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ON THE μ^i IN A MINIMAL INJECTIVE RESOLUTION II

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Let M be a module over a local (noetherian, commutative) ring A (with a multiplicative identity). The ith injective module I_M^i is a minimal injective resolution

$$0 \to M \to I_M^0 \to I_M^1 \to \ldots \to I_M^i \to \ldots$$

of M decomposes into the direct sum of indecomposable injective modules and for each prime ideal $\mathfrak{p}$ in A the cardinal number $\mu_A^i(\mathfrak{p}, M)$ denotes the number of copies of $E_A(A/\mathfrak{p})$ (=the injective hull of $A/\mathfrak{p}$) in this decomposition. This number $\mu_A^i(\mathfrak{p}, M)$ is only depending on i, $\mathfrak{p}$, and M, and it is finite if M is finitely generated, see Bass [2].

The previous paper [6] dealt with the question: For which integers i does $E_A(A/\mathfrak{p})$ appear in the decomposition of I_M^i (that is, $\mu^i(\mathfrak{p}, M) > 0$)? In the first part of the present paper we are interested in what happens if there is exactly one copy of $E_A(A/\mathfrak{p})$ in the decomposition of I_M^i for a certain i (that is, $\mu_A^i(\mathfrak{p}, M) = 1$). The theme is that if this happens for an i less or equal to the (Krull) dimension of A, then i is equal the depth of the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ (provided M is finitely generated).

Before these results are discussed more precisely it should be mentioned that

$$\mu_A^i(\mathfrak{p}, M) = \mu_{A_\mathfrak{p}}^i(\mathfrak{p}A_\mathfrak{p}, M_\mathfrak{p}),$$

so it suffices to consider the number $\mu^i(M) = \mu_A^i(\mathfrak{m}, M)$ for the maximal ideal $\mathfrak{m}$ in A. If M is finitely generated then it turns out that the least i for which $\mu^i(M) > 0$ is the depth of M, while the largest i with $\mu^i(M) > 0$ is the injective dimension $\mathrm{id}_A M$ of M (this might be infinite). The question of [6] was: Is $\mu^i(M) > 0$ for all i with $\operatorname{depth} M \leqq i \leqq \mathrm{id}_A M$ $(\leqq \infty)$ (when M is finitely generated)? and affirmative answers were given in special cases. In the more recent papers [5] (by Fossum, Foxby, Griffith, and Reiten) and [23] (by P. Roberts) this question has been answered in general (in the affirmative).

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In the present paper stronger results are obtained. Namely, if M is a non-zero finitely generated A-module we show that $\mu^i(M) \geqq 2$ if $\operatorname{depth} M < i < d-1$ where d is the (Krull) dimension of A, and even $\mu^i(M) \geqq 2$ if $\operatorname{depth} M < i < d$ provided A is (essentially) equicharacteristic (for definition see the beginning of Section 1). This has connection to the following conjecture of Vasconcelos [28, p. 53]: A is a Gorenstein ring if (and only if) A is of type one (that is, $\mu^d(A) = 1$ for $d = \dim A$). We prove that this conjecture — as well as a corresponding conjecture for modules (Conjecture B of Section 3) — holds for (essentially) equicharacteristic rings. This is Corollary (3.7), and the proof relies on a version of Peskine's and Szpiro's socalled New Intersection Theorem. This version is proved in Section 1. Also Grothendieck's Local Duality Theorem (see Section 2) is used.

We include also results in mixed characteristics, e.g. Vasconcelos' conjecture holds if A is a complete local ring without embedded prime divisors of 0, or if the dimension of A is at most two (see Section 4).

Section 5 contains the following result: $\mu_A^i(\mathfrak{p}, M) \leqq \mu_A^{i+l}(\mathfrak{q}, M)$ for prime ideals $\mathfrak{p} \subseteqq \mathfrak{q}$ in A and all i, provided l is the dimension of the local ring $A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q}$ and M is a finitely generated A-module. The proof uses Grothendieck's theory of dualizing complexes (which is discussed in Section 2) and a formula from [8] which allows passage to completions.

In Section 6 we give a lower bound for the injective dimension of a (not necessarily finitely generated) module of depth zero, provided A is essentially equicharacteristic. This result is known for finitely generated modules.

Notation and conventions.

We will use the notation of the previous paper [6] with the following three exceptions.

$\operatorname{id}_A M$ = the injective dimension of the A-module M.
$\operatorname{pd}_A M$ = the projective (= homological) dimension of M.
$\operatorname{fd}_A M$ = the flat (= weak homological) dimension of M.

By the dimension $\dim_A M$ of an A-module M we mean the dimension of the support $\operatorname{Supp}_A M$ of M, so if M is finitely generated then $\dim_A M$ is the (Krull) dimension of the local ring $A/\operatorname{Ann}_A M$, and in general, if M is the union of the submodules M_i (for i in an index set I) then

$$\dim_A M = \sup \{\dim_A M_i \mid i \in I\} .$$

Convention: $\dim_A 0 = -1$.

Though references in general will be given, the reader is assumed to be familiar with the basic facts about minimal injective resolutions, the numbers

$\mu_A^i(\mathfrak{p}, M)$, and Gorenstein rings (as in §§ 1–4 of Bass' paper [2]). Some of these facts are summarized in Section 1 of [6]. In particular we mention that $\mu_A^i(\mathfrak{p}, M)$ is the dimension of

$$\operatorname{Ext}_A^i (A/\mathfrak{p}, M)_\mathfrak{p} = \operatorname{Ext}_{A_\mathfrak{p}}^i ((k(\mathfrak{p}), M_\mathfrak{p})$$

considered as a vector space over $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, and we write $\mu^i(M) = \mu^i(\mathfrak{m}, M)$.

Finally let us repeat the standing assumption thoughout the entire paper:

$(A, \mathfrak{m}, k)$ *is a local ring of dimension* d.

