

ON THE μ^i IN A MINIMAL INJECTIVE RESOLUTION

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A will always denote a (commutative) noetherian local ring with maximal ideal \mathfrak{m} and residue class field $k = A/\mathfrak{m}$.

Introduction.

Let M be a finitely generated A -module, and let

$$0 \rightarrow M \rightarrow E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \dots \rightarrow E^i \xrightarrow{d^i} \dots$$

be a minimal injective resolution of M , that is, E^i is the injective envelope of $\text{Ker } d^i = \text{Im } d^{i-1}$ for all i . Then the cardinal number $\mu^i(\mathfrak{p}, M)$, where \mathfrak{p} is a prime ideal, denotes the number of indecomposable components of E^i isomorphic to $E(A/\mathfrak{p})$. We know that $\mu^i(\mathfrak{p}, M)$ is finite and is only depending on i , \mathfrak{p} , and M (cf. Hyman Bass [1, §2]).

The depth (or co-dimension) of M turns out to be the smallest integer i for which $\mu^i(\mathfrak{m}, M) > 0$, while the injective dimension is the greatest i for which $\mu^i(\mathfrak{m}, M) > 0$ (if such an i exists). This means that if $E(k)$ is a direct summand in E^i (that is, $\mu^i(\mathfrak{m}, M) > 0$), then i must lay between the depth and the injective dimension of M .

The first part of this paper deals with the question: is $E(k)$ a direct summand in E^i for all i between the depth and the injective dimension? Since $\mu^i(\mathfrak{p}, M) = \mu^i(\mathfrak{p}A_{\mathfrak{p}}, M_{\mathfrak{p}})$, the answer of this question also tells us for which i , $E(A/\mathfrak{p})$ is a direct summand in E^i . Unfortunately I have only been able to answer the question, in the affirmative, in the following cases:

- 1) A is Cohen-Macaulay,
- 2) $\text{depth } M \geq \text{depth } A$,
- 2') $M = A$,
- 3) M is Cohen-Macaulay,
- 4) $\text{inj dim } M$ finite.

In case 4) holds $\varphi(M) = \sum_i (-1)^{s-i} \mu^i(\mathfrak{m}, M) \geq 0$, where $s = \text{inj dim } M$, and the following statements are equivalent:

- (i) $\varphi(M) > 0$, (ii) $\text{Ann } M = 0$, (iii) $\text{grad } M = 0$.

This is the analogue to the corresponding result about the Euler-characteristic of a finitely generated module of finite projective dimension.

In the second part (section 3) we will assume that there exists a Gorenstein module ([11] and [12]). Then μ^i of a Cohen–Macaulay module or a finitely generated module of finite injective dimension can be expressed by some Betti-numbers.

1. Notation and previous results.

NOTATION 1.1. Let L and M be finitely generated non-zero A -modules. Then the following notation will be used:

$\text{grade}_M L$ = maximal length of a M -regular sequence in $\text{Ann } L$. (Elements $a_1, \dots, a_s \in \mathfrak{m}$ are called a M -regular sequence, if each a_i is not a zero-divisor in $M/(a_1, \dots, a_{i-1})M$.)

$\text{depth } M = \text{grade}_M k$.

$\text{dim } M$ = the Krull-dimension of M , that is (in this case), the Krull-dimension of the local ring $A/\text{Ann } M$.

M Cohen–Macaulay, if $\text{depth } M = \text{dim } M$ (and M non-zero and finitely generated).

zM = the zero-divisors on M .

$\mu^i(M) = \mu^i(\mathfrak{m}, M)$ as defined in the introduction.

$\beta_i(M)$ = the i th Betti-number of M , that is, the dimension of $\text{Tor}_i(k, M)$ considered as a vector space over k .

The results in Bourbaki [2, Chap. IV] are considered as well known and will hence be used without reference.

