$X_{\begin{pmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{pmatrix}}^{\begin{pmatrix} i & s \end{pmatrix}}(A)$$
$$= \{\langle x, [a_1 \inf M_A(x) + b_1(1 - \inf M_A(x) - c_1 \inf N_A(x)),$$
$$a_2 \sup M_A(x) + b_2(1 - \sup M_A(x) - c_2 \sup N_A(x))],$$
$$[d_1 \inf N_A(x) + e_1(1 - f_1 \inf M_A(x) - \inf N_A(x)),$$
$$d_2 \sup N_A(x) + e_2(1 - f_2 \sup M_A(x) - \sup N_A(x))]\rangle | x \in E\},$$
$$= \{\langle x, [(a_1 - b_1) \inf M_A(x) - b_1c_1 \inf N_A(x) + b_1,$$
$$(a_2 - b_2) \sup M_A(x) - b_2c_2 \sup N_A(x) + b_2],$$
$$[(d_1 - e_1) \inf N_A(x) - e_1f_1 \inf M_A(x) + e_2, b$$
$$(d_2 - e_2) \sup N_A(x) - e_2f_2 \sup M_A(x) + e_2]\rangle | x \in E\}$$
$$= \{\langle x, [\alpha_1 \inf M_A(x) - \varepsilon_1 \inf N_A(x) + \gamma_1, \alpha_2 \sup M_A(x) - \varepsilon_2 \sup N_A(x) + \gamma_2]\,,$$
$$[\beta_1 \inf N_A(x) - \zeta_1 \inf M_A(x) + \delta_1, \beta_2 \sup N_A(x) - \zeta_2 \sup M_A(x) + \delta_2]\rangle | x \in E\}$$
$$= \boxdot_{\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_1 & \delta_2 & \varepsilon_1 & \zeta_1 \end{pmatrix}}^{\begin{pmatrix} i & s \end{pmatrix}}(A).$$

Therefore, the two operators are equivalent. □

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References

[1] Atanassov, K. (1988). Review and new results on intuitionistic fuzzy sets. *Preprint IM-MFAIS-1-88*, Sofia, 1988.

[2] Atanassov, K. (1999). *Intuitionistic Fuzzy Sets*, Springer, Heidelberg.

[3] Atanassov, K. (2018). On the Most Extended Modal Operator of First Type over Interval-Valued Intuitionistic Fuzzy Sets. *Mathematics*, 6, 123; doi:10.3390/math6070123

[4] Atanassov, K. (2018). On the two most extended modal types of operators defined over interval-valued intuitionistic fuzzy sets. *Annals of Fuzzy Mathematics and Informatics*, 16 (1), 1–12.

[5] Atanassov, K. & Gargov, G. (1989). Interval valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 31 (3), 343–349.