1. The New Intersection Theorem.

A complex of A-modules is said to be *non-trivial* if at least one of the homology modules is non-zero. In this section of the paper we discuss — for a given bounded non-trivial complex of finitely generated free A-modules — a relation between the dimension of A, the dimensions of the homology modules, and the length of the complex. The best relation is obtained for the rings covered by the following definition.

DEFINITION. The ring A is said to be *essentially equicharacteristic*, if there exists an ideal $\mathfrak{a}$ in A with $\dim A/\mathfrak{a} = \dim A$ such that $A/\mathfrak{a}$ is equicharacteristic (that is, $A/\mathfrak{a}$ contains a field (as a subring)).

The *New Intersection Theorem* of Peskine and Szpiro [22], and independently of P. Roberts [23], asserts:

THEOREM (1.1). *Let* $0 \to F_s \to \ldots \to F_0 \to 0$ *be a non-trivial complex of finitely generated free modules and let* t *be an integer such that*

$$\dim H_i(F.) \leqq t \quad \textit{for all } i\,.$$

Then $\dim A \leqq s+t$, *provided* A *is essentially equicharacteristic.*

As it is the case with the original Intersection Theorem (see [2, Théorème (2.1)] and [15, p. 8]) also the New Intersection Theorem follows from Hochster's remarkable theorem on the existence of (socalled) big maximal Cohen–Macaulay modules over essentially equicharacteristic rings (cf. [15] and [16, (6.1) Theorem]).

For our applications we need the slightly stronger version of the New Intersection Theorem stated below.

THEOREM (1.2). *Let* $0 \to F_s \to \ldots \to F_0 \to 0$ *be a non-trivial complex of finitely generated free A-modules and let t be an integer such that*

$$\dim H_i(F.) \leqq t+i \quad \textit{for all } i\,.$$

Then $\dim A \leqq s+t+1$.

If moreover A is essentially equicharacteristic, then even $\dim A \leqq s+t$.

Though this version of the New Intersection Theorem is stronger than the original version (1.1) in two ways (the bounds on the dimensions of the homology modules have been weakened and there is also a conclusion in mixed characteristics) a proof of the original version could — after only minor modifications — give a proof of this version. However there is no complete proof of (1.1) in the literature and this is one reason for including a proof here. This proof relies on Hochster's result on the existence of big maximal Cohen–Macaulay modules (stated as Theorem (1.6) below) and gives — as a biproduct — a sufficient condition for the finiteness of (homological) depth of non-finitely generated modules, a fact to be used in Section 6. This is another reason for including a proof of (1.2) here.

Recall that for a finitely generated module M the maximal length of an M-regular sequence in $\mathfrak{m}$ is the depth of M, and

$$\operatorname{depth} M = \inf\{i \mid \operatorname{Ext}^i(k, M) \neq 0\}$$

(cf. [2]). For non-finitely generated modules the latter expression is taken as the definition of depth.

DEFINITION. $\operatorname{depth}_A M = \inf\{i \mid \operatorname{Ext}^i(k, M) \neq 0\}$ for any A-module M.

Note that even if $M \neq 0$ then $\operatorname{Ext}^i(k, M)$ might be zero for all i, so $\operatorname{depth} M = \infty$ might occur. However, if $a_1, \ldots, a_r$ is an M-regular sequence such that $\mathfrak{m}$ is in $\operatorname{Ass}(M/(a_1, \ldots, a_r)M)$ then it is easy to see that $\operatorname{depth} M = r$. Note finally that if n is an integer, and if $0 \to K \to L \to M \to 0$ is a short exact sequence of A-modules, then

(I) $\operatorname{depth} M \geqq n$ if $\operatorname{depth} K > n$ and $\operatorname{depth} L \geqq n$.

Here — and in what follows — we have not assumed that the depths of the involved modules are finite (unless explicitely stated).

The following useful lemma, which is known as the *Acyclicity Lemma*, has been stated and proved by Peskine and Szpiro (see [21, chapter I, (1.8)]) for finitely generated modules. With the above definition of the depth of a non-finitely generated module this lemma holds also for general modules. The proof in the general case is exactly the same as in [21] using the remark (I) above. We want to state the Acyclicity Lemma as follows.

LEMMA (1.3). *Let* $0 \to C_s \to \dots \to C_0 \to 0$ *be a complex of A-modules satisfying for all i,* $0 \leqq i \leqq s$, *the following two conditions.*

1) depth $C_i > i$.
2) depth $H_i(C.) = 0$ *or* $H_i(C.) = 0$.

Then $H_i(C.) = 0$ *for all i (so the complex C. is trivial).*

PROOF. As in [21, p. 55] we get $H_i(C.) = 0$ for $i > 0$, so the sequence

$$0 \to C_s \to \dots \to C_0 \to H_0(C.) \to 0$$

is exact. This long exact sequence breaks up into short exact sequences, and if the remark (I) above is applied to each of these, then we obtain depth $H_0(C.) > 0$, that is $H_0(C.) = 0$.

COROLLARY (1.4). *Let* $0 \to F_s \to \dots \to F_0 \to 0$ *be a non-trivial complex of finitely generated free modules such that* $H_i(F.)$ *is of finite length for all* $i \geqq 0$. *Let C be any A-module with* $\mathfrak{m}C \neq C$. *Then* depth $C \leqq s$.

PROOF. Since the complex $F.$ is non-trivial we can assume (by splitting off irrelevant free modules) that $H_0(F.) \neq 0$. Each of the homology modules of the complex $F. \otimes C$ satisfies

$$\operatorname{Supp} H_i(F. \otimes C) \subseteqq \{\mathfrak{m}\}$$

(since for each non-maximal prime ideal $\mathfrak{p}$ the complex $(F.)_\mathfrak{p}$ of free $A_\mathfrak{p}$-modules will be completely split, so $(F. \otimes_A C)_\mathfrak{p} = (F.)_\mathfrak{p} \otimes_A C$ becomes completely split too). In particular we have for all i that depth $H_i(F. \otimes C) = 0$ or $H_i(F. \otimes C) = 0$. Now suppose depth $C > s$, and thereby depth $(F_i \otimes C) > s \geqq i$ for all i, $0 \leqq i \leqq s$. By the Acyclicity Lemma applied to the complex $F. \otimes C$ we get $H_i(F. \otimes C) = 0$ for all i. However, $H_0(F. \otimes C) = H_0(F.) \otimes C$ is a non-zero module (by Nakayama's Lemma using that $H_0(F.)$ is non-zero and finitely generated, and using the assumption $k \otimes_A C \neq 0$). Thus the desired contradiction is obtained.