Some of the previous results, mainly from Bass [1, §§ 2–3], can be summarized in

PROPOSITION 1.2. *Let L and M be finitely generated non-zero A -modules. Then:*

(a) $\mu^i(M)$ is the dimension of $\text{Ext}^i(k, M)$ considered as a vector space over k . Further, $\mu^i(\mathfrak{p}, M) = \mu_{A_{\mathfrak{p}}}^i(M_{\mathfrak{p}})$ for all prime ideals \mathfrak{p} .

(b) $\text{depth } M = \inf \{i \mid \mu^i(M) > 0\} \leq \text{dim } M$
 $\leq \sup \{i \mid \mu^i(M) > 0\} = \text{inj dim } M (\leq \infty)$.

(c) $\mu_A^i(M) = \mu_{A/(a)}^{i-1}(M/aM)$ for all i , if $a \in \mathfrak{m} - (zA \cup zM)$.

(d) If $\text{Ext}^i(L, M) \neq 0$ for some i and length $L < \infty$, then $\mu^i(M) > 0$.

(e) If $\text{inj dim } M = \infty$, then $\mu^i(M) > 0$ for $i \geq \text{dim } A$.

- (f) If $\text{inj dim } M < \infty$, then $\text{inj dim } M = \text{depth } A$, and
- (g) $\text{depth } A = \text{depth } L + \sup \{i \mid \text{Ext}^i(L, M) \neq 0\}$.
- (h) $\text{grade}_M L = \inf \{i \mid \text{Ext}^i(L, M) \neq 0\} = (\text{length of a maximal } M\text{-regular sequence in } \text{Ann } L) = \text{grade}_M A/\mathfrak{a}$, where \mathfrak{a} is the radical of $\text{Ann } L$.
- (i) $\text{depth } A \leq \text{grade}_A M + \dim M \leq \dim A$.

For (g) and (i) we refer to [7], [8] and [9]:

(g): see [7, § 2] or [8, Theorem 3.1] (= [9, Theorem 1.1]).

(i): see [8, Lemma 3.8] (= [9, Lemme 1.4]).

Also the following results will be useful:

PROPOSITION 1.3. *Let M be a finitely generated non-zero A -module. Then:*

(a) If
$$\dots \rightarrow F_i \xrightarrow{d_i} \dots \rightarrow F_1 \xrightarrow{d_1} F_0 \rightarrow M \rightarrow 0$$

is a minimal free resolution of M (that is, $\text{Im } d^i \subseteq \mathfrak{m}F_{i-1}$ for all i), then the rank of F_i is $\beta_i(M)$ for all i .

(b) $\text{proj dim } M = \sup \{i \mid \beta_i(M) > 0\} = \inf \{i \geq 0 \mid \beta_{i+1}(M) = 0\}$.

(c) If $\text{proj dim } M < \infty$, then the Euler-characteristic $\chi(M) \geq 0$ (where $\chi(M) = \sum_i (-1)^i \beta_i(M)$), and the following statements are equivalent:

- (i) $\chi(M) > 0$,
- (ii) $\text{Ann } M = 0$,
- (iii) $\text{grade}_A M = 0$.

PROOF. (a) and (b): see Serre [10, IV appendix I]. (c): see Kaplansky [6, section 4-3].

2. Non-vanishing of μ^i .

PROPOSITION 2.1. *Let A be Cohen-Macaulay, and let M be a non-zero finitely generated A -module. Then*

$$\mu^i(M) > 0 \quad \text{for} \quad \text{depth } M \leq i \leq \text{inj dim } M (\leq \infty).$$

PROOF. By Proposition 1.2 (e) and (f) it is enough to show $\mu^i(M) > 0$ for $\text{depth } M \leq i < s = \text{depth } A = \dim A$. The proof will be by induction on s :

$s = 0$: nothing to prove.

$s - 1 \rightarrow s, s \geq 1$ (the inductive step): Divide in two cases:

1°. $\text{depth } M > 0$. With $a \in \mathfrak{m} - (zA \cup zM)$ one obtains by 1.2 (c) that $\mu_A^i(M) = \mu_{A/(a)}^{i-1}(M/aM)$. The inductive hypothesis gives the desired assertion.

2°. $\text{depth } M = 0$. Choose F finitely generated free and a submodule K of F such that $F/K = M$. The corresponding long-exact sequence for $\text{Ext}(k, -)$ gives us $\text{Ext}^i(k, M) = \text{Ext}^{i+1}(k, K)$ for $i \leq s-2$, and hence $\text{depth } K = 1 > 0$ and $\mu^i(M) = \mu^{i+1}(K) > 0$ for $0 \leq i \leq s-2$ (by case 1°).

The only thing left to show is now $\mu^{s-1}(M) > 0$.

Let a_1, \dots, a_s be an A -regular sequence such that $\text{depth } M_i = 0$ for all i , $0 \leq i \leq s$, where $M_i = M/a_iM$ and $a_i = (a_1, \dots, a_i)$, $a_0 = 0$. To see that this is possible assume that $a_{i-1} = (a_1, \dots, a_{i-1})$ has been chosen. Pick $a \in \mathfrak{m} - z(A/a_{i-1})$, $x \in M_{i-1}$ such that $\text{Ann } x = \mathfrak{m}$ (since $\text{depth } M_{i-1} = 0$), and $t > 0$ such that $x \notin a^t M_{i-1}$. Put $a_i = a^t \in \mathfrak{m} - z(A/a_{i-1})$. Then

$$\mathfrak{m} = \text{Ann } x \subseteq \text{Ann } \bar{x} \subseteq \mathfrak{m},$$

where \bar{x} is the residue class of x in $M_{i-1}/a_i M_{i-1} = M_i$. This gives $\text{Ann } \bar{x} = \mathfrak{m}$, that is, $\text{depth } M_i = 0$.

Since $\text{projdim } A/a_{i-1} = i-1$, there is an exact sequence

$$\text{Ext}^{i-1}(A/a_{i-1}, M) \xrightarrow{a_i} \text{Ext}^{i-1}(A/a_{i-1}, M) \rightarrow \text{Ext}^i(A/a_i, M) \rightarrow 0.$$

Now induction on i shows: $\text{Ext}^i(A/a_i, M) = M_i$. In particular

$$a_s \in zM_{s-1} = z\text{Ext}^{s-1}(A/a_{s-1}, M),$$

so from the exact sequence

$$\text{Ext}^{s-1}(A/a_s, M) \rightarrow \text{Ext}^{s-1}(A/a_{s-1}, M) \xrightarrow{a_s} \text{Ext}^{s-1}(A/a_{s-1}, M)$$

it follows that $\text{Ext}^{s-1}(A/a_s, M) \neq 0$, and hence $\mu^{s-1}(M) > 0$ (by Proposition 1.2 (d)), since the length of A/a_s is finite.

LEMMA 2.2. *If $\text{depth } A = 0$, there exists $t \geq 0$ such that k is a direct summand in \mathfrak{m}^t .*

PROOF. Since $\text{depth } A = 0$, $\text{Ann } \mathfrak{m} \neq 0$. Choose $a_0 \in \text{Ann } \mathfrak{m} - 0$ and $t \geq 0$ such that $a_0 \in \mathfrak{m}^t - \mathfrak{m}^{t+1}$. Expand $\bar{a}_0 \in \mathfrak{m}^t/\mathfrak{m}^{t+1}$ to a basis $(\bar{a}_0, \bar{a}_1, \dots, \bar{a}_p)$ for the vector space $\mathfrak{m}^t/\mathfrak{m}^{t+1}$ over k . Then $(a_0, a_1, \dots, a_p) = \mathfrak{m}^t$ (by Nakayama's lemma) and $\sum_i b_i a_i = 0$ implies that all $b_i \in \mathfrak{m}$.