The following immediate consequence of this Corollary will play a crucial role in Section 6.

COROLLARY (1.5). *If the A-module C satisfies* $\mathfrak{m}C \neq C$ *then* depth $C \leqq \dim A$ $(< \infty)$.

PROOF. Let $F.$ be the Koszul complex for a system of parameters for A, and apply (1.4).

Now we state Hochster's result on the existence of big maximal Cohen–Macaulay modules (cf. [15]).

THEOREM (1.6). *If A is essentially equicharacteristic and if $(a_1,\ldots,a_d)$ is a system of parameters for A, then there exists an A-module C such that $a_1,\ldots,a_d$ is a C-regular sequence.*

∎

Such a module C, which need not be finitely generated, is called a *big maximal Cohen–Macaulay module*. Note that the term "$a_1,\ldots,a_d$ is a C-regular sequence" includes that $(a_1,\ldots,a_d)C \neq C$, so $\operatorname{depth} C = d$ (by a remark made earlier). We state this as a Corollary to Hochster's Theorem (1.6).

COROLLARY (1.7). *If A is essentially equicharacteristic, then there exists a module C such that $\mathfrak{m}C \neq C$ and* $\operatorname{depth} C = \dim A$.

For general rings (of possibly mixed characteristics) we can only do almost as good.

COROLLARY (1.8). *If $d = \dim A > 0$ then there exists a module D with $\mathfrak{m}D \neq D$ and* $\operatorname{depth} D = d-1$.

PROOF. If A is essentially equicharacteristic then let $D = C/a_1C$ (with C and a_1 as above).

Now suppose that A is not essentially equicharacteristic and let p be the characteristic of the residue field k (so p is necessarily a prime number). Then the ring $\bar{A} = A/pA$ is equicharacteristic and of dimension $d-1$, and hence there exists (for a given system $(\bar{a}_1,\ldots,\bar{a}_{d-1})$ of parameters for $\bar{A}$) a big maximal Cohen–Macaulay module D over $\bar{A}$. We have $\operatorname{depth}_A D = d-1$. (Pick $a_1,\ldots,a_{d-1} \in A$ such that the image in $\bar{A}$ of each a_i is $\bar{a}_i$. Then $a_1,\ldots,a_{d-1}$ is a D-regular sequence and $\mathfrak{m}$ belongs to $\operatorname{Ass}(D/(a_1,\ldots,a_{d-1})D)$.)

PROOF OF THEOREM (1.2). Without any loss of generality we may assume that A is complete (in the $\mathfrak{m}$-adic topology) and thereby catenary.

The next step is to reduce to the case where A is a domain by replacing A by $A/\mathfrak{p}$ for a suitable prime ideal $\mathfrak{p}$. Namely, pick $\mathfrak{p}$ in $\operatorname{Spec} A$ with $\dim A/\mathfrak{p} = \dim A$ (such that if A is essentially equicharacteristic, then $A/\mathfrak{p}$ is equicharacteristic). Let for a short while $\tilde{F}.$ denote the complex $F.\otimes_A A/\mathfrak{p}$ which is a bounded complex of finitely generated free $A/\mathfrak{p}$-modules. It is a non-trivial complex since

$$H_j(\tilde{F}.) = H_j(F.) \otimes A/\mathfrak{p} \neq 0 \quad \text{when } j = \inf\{i \mid H_i(F.)\neq 0\}\ .$$

Now to justify that A can be replaced by $A/\mathfrak{p}$ it only remains to prove that $\dim H_i(\tilde{F}.) \leqq t+i$ for all $i \geqq 0$. This is easy to see, because if for a fixed number l the prime ideal $\mathfrak{q}$ contains $\mathfrak{p}$ and has $\dim A/\mathfrak{q} > l+t$, then

$$H_i(F.\otimes A_\mathfrak{q}) = H_i(F.)_\mathfrak{q} = 0 \quad \text{for } i \leqq l\ ,$$

and hence

$$H_i(\tilde{F}.)_\mathfrak{q} = H_i(F\otimes A_\mathfrak{q} \otimes A/\mathfrak{p}) = 0 \quad \text{for } i \leqq l\ .$$

These first two steps shows that we in the rest of this proof can assume that A is a catenary domain. Actually, the only thing we need is

(II) $$\dim A_\mathfrak{p} = ht\,\mathfrak{p} = \dim A - 1$$

for all $\mathfrak{p}$ in $\operatorname{Spec} A$ with $\dim A/\mathfrak{p} = 1$.

The proof is continued by induction on $d = \dim A$.

The case $d=0$ is trivial since necessarily $t \geqq -s$.

The inductive step. The inductive hypothesis is that the result holds for rings of dimension $d-1$. Divide into two cases.

1°. $\dim H_i(F.) \leqq 0$ *for all* i. Let $j = \inf\{i \mid H_i(F.) \neq 0\}$. Then $t \geqq -j$ and Z_j = the kernel of $F_j \to F_{j-1}$ is a free module. Now applying (1.4) and (1.8), respectively (1.4) and (1.7), to the complex

$$0 \to F_s \to \dots \to F_{j+1} \to Z_j \to 0$$

we get $d-1 \leqq s-j \leqq s+t$, respectively $d \leqq s+t$.

2°. $\dim H_k(F.) > 0$ *for a suitable* k. Pick $\mathfrak{p}$ in $\operatorname{Supp} H_k(F.)$ with $\dim A/\mathfrak{p} = 1$. The complex $L. = (F.)_\mathfrak{p}$ of finitely generated $A_\mathfrak{p}$-modules in non-trivial and has homology modules $H_i(L.) = H_i(F.)_\mathfrak{p}$, and

$$\dim_{A_\mathfrak{p}} H_i(F.)_\mathfrak{p} \leqq \dim_A H_i(F.) - 1 \leqq t-1+i$$

for all i. By the inductive hypothesis and (II) we get

$$d-1 = \dim A_\mathfrak{p} \leqq s+(t-1)+1 + s+t$$

in general, and even $d-1 \leqq s+t-1$ if A is essentially equicharacteristic.

REMARK (1.9). Let $\mathscr{C}$ be a class of local rings such that $\mathscr{C}$ is closed under completion and localization at prime ideals. and such that there to each complete local ring $(A, \mathfrak{m}, k)$ in this class $\mathscr{C}$ exists an A-module C and an ideal $\mathfrak{a}$ in A such that $\mathfrak{m}C \neq C$, $\operatorname{depth} C = \dim A = \dim A/\mathfrak{a}$, and $A/\mathfrak{a}$ is an equidimensional ring (that is, all the minimal prime ideals in the ring have the same coheight, namely the dimension of the ring). Besides the example where $\mathscr{C}$ is the class of essentially equicharacteristic local rings (cf. Hochster's Theorem (1.6)) there are two immediate examples: 1) $\mathscr{C}$ = the class of Cohen–Macauly local rings, and 2) $\mathscr{C}$ = the class of local rings of dimension at most two. (If A is complete and $\dim A \leqq 2$ then the depth of the finitely generated A-module Ω_A^0 is $d = \dim A$, see Section 2 and [17].) It follows from the proof of (1.2) that the condition "A is essentially equicharacteristic" in the last paragraph of the statement in (1.2) could be replaced by "A is in such a class $\mathscr{C}$ of local rings".

2. Local cohomology and dualizing complexes.

In the remainder of the paper E denotes the injective hull $E_A(k)$ of the residue field k, and $\check{}$ denotes Matlis duality, that is $M\check{} = \operatorname{Hom}_A(M, E)$ for an A-module M.

The section functor $\Gamma_{\mathfrak{m}}$ with support in the closed point $\mathfrak{m}$ is defined by

$$\Gamma_{\mathfrak{m}}(M) = \{x \in M \mid \mathfrak{m}^j x = 0 \textit{ for } j \gg 0\}$$
$$= \varinjlim_j \operatorname{Hom}(A/\mathfrak{m}^j, M)$$

for an A-module M. The ith right derived functor of this left exact functor is called th ith *local cohomology functor*, it is denoted by $H_{\mathfrak{m}}^i$, and for each module M and each i we have

$$H_{\mathfrak{m}}^i(M) = \varinjlim_j \operatorname{Ext}^i(A/\mathfrak{m}^j, M) .$$

For this, and for other details about local cohomology, see [11] or the more elementary accounts [26] and [19].

For future reference we list the first two basic properties of the local cohomology functors, namely the relation to dimension and depth.

LEMMA (2.1). *Let M be any A-module. Then*

(1) $H_{\mathfrak{m}}^i(M) = 0$ *for* $i > \dim M$ *and*

$H_{\mathfrak{m}}^n(M) \neq 0$ *if* $n = \dim M$ *and M is finitely generated.*

(2) $H_{\mathfrak{m}}^i(M) = 0$ *for* $i < \operatorname{depth} M$ *and*

$H_{\mathfrak{m}}^t(M) \neq 0$ *if* $t = \operatorname{depth} M < \infty$.

PROOF. (1) is [11, Proposition 6.4] and [26, 6.1. Theorem] and (2) follows as the corresponding result for finitely generated modules, see [19, p. 20 lines 13–18].

COROLLARY (2.2). *If* $\mathfrak{m}M \neq M$ *then*

$$\operatorname{depth} M \leqq \dim M \ (<\infty) .$$

PROOF. $t = \operatorname{depth} M$ is finite by (1.5), and hence $H_{\mathfrak{m}}^t(M) \neq 0$ by (2) of the Lemma, so $t \leqq \dim M$ follows from (1) of the Lemma.

Dualizing complexes. In order to facilitate the discussion of dualizing complexes we will in the rest of this Section assume that A is a homomorphic image of a Gorenstein local ring R of dimension n ($\geqq d = \dim A$). Let $Q^{\cdot}$ be the minimal injective resolution of R (as an R-module). Then

$$Q^i = \coprod_{\operatorname{ht} \mathfrak{q} = i} E_R(R/\mathfrak{q}) \qquad \text{for all } i \text{ (cf. [2])} .$$

So in particular $Q^i = 0$ if $i > n$ ($= \dim R$) (or $i < 0$).

In general for a complex $X^{\cdot}$ and an integer l let $X^{\cdot}[l]$ denote the complex $X^{\cdot}$ shifted l places to the left, that is X^i sits degree $i-l$ in the complex $X^{\cdot}[l]$.

The complex $D^{\cdot} = \operatorname{Hom}_R(A, Q^{\cdot})\ [n-d]$ is called the *dualizing complex* of A (with respect to R). Note that $H^i(D^{\cdot}) = \operatorname{Ext}_R^{n-d+i}(A, R)$ is finitely generated and that

$$D^i = \coprod_{\dim A/\mathfrak{p} = d-i} E_A(A/\mathfrak{p}) .$$

So in particular $D^i = 0$ for $i > d$ (or $i < 0$). Hartshorne's notes [13, chapter V] serves as the basic reference for dualizing complexes, but a more elementary account can be found in Sharp's paper [27]. Hartshorne's account works in the derived category (here denoted by $\mathscr{D}$) of the category of A-modules (cf. [13, chapter I]), but the only thing we need to know is that isomorphic complexes in $\mathscr{D}$ have isomorphic cohomology modules. For complexes $X^{\cdot}$ and $Y^{\cdot}$ the notation $\operatorname{Hom}^{\cdot}(X^{\cdot}, Y^{\cdot})$ denotes the complex in which the module $\prod_p \operatorname{Hom}(X^p, Y^{p+i})$ is sitting in degree i (and with the obvious differentiations).