If $b_0 a_0 = \sum_{i>0} b_i a_i \in (a_0) \cap (a_1, \dots, a_p)$, then $b_0 \in \mathfrak{m}$, that is, $b_0 a_0 = 0$. This shows $(a_0) \cap (a_1, \dots, a_p) = 0$, and hence

$$\mathfrak{m}^t = (a_0) \oplus (a_1, \dots, a_p) = k \oplus (a_1, \dots, a_p).$$

PROPOSITION 2.3. *Let M be a finitely generated non-zero A -module with $\text{depth } M \geq \text{depth } A$. Then*

$$\mu^i(M) > 0 \quad \text{for} \quad \text{depth } M \leq i \leq \text{inf dim } M (\leq \infty),$$

in particular,

$$\mu^i(A) > 0 \quad \text{for} \quad \text{depth } A \leq i \leq \text{inj dim } A (\leq \infty).$$

PROOF. By Proposition 1.2 (c), assume $\text{depth } A = 0$ and use induction on i :

$i - 1 \rightarrow i, i \geq 2$: From the exact sequence

$$0 \rightarrow \mathfrak{m}^t \rightarrow A \rightarrow A/\mathfrak{m}^t \rightarrow 0,$$

with $t \geq 0$ such that $\mathfrak{m}^t = k \oplus \mathfrak{a}$ (by Lemma 2.2), it follows for $i \geq 2$ that

$$\text{Ext}^i(A/\mathfrak{m}^t, M) = \text{Ext}^{i-1}(\mathfrak{m}^t, M) = \text{Ext}^{i-1}(k, M) \oplus \text{Ext}^{i-1}(\mathfrak{a}, M).$$

By Proposition 1.2 (d) this gives $\mu^i(M) > 0$, if $\mu^{i-1}(M) > 0$ and $i \geq 2$.

$i = 1$: By Proposition 1.2 (b), (f) and (e) we can assume:

$$\text{depth } M = 0, \quad \text{inj dim } M = \infty, \quad \text{dim } A > 0.$$

Choose a prime ideal \mathfrak{p} with $\text{dim } A/\mathfrak{p} = 1$ and $a \in \mathfrak{m} - \mathfrak{p}$. From the exact sequence

$$0 \rightarrow A/\mathfrak{p} \xrightarrow{a} A/\mathfrak{p} \rightarrow A/(\mathfrak{p} + (a)) \rightarrow 0$$

it follows that also

$$\text{Hom}(A/\mathfrak{p}, M) \xrightarrow{a} \text{Hom}(A/\mathfrak{p}, M) \rightarrow \text{Ext}^1(A/(\mathfrak{p} + (a)), M)$$

is exact. Here $\text{Hom}(A/\mathfrak{p}, M) \neq 0$, since $\text{depth } M = 0$, and thus

$$\text{Ext}^1(A/(\mathfrak{p} + (a)), M) \neq 0$$

(by Nakayama's lemma), that is, $\mu^1(M) > 0$ (by Proposition 1.2 (d)).

PROPOSITION 2.4. *If M is a Cohen-Macaulay A -module, then $\mu^i(M) > 0$ for $\text{depth } M = \text{dim } M \leq i \leq \text{inj dim } M (\leq \infty)$.*

PROOF. Induction on $d = \text{depth } M = \text{dim } M$:

$d = 0$ is equivalent to $\text{Supp } M = \{\mathfrak{m}\}$, that is, $\mu^i(\mathfrak{p}, M) = 0$ for all i and

all prime ideals $\mathfrak{p} \neq \mathfrak{m}$. This gives $E(k)^{\mu^i(M)} = E^i \neq 0$ (that is, $\mu^i(M) > 0$) for $0 \leq i \leq \text{inj dim } M$, when

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^i \rightarrow \dots$$

is a minimal injective resolution of M .

$d - 1 \rightarrow d, d \geq 1$: By Proposition 2.3 we can assume $\text{depth } A > 0$. If $a \in \mathfrak{m} - (zA \cup zM)$, then M/aM is a Cohen–Macaulay $A/(a)$ -module of dimension $d - 1$ (see Serre [10, IV B]). The inductive hypothesis and Proposition 1.2 (c) now gives us the desired result.