Now we state the first fundamental property of the dualizing complex $D^{\cdot}$.

PROPOSITION (2.3). *For each bounded complex* $X^{\cdot}$ *with finitely generated cohomology modules there is an isomorphism in* $\mathscr{D}$:

$$X^{\cdot} \cong \operatorname{Hom}^{\cdot}(\operatorname{Hom}^{\cdot}(X^{\cdot}, D^{\cdot}), D^{\cdot}) .$$

See [13, chapter V, Proposition 2.1, p. 258] and [27, (3.6) Theorem]. Actually the duality of (2.3) characterizes the dualizing complexes among the bounded complexes of injective modules having finitely generated cohomology modules. Also the dualizing complex is unique (up to isomorphism in $\mathscr{D}$ and translation) (cf. [13] or [27]).

DEFINITION. For a finitely generated A-module M and an integer i write

$$\Omega_M^i = \mathrm{Ext}_R^{n-d+i}(M, R)$$

(with R as above).

Note that $\Omega_M^i = H^i(\mathrm{Hom}_A(M, D^\cdot))$ (since $\mathrm{Hom}_A(M, D^\cdot) = \mathrm{Hom}_R(M, Q^\cdot)$). That Ω_M^i is independent of the choice of the Gorenstein local ring R is one consequence of the next fundamental property of dualizing complexes. This result is known as the *Local Duality Theorem* (cf. [13, chapter V, Theorem 6.2, p. 278]).

THEOREM (2.4). *For each finitely generated module M with the minimal injective resolution $I_M^\cdot$ there is an isomorphism in $\mathscr{D}$:*

$$\Gamma_{\mathfrak{m}}(I_M^\cdot) \cong \mathrm{Hom}_A(M, D^\cdot)\check{}\,[-d] .$$

In particular for all i:

$$H_{\mathfrak{m}}^i(M) \cong (\Omega_M^{d-i})\check{} \quad \text{(as modules)} .$$

COROLLARY (2.5). *For a finitely generated non-zero A-module M the following hold.*

(a) $\Omega_M^{d-i} = 0$ *if* $i > \dim M$ *or* $i < \operatorname{depth} M$, *and*

$\Omega_M^{d-i} \neq 0$ *if* $i = \dim M$ *or* $i = \operatorname{depth} M$.

(b) *M is a Cohen–Macaulay module of dimension n if and only if*

$\Omega_M^{d-i} = 0$ *for* $i \neq n$.

(c) $\dim \Omega_M^{d-i} \leqq i$ *for all* i $(\geqq 0)$.

(d) *If* $n = \dim M$ *then* $\mathrm{Ass}\,\Omega_M^{d-n}$ *consists of the* $\mathfrak{p}$ *in* $\mathrm{Supp}\,M$ *with* $\dim A/\mathfrak{p} = n$. *In particular* $\Omega_M^{d-n} \neq 0$ *and* $\dim \Omega_M^{d-n} = n = \dim M$.

PROOF. (a) follows directly from (2.1) and (2.4), and (b) follows from (a). (c) and (d) are [11, Propositions 6.4 and 6.6].

If A is a Cohen–Macaulay ring, then the dualizing complex $D^\cdot$ has only one non-vanishing cohomology module, namely Ω_A^0 in degree zero. Ω_A^0 is in this case called the *dualizing module*, and we have $\operatorname{depth} \Omega_A^0 = \operatorname{id} \Omega_A^0 = d$ (since $D^\cdot$ is the minimal injective resolution of Ω_A^0) and $\mathrm{Hom}_A(\Omega_A^0, \Omega_A^0) \cong A$. Other names for this module are *canonical module* (in [14]) and *Gorenstein module of rank one* (in [25] and [7]).

Note furthermore that it follows from the definition of dualizing complexes, that if $\mathfrak{p}$ is a prime ideal then $(D^\cdot)_\mathfrak{p}[d-\dim A/\mathfrak{p}-\operatorname{ht}\mathfrak{p}]$ is a dualizing complex for the ring $A_\mathfrak{p}$ (which is a homomorphic image of the Gorenstein local ring $R_\mathfrak{q}$ where $\mathfrak{q}$ is the prime ideal in R with $\mathfrak{q}A=\mathfrak{p}$). In particular

$$(\Omega_A^{d-i})_\mathfrak{p} = \Omega_{A_\mathfrak{p}}^{u-i} \quad \text{where } u = \dim A/\mathfrak{p}+\operatorname{ht}\mathfrak{p} \ (\leqq d) .$$

Thus we have obtained the following result.

REMARK (2.6). If $\mathfrak{p}$ is a minimal prime ideal in A (so $A_\mathfrak{p}$ is an artinian ring), then $(\Omega_A^{d-i})_\mathfrak{p}$ is zero for $i \neq \dim A/\mathfrak{p}$, while

$$(\Omega_A^{d-\dim A/\mathfrak{p}})_\mathfrak{p} = E_{A_\mathfrak{p}}(k(\mathfrak{p}))$$

which is an $A_\mathfrak{p}$-module of length equal to the length of $A_\mathfrak{p}$ itself (since Matlis duality preserves the length of $A_\mathfrak{p}$).

Now let (as in (2.4)) $I_M^\cdot$ denote the minimal injective resolution of the finitely generated A-module M. Then $\Gamma_\mathfrak{m}(I_M^j)=E^{\mu^j(M)}$ for all j (cf. [11, Corollary 4.8]), so if $F.$ denotes the complex $\Gamma_\mathfrak{m}(I_M^\cdot)^\vee$ then for each j we have

$$F_j = \Gamma_\mathfrak{m}(I_M^j)^\vee = (E^\vee)^{\mu^j(M)} = \hat{A}^{\mu^j(M)}$$

(where $\hat{}$ denotes the completion with respect to the $\mathfrak{m}$-adic topology) and

$$H_j(F.) = H_\mathfrak{m}^j(M)^\vee \cong \hat{\Omega}_M^{d-j}$$

(by (2.4)). We collect these observations in the following remark.