The next result is a corollary of Theorem 3.4 of Peskine and Szpiro [8]. The equivalence between (ii) and (iii) in (c) is in fact well known [8, Cor. 3.5 of Th. 3.4] and is also proved by Levin and Vasconcelos [7, Th. 4.1] using different methods.

THEOREM 2.5. *Let M be a non zero finitely generated A -module of finite injective dimension s . Then:*

- (a) $\mu^i(M) > 0$ for $\text{depth } M \leq i \leq \text{inj dim } M = s$,
- (b) $\varphi(M) = \sum_i (-1)^{s-i} \mu^i(M) \geq 0$.
- (c) *The following statements are equivalent:*
 - (i) $\varphi(M) > 0$,
 - (ii) $\text{Ann } M = 0$,
 - (iii) $\text{grade}_A M = 0$.

REMARK. As pointed out before, this is the analogue of Proposition 1.3, the theorem about the Euler-characteristic of a finitely generated module of finite projective dimension. In fact we will use this proposition to prove Theorem 2.5.

PROOF. Let $\hat{}$ denote completion with respect to the \mathfrak{m} -adic topology. Then \hat{A} is again noetherian local with residue class field k , and further we know:

$$(\text{Ann}_A M)^\wedge = \text{Ann}_{\hat{A}} \hat{M}, \quad zA = A \cap z\hat{A}, \quad \hat{M} = M \otimes_A \hat{A}$$

and \hat{A} is A -flat (see Bourbaki [2, Chap. III, § 3, no. 4, th. 3 et cor. 1]). Let also L be a finitely generated A -module. Then $\text{Hom}_A(L, M)^\wedge = \text{Hom}_{\hat{A}}(\hat{L}, \hat{M})$ (by Bourbaki [2, Chap. I, § 2, no. 10, prop. 11]) and hence (by an easy induction on i) $\text{Ext}_A^i(L, M)^\wedge = \text{Ext}_{\hat{A}}^i(\hat{L}, \hat{M})$, in particular $\mu_A^i(M) = \mu_{\hat{A}}^i(\hat{M})$, since $\hat{k} = k$, and thus

$$\text{depth}_A M = \text{depth}_{\hat{A}} \hat{M} \quad \text{and} \quad \text{inj dim}_A M = \text{inj dim}_{\hat{A}} \hat{M}.$$

Therefore we may assume that A is complete.

Peskine and Szpiro ([8, Th. 3.4]) have proved that $\text{Ext}^i(E(k), N) = 0$ for $i \neq s$, and that $N = \text{Ext}^s(E(k), M)$ is finitely generated with $\text{Supp } M = \text{Supp } N$ and $\text{projdim } N < \infty$.

We want to show $\mu^i(M) = \beta_{s-i}(N)$ for all i . Assume that this is done. Then by Proposition 1.3 (b)

$$\text{projdim } N = s - \text{depth } M \quad \text{and} \quad \mu^i(M) = \beta_{s-i}(N) > 0$$

for $\text{depth } M \leq i \leq s$. By Proposition 1.3 (c) we get

$$\varphi(M) = \sum_i (-1)^{s-i} \mu^i(M) = \sum_j (-1)^j \beta_j(N) = \chi(N) \geq 0.$$

Now (a) and (b) are established. We prove (c) as follows.

(i) \Rightarrow (ii): By Proposition 1.3 (c), $\chi(N) = \varphi(M) > 0$ implies that $0 = \text{Ann } N \supseteq \text{Ann } M$.

(ii) \Rightarrow (iii) is obvious.