REMARK (2.7). To each finitely generated module M there exists a complex

$$F. = \ldots \to F_j \to \ldots \to F_1 \to F_0 \to 0$$

such that each F_j is a free $\hat{A}$-module of rank $\mu^j(M)$, and such that

$$H_j(F.) = \begin{cases} \hat{\Omega}_M^{d-j} & \text{for depth } M \leqq j \leqq \dim M \\ 0 & \text{otherwise} . \end{cases}$$

Therefore $\dim H_i(F.) \leqq i$ for all i and $\dim H_n(F.) = \dim M$ if $n=\dim M$ (by (2.6.c) and (2.6.d)).

3. Essentially equicharacteristic rings.

Two of the best characterizations of Gorenstein rings contained in Bass [2] is that A is a Gorenstein ring if and only if A is a Cohen–Macaulay ring with some system of parameters generating an irreducible ideal (that is, an ideal that is not the intersection of two strictly larger ideals) or equivalently: A is a Cohen–Macaulay ring with $\mu^d(A)=1$ ($d=\dim A$). Vasconcelos has conjectured that in the latter characterization the condition "A is a Cohen–Macaulay ring" can be omitted (see [28, p. 53]), and he proves that this is the case if $\dim A = 1$ and $A_{\mathfrak{p}}$ is Gorenstein for all minimal prime ideal $\mathfrak{p}$.

CONJECTURE A. If $\mu^d(A) = 1$ $(d = \dim A)$ then A is a Gorenstein ring.

Using our version of the New Intersection Theorem and the Local Duality Theorem it is possible to prove that this Conjecture holds for essentially equicharacteristic rings. This is done in this Section. In the next Section we shall see that the Conjecture holds for special rings of mixed characteristics, e.g. complete local rings without embedded prime divisors of 0. Conjecture A has also been touched by Peskine and Szpiro in Proposition 5.7 in the unpublished notes "Notes sur un air de H. Bass".

The following simple example shows that the condition "A is a Cohen–Macaulay ring" can not be omitted in the first of the two characterizations of Gorenstein rings mentioned above.

EXAMPLE. Let R be a 2-dimensional Gorenstein local ring and let x, y be an R-regular sequence. Though the ring $A = R/x(x, y)$ is not Cohen–Macaulay, and hence not Gorenstein, the image $\bar{y}$ of y in A forms a system of parameters and the ideal $(\bar{y})$ in A is irreducibel (since $A/(\bar{y}) \cong R/(x^2, y)$ is Gorenstein).

The following result is a generalization to modules of the second of the characterizations of Gorenstein rings mentioned in the beginning of this Section. See also [14, Korollar 6.12] and [7, Corollary 4.3].

PROPOSITION (3.1). *Let M be an n-dimensional Cohen–Macaulay module such that $\mu^n(M) = 1$. Then the local ring $B = A/\operatorname{Ann} M$ is Cohen–Macaulay (of dimension n) and M is a dualizing B-module.*

This proposition (which will be proved below) leads to the following conjecture (strengthening Conjecture A).

CONJECTURE B. If M is an n-dimensional finitely generated module with $\mu^n(M) = 1$, then both M and $B = A/\operatorname{Ann} M$ are Cohen–Macaulay and M is a dualizing B-module.

We shall (in this section) see that this conjecture holds if either $n < d = \dim A$ or $n = \dim A$ and A is essentially equicharacteristic. In the next section other partial results are given.

PROOF OF (3.1). As usual (see for example the beginning of the proof of [6, Theorem 2.5]) we will assume that A is complete, or just a homomorphic image of Gorenstein local ring R (cf. [4]). We may furthermore assume $\dim R = \dim A = d$ (by taking R modulo a maximal R-regular sequence in $\operatorname{Ann}_R A$). Note that M is a Cohen–Macaulay module of dimension n also as an R-module. From [6, Corollary 3.6] (or from the corresponding results in [14]) it follows that $\Omega_M^{d-n} = \operatorname{Ext}_R^{d-n}(M,R)$ is a Cohen–Macaulay module of dimension n and with $\mu_R^n(M)$ as the minimal number of generators. Now $\mu_R^n(M) = \mu_A^n(M) = 1$ (by [14, 1.22 b)]), so Ω_M^{d-n} is a cyclic module, that is

$$\Omega_M^{d-n} = A/\operatorname{Ann}_A \Omega_M^{d-n} .$$

From [6, Corollary 3.6] it follows also that $\operatorname{Ext}_R^{d-n}(\Omega_M^{d-n}, R) = M$, so in particular $\operatorname{Ann}_A \Omega_M^{d-n} = \operatorname{Ann}_A M$, that is $\Omega_M^{d-n} = A/\operatorname{Ann}_A M = B$, and hence B is Cohen–Macaulay of dimension n and $M = \operatorname{Ext}_R^{d-n}(B,R)$ is a dualizing B-module.

Together with Remark (2.7) the following consequence of our version of the New Intersection Theorem will be essential to the results in this section.