(iii) \Rightarrow (i): Assume $\text{grade}_A M = 0$. Since $\text{Supp } N = \text{Supp } M$, also $\text{grade}_A N = 0$ by Proposition 1.2 (h). By Proposition 1.3 (c) this gives

$$0 < \chi(N) = \varphi(M).$$

Now follows the proof of $\mu^i(M) = \beta_{s-i}(N)$ for all i : Let X, Y and Z be A -modules. Then there is a homomorphism

$$\sigma: X \otimes \text{Hom}(Y, Z) \rightarrow \text{Hom}(\text{Hom}(X, Y), Z)$$

defined by

$$\begin{aligned} \sigma(x \otimes f)(g) &= fg(x), \quad x \in X, \\ X &\xrightarrow{g} Y \xrightarrow{f} Z. \end{aligned}$$

The homomorphism σ is natural in X and Z , and σ is an isomorphism, if X is finitely generated and free (cf. [3, VI, Prop. 5.2]). Assume X finitely generated and let

$$\dots \rightarrow F_p \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow X \rightarrow 0$$

be a free resolution of X with each F_i finitely generated. Let

$$0 \rightarrow Z \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^q \rightarrow \dots$$

be an injective resolution of Z and define a double complex $K \cdots$ by

$$K^{pq} = F_{-p} \otimes \text{Hom}(Y, E^q) = \text{Hom}(\text{Hom}(F_{-p}, Y), E^q)$$

(and obvious differentiations). By standard filtrations one gets two spectral sequences with the same limit and initial terms:

$$\begin{aligned}
 (*) \quad I_2^{p^q} &= \text{Tor}_{-p}(X, \text{Ext}^q(Y, Z)), \\
 II_2^{p^q} &= \text{Ext}^p(\text{Ext}^{-q}(X, Y), Z).
 \end{aligned}$$

Let $X = k$, $Y = E(k)$, and $Z = M$ in these spectral sequences:

$$\begin{aligned}
 I^{p^q} &= \text{Tor}_{-p}(k, N) && \text{for } q = s, \\
 &= 0 && \text{otherwise,} \\
 II^{p^q} &= \text{Ext}^p(k, M) && \text{for } q = 0, \\
 &= 0 && \text{otherwise,}
 \end{aligned}$$

and hence we have $\text{Tor}_{s-i}(k, N) = \text{Ext}^i(k, M)$ for all i .

This ends the proof of Theorem 2.5.

REMARK. There are more direct ways to prove $\mu^i(M) = \beta_{s-i}(N)$, but we shall use the spectral sequences (*) later.

PROPOSITION 2.6. *If M is a non-zero finitely generated A -module, then:*

$$\mu^i(M) > 0 \quad \text{for} \quad \dim M \leq i \leq \text{injdim } M \ (\leq \infty).$$

PROOF. It is enough to show: $\mu^{i+1}(M) = 0$, if $\mu^i(M) = 0$ and $i \geq \dim M$. Assume therefore $\mu^i(M) = 0$ and $i \geq \dim M$, and hence also $\text{Ext}^i(L, M) = 0$ for all A -modules L of finite length (by Proposition 1.2 (d)). This gives that $\text{Ext}^{i+1}(\cdot, M)$ is a left-exact contra-variant functor in the category of A -modules of finite length and hence

$$\begin{aligned}
 \text{Ext}^{i+1}(k, M) &= \text{Hom}(k, \varinjlim_n \text{Ext}^{i+1}(A/\mathfrak{m}^n, M)) \\
 &= \text{Hom}(k, H_{\mathfrak{m}}^{i+1}(M)) = 0,
 \end{aligned}$$

where $H_{\mathfrak{m}}^i$ is the i th derived of the local cohomology functor (see Hartsorne [4, Prop. 4.5, Th. 2.8, and Prop. 6.4]).

REMARK. Proposition 2.4 follows of course also from Proposition 2.6.

3. μ^i and Gorenstein modules.

Assume in this section that G is a Gorenstein module of dimension s , that is (Sharp [11]), G is a finitely generated non-zero A -module with

$$s = \text{depth } G = \text{injdim } G \ (< \infty).$$

Then G is a Cohen-Macaulay module of dimension s (by Proposition 1.2 (b)).

Also the ring A is Cohen-Macaulay of dimension s (Sharp [11, 3.9]).