LEMMA (3.2). *Let t be an integer, $0 \leqq t \leqq \dim A$, and let $F.$ be a complex of finitely generated free A-modules, such that $F.$ is bounded to the right by t, that is*

$$F. = \ldots \to F_i \to \ldots \to F_{t+1} \to F_t \to 0 .$$

Let r_i denote the rank of F_i, and let s_i denote the alternating sum

$$s_i = r_i - r_{i-1} + \ldots + (-1)^{i-t} r_t .$$

Assume furthermore that

$$\dim H_i(F.) \leqq i \quad \textit{for all } i \leqq \dim A .$$

Then $s_m \geqq 0$ for all $m \leqq \dim A$, and even $s_m > 0$ provided $m \geqq t$, $F_t \neq 0$, and

either 0) $m < \dim A - 1$,
or 1) $\dim H_m(F.) \geqq m$,
or 2) *$m < \dim A$ and A is essentially equicharacteristic.*

PROOF. Let Z_m denote the kernel of $F_m \to F_{m-1}$ and let $F.|m$ denote the truncated complex

$$F.|m = 0 \to F_m \to \ldots \to F_{t+1} \to F_t \to 0 .$$

Then

$$H_i(F.|m) = \begin{cases} Z_m & \text{for } i = m \\ H_i(F.) & \text{for } i < m . \end{cases}$$

For a prime ideal $\mathfrak{p}$ with $\dim A/\mathfrak{p} \geqq m$ (and such exists) the assumption $\dim H_i(F.) \leqq i$ for $i \leqq \dim A$ gives that $(Z_m)_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module of rank s_m, so in particular $s_m \geqq 0$ for $m \leqq \dim A$. Now suppose $s_m = 0$, that is $(Z_m)_\mathfrak{p} = 0$ for all $\mathfrak{p}$ in $\operatorname{Spec} A$ with $\dim A/\mathfrak{p} \geqq m$, or in other words $\dim Z_m < m$. This gives

$$\dim H_m(F.) \leqq \dim Z_m < m$$

(contrary to assumption 1)). Furthermore if we apply the New Intersection Theorem (1.2) to the bounded complex $F.|m$ we get that $\dim A \leqq m+1$, and even $\dim A \leqq m$ if A is essentially equicharacteristic (contrary to assumptions 0) and 2) respectively).

Using Remark (2.7) one can translate this Lemma into a result on the $\mu^i(M)$'s.

THEOREM (3.3). *Let M be a finitely generated non-zero A-module and let m be an integer such that* $\operatorname{depth} M \leqq m \leqq \dim A$. *Write*

$$\sigma^m(M) = \mu^m(M) - \mu^{m-1}(M) + \ldots + (-1)^{m-t}\mu^t(M) \quad \textit{where } t = \operatorname{depth} M .$$

Then $\sigma^m(M) \geqq 0$, and even $\sigma^m(M) > 0$ provided

either 0) $m < \dim A - 1$,
or 1) $m = \dim M$,
or 2) $m < \dim A$ *and A is essentially equicharacteristic.*

COROLLARY (3.4). *If M is a finitely generated non-zero A-module then the following hold* (with $d = \dim A$).

a) $\mu^m(M) \geqq 2$ *for all m with* $\operatorname{depth} M < m < d-1$, *and* $\mu^{d-1}(M) \geqq 1$.
b) $\mu^{d-1}(M) \geqq 2$ *if* $\operatorname{depth} M < d-1$ *and either*

1) $\dim M = d-1$, *or*
2) *A is essentially equicharacteristic.*

c) $\mu^d(M) \geqq 2$ *if* $\operatorname{depth} M < \dim M = d$ *and A is essentially equicharacteristic.*

PROOF. Use the Theorem and the fact that if $\sigma^{m-1}(M) > 0$ and $\sigma^m(M) > 0$ then

$$\mu^m(M) > \mu^m(M) - \sigma^{m-1}(M) = \sigma^m(M) > 0 ,$$

and hence $\mu^m(M) \geqq 2$.

REMARK (3.5). Point a) of this Corollary gives an affirmative answer to the question of [6]: If M is non-zero and finitely generated then $\mu^m(M) > 0$ for all m with $\operatorname{depth} M \leqq m \leqq \operatorname{id} M$ ($\leqq \infty$) (cf. also [2, (3.5) Lemma] or [6, Proposition

2.6]). However, this result is known, see Fossum, Foxby, Griffith and Reiten [5, Theorem (1.1)] (where the proof also uses Hochster's big maximal Cohen–Macaulay modules in equal characteristics to get the result in general). P. Roberts has given a different proof (see [23]) which relies on dualizing complexes and local duality. We shall use these ideas in Section 5.

REMARK (3.6). If M is a finitely generated non-zero A-module of finite injective dimension $\operatorname{id} M = \operatorname{depth} A$ (cf. [2, (3.3) Lemma]) then (3.4.a) and c) give $\operatorname{depth} A = \operatorname{id} M \geqq \dim A - 1$, and even $\operatorname{depth} A = \dim A$ (that is, A is Cohen–Macaulay) if A is essentially equicharacteristic, and thus we have rediscovered the fact that the socalled Bass' conjecture holds for essentially equicharacteristic rings, see [21, chaptitre II, Théorème (5.1)] and [15, p. 10].

COROLLARY (3.7). *If M is a finitely generated A-module of dimension n such that $\mu^n(M) = 1$, then M is a dualizing $A/\operatorname{Ann} M$-module provided either*

1) $n < \dim A$, *or*

2) $n = \dim A$ *and A is essentially equicharacteristic.*

In particular, if A is essentially equicharacteristic and $\mu^d(A) = 1$ $(d = \dim A)$, then A is a Gorenstein ring.

In view of Remark (1.9) we get (with the notation of Lemma (3.2)) $s_m > 0$ provided $t \leqq m < \dim A$ and A is Cohen–Macaulay and from this we get the following result.

PROPOSITION (3.8). *If A is Cohen–Macaulay and M is a finitely generated A-module, then*

$$\mu^m(M) \geqq 2 \quad \text{if } \operatorname{depth} M < m < \dim A ,$$

and

$$\mu^d(M) \geqq 2 \quad \text{if } \operatorname{depth} M < d = \dim M = \dim A .$$

REMARK (3.9). This shows that Conjecture B holds for modules over a Cohen–Macaulay ring, a fact that is also covered by Proposition (4.2) in the next section.