PROPOSITION 3.1. *Let M be a finitely generated non-zero A -module of finite injective dimension. Then:*

- (a) $\mu^i(M)\mu^s(G) = \beta_{s-i}(\text{Hom}(G, M))$ for all i , and $\text{proj dim Hom}(G, M) = s - \text{depth } M < \infty$,
- (a') $\text{Hom}(G, G)$ is free of rank $\mu^s(G)^2$,
- (b) $\text{grade}_M L = \text{grade}_{\text{Hom}(G, M)} L$ for each non-zero finitely generated module L , and $\text{Ass } M = \text{Ass Hom}(G, M)$.

REMARK. The first part of (b) follows, in the case $M = G$, from Sharp [11, 4.11].

PROOF. Consider the spectral sequences (*) from the proof of Theorem 2.5 and let $X = k$, $Y = G$ and $Z = M$. Since $\text{Ext}^q(G, M) = 0$ for $q \neq 0$ (by Proposition 1.2 (g)) and $\text{Ext}^{-q}(k, G) = 0$ for $q \neq -s$, the spectral sequences degenerate to isomorphisms:

$$\text{Tor}_{s-i}(k, \text{Hom}(G, M)) = \text{Ext}^i(\text{Ext}^s(k, G), M) \quad \text{for all } i.$$

This shows (a) and (a').

Since $\text{Ann } G \subseteq \text{Ann Hom}(G, G) = \text{Ann } A = 0$, it follows that

$$\text{Ass Hom}(G, M) = \text{Supp } G \cap \text{Ass } M = \text{Ass } M.$$

For $a \in \text{Ann } L - zM$ we have

$$\text{Hom}(G, M/aM) = \text{Hom}(G, M)/a \text{ Hom}(G, M)$$

and therefore the rest of (b) is an easy induction.

COROLLARY 3.2. *Let A be Gorenstein of dimension s , and let M be a non-zero finitely generated A -module. Then*

- (i) $\text{inj dim } M < \infty$,

if and only if

- (ii) $\text{proj dim } M < \infty$,

and when these conditions are satisfied, then

$$\mu^i(M) = \beta_{s-i}(M) \quad \text{for all } i.$$

REMARK. The equivalence between (i) and (ii) is well-known [7, Th. 2.2].

A stronger result is (cf. Jensen [5, Cor. 5]): A is Gorenstein, if and only if every (not necessarily finitely generated) A -module M of finite injective dimension has finite projective dimension. In fact, for a module M over a Gorenstein ring, (i) and (ii) are equivalent with

(iii) $\text{weak dim } M < \infty.$

LEMMA 3.3. *Let M be a finitely generated non-zero A -module. Then M is Cohen-Macaulay of dimension d , if and only if $\text{Ext}^i(M, G) = 0$ for $i \neq s - d$.*

PROOF. $\text{Ext}^i(M, G) = 0$ for $i \neq s - d$, if and only if $\text{depth } M = d$ and $\text{grade}_G M = s - d$ (by Proposition 1.2 (g) and (h)). By Proposition 3.1 (b) it follows that $\text{grade}_A M = \text{grade}_G M$, and hence $\text{depth } M = d$ and $\text{grade}_G M = s - d$, if and only if M is Cohen-Macaulay of dimension d (by Proposition 1.2 (i), since A is Cohen-Macaulay).

PROPOSITION 3.4. *If M is a Cohen-Macaulay A -module of dimension d , then:*

- (a) $\text{Ext}^{s-d}(M, G)$ is Cohen-Macaulay of dimension d too, and $\text{Ext}^{s-d}(\text{Ext}^{s-d}(M, G), G) = M \otimes \text{Hom}(G, G) = M^{\mu^s(G)^2}$.
- (b) $\text{Ass } M = \text{Ass } \text{Ext}^{s-d}(M, G)$.

REMARK. From Theorem 3.11 (v) of Sharp [11], (a) follows in the case $d = 0$.

PROOF. Consider the spectral sequences (*) from the proof of Theorem 2.5 with $X = M$ and $Y = Z = G$. By Proposition 3.1 and Lemma 3.3 they degenerate to

$$\text{Ext}^i(\text{Ext}^{s-d}(M, G), G) = \begin{cases} M \otimes \text{Hom}(G, G) & \text{for } i = s - d \\ 0 & \text{otherwise,} \end{cases}$$

and thus (a) follows from Lemma 3.3.

Let us now show $zM = z\text{Ext}^{s-d}(M, G)$. Assume $a \in \mathfrak{m} - zM$. Then M/aM is Cohen-Macaulay of dimension $d - 1$ (by Serre [10, IV B]), and hence $\text{Ext}^{s-d}(M/aM, G) = 0$. We have therefore the exact sequence

$$0 \rightarrow \text{Ext}^{s-d}(M, G) \xrightarrow{a} \text{Ext}^{s-d}(M, G),$$

and thus $a \notin z\text{Ext}^{s-d}(M, G)$. Now $zM \subseteq z\text{Ext}^{s-d}(M, G)$ is shown. The other inclusion follows by the duality of (a).

To show (b), it is enough to show $\text{Ass } M \subseteq \text{Ass Ext}^{s-d}(M, G)$ (again by the duality of (a)). Assume therefore $\mathfrak{p} \in \text{Ass } M$, that is,

$$\mathfrak{p} \subseteq zM = z\text{Ext}^{s-d}(M, G),$$

and hence there exist $\mathfrak{p}' \in \text{Ass Ext}^{s-d}(M, G)$ such that $\mathfrak{p} \subseteq \mathfrak{p}'$ and $\mathfrak{q} \in \text{Ass } M$ such that $\mathfrak{p}' \subseteq \mathfrak{q}$. We have $\mathfrak{p} \subseteq \mathfrak{p}' \subseteq \mathfrak{q}$, that is,

$$\mathfrak{p} = \mathfrak{p}' \in \text{Ass Ext}^{s-d}(M, G),$$

since $\text{Ass Ext}^{s-d}(M, G)$ is without embedded primes (see Prop. 13 of [10, IV B]).

PROPOSITION 3.5. *Let M be a Cohen–Macaulay A -module of dimension d . Then*

- (a) $\mu^i(M)\mu^s(G) = \beta_{i-d}(\text{Ext}^{s-d}(M, G))$ and $\beta_i(M)\mu^s(G) = \mu^{i+d}(\text{Ext}^{s-d}(M, G))$ for all i ,
- (b) $\text{inj dim } M < \infty \Leftrightarrow \text{proj dim Ext}^{s-d}(M, G) < \infty$ and $\text{proj dim } M < \infty \Leftrightarrow \text{inj dim Ext}^{s-d}(M, G) < \infty$.

PROOF. Consider the spectral sequences (*) with $X = k$, $Y = M$, and $Z = G$. They degenerate to isomorphisms:

$$\text{Tor}_{i-d}(k, \text{Ext}^{s-d}(M, G)) = \text{Ext}^s(\text{Ext}^i(k, M), G) \quad \text{for all } i.$$

This gives the first part of (a). The second part follows from the first and Proposition 3.4 (a).

(b) is now obvious.

COROLLARY 3.6. *Let A be a Gorenstein ring of dimension s , and $l_A M$ be a Cohen–Macaulay A -module of dimension d . Then:*

- (a) $\text{Ext}^{s-d}(M, A)$ is Cohen–Macaulay of dimension d and $\text{Ext}^{s-d}(\text{Ext}^{s-d}(M, A), A) = M$.
- (b) $\mu^i(M) = \beta_{i-d}(\text{Ext}^{s-d}(M, A))$ and $\beta_i(M) = \mu^{i+d}(\text{Ext}^{s-d}(M, A))$ for all i .
- (c) $\text{inj dim } M < \infty \Leftrightarrow \text{inj dim Ext}^{s-d}(M, A) < \infty \Leftrightarrow \text{proj dim } M < \infty \Leftrightarrow \text{proj dim Ext}^{s-d}(M, A) < \infty$.

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