So far we have only been concerned with the numbers $\mu^m(M)$ for m between $\operatorname{depth} M$ and $\dim A$ (Proposition (4.3) will be the only exception). The last result of this Section gives the reason for this: For $m > \dim A$ (even for $m \geqq \dim M$) $\mu^m(M)$ can be expressed as some Betti number of a certain module, and problems on Betti numbers seem to require different methods.

PROPOSITION (3.10). *To each finitely generated module M of dimension n there exists a finitely generated $\hat{A}$-module N such that*

$$\mu_A^m(M) = \beta_{m-n}^{\hat{A}}(N) \quad \textit{for all } m \geqq n .$$

PROOF. The truncation

$$\ldots \to F_i \to \ldots \to F_{n+1} \to F_n \to 0$$

of the complex from Remark (2.7) is a free resolution of N = the cokernel of $F_{n+1} \to F_n$. This is even a minimal free resolution, since for all i the map $k \otimes F_{i+1} \to k \otimes F_i$ is zero, because it is the Maltis dual of the map

$$\operatorname{Hom}(k, \Gamma_m(I_M^i)) \to \operatorname{Hom}(k, \Gamma_m(I_M^{i+1})) ,$$

which is nothing but the map $\operatorname{Hom}(k, I_M^i) \to \operatorname{Hom}(k, I_M^{i+1})$ and thus zero (when $I_M^{\cdot}$ is a minimal injective resolution of M).

REMARK (3.11). Note that it follows from the proof that Ω_M^{d-n} is a submodule of N and that the homomorphisms $F_{j+1} \to F_j$ are non-zero for $j \geqq n = \dim M$ if $\operatorname{id} M = \infty$.

4. Unmixedness conditions and rings of low dimension.

In this Section we prove that Conjecture B holds if A is complete and 0 is unmixed in A in the sense of Nagata [20, p. 82], that is A is equidimensional and 0 is without embedded prime divisors in A, or in other words

$$\operatorname{Ass} A = \{\mathfrak{p} \in \operatorname{Spec} A \mid \dim A/\mathfrak{p} = d\} ,$$

so if Z is a non-zero submodule of a finitely generated free module then $\dim Z = d$ $(= \dim A)$.

LEMMA (4.1). *In the situation of Lemma* (3.2) *assume that* $s_m = 0$ *for a fixed* $m \leqq \dim A$ *and that* $\dim A/\mathfrak{p} \geqq m$ *for all* $\mathfrak{p}$ *in* $\operatorname{Ass} A$. *Then the homomorphism* $F_{m+1} \to F_m$ *is zero.*

PROOF. With the notation of the proof of (3.2) we get $\dim Z_m < m$ (as there), and hence $Z_m = 0$ (by assumption).

PROPOSITION (4.2). *Conjecture* B *holds if* $\dim \hat{A}/\mathfrak{q} \geqq \dim A - 1$ *for all* $\mathfrak{q}$ *in* $\operatorname{Ass} \hat{A}$. *In particular, Conjecture* B *holds if* 0 *is unmixed in* $\hat{A}$.

PROOF. As usual we will assume that A is complete. Since Conjecture B holds when $\dim M < d$ $(= \dim A)$, we will also assume that $\dim M = d$. By Proposition

(3.1) it suffices to show that M is Cohen–Macaulay (provided $\mu^d(M)=1$), so we assume moreover that depth $M<d=\dim M$, and thus we are required to prove $\mu^d(M)\geqq 2$.

Now $\sigma^d(M)>0$ by Theorem (3.3.1), and hence "$\sigma^{d-1}(M)>0$" is the only thing left to justify, so assume $\sigma^{d-1}(M)=0$. Lemma (4.1) gives that $F_d \to F_{d-1}$ is the zero homomorphism. From Roberts' proof of the non-vanishing of the μ^i (mentioned in Remark (3.5)) (cf. [23, p. 106]) we get $M=M'\oplus M''$ where id M' $<d$ and depth $M''\geqq d$. By [21, Corollaire (4.9)] one of the modules M' and M'' must be zero. We have assumed depth $M<d$, so $M''=0$, and hence depth A $=$ id $M<d$. But on the other hand $\dim M=d$ and id $M<\infty$ implies that id M $\geqq d$ (by [2, (3.2) Corollary]), and thus we have obtained the desired contradiction.

Even if we drop the condition that $\hat{A}$ is equidimensional Conjecture A holds. If we let Min $\hat{A}$ denote the set of minimal prime ideals in $\hat{A}$, then the only condition we need is Min $\hat{A}=$ Ass $\hat{A}$, that is, 0 is without embedded prime divisor in $\hat{A}$. Actually we prove more.

PROPOSITION (4.3). *If there exists an integer $m\geqq d$ $(=\dim A)$ such that $\mu^m(A)\leqq 1$, and if 0 is without embedded prime divisors in $\hat{A}$, then A is a Gorenstein ring.*

PROOF. Again we assume that A is complete. Furthermore let us assume that there exists an $m\geqq d$ such that $\mu^m(A)=1$ ($\mu^d(A)=0$ is impossible and $\mu^m(A)=0$ for some $m>d$ implies that A is Gorenstein, cf. [2, (3.5) Lemma]). So $F_m=A$ when $F.=\Gamma_{\mathfrak{m}}(I^{\cdot}_M)^{\vee}$ as usual.

Let $L.$ denote the truncated complex

$$0 \to A \to F_{m-1} \to \dots \to F_t \to 0$$

where $t=$ depth A, and let $\mathfrak{b}=Z_m=$ the kernel of $A \to F_{m-1}$. We have

$$H_i(L.) = \begin{cases} \Omega_A^{d-i} & \text{for } i<m \\ \mathfrak{b} & \text{for } i=m\,. \end{cases}$$

Now let $\mathfrak{p}$ be a minimal prime ideal. For $\dim A/\mathfrak{p}<m$ we have (by (2.6))

$$H_i((L.)_{\mathfrak{p}}) = \begin{cases} E_{A_{\mathfrak{p}}}(k(\mathfrak{p})) & \text{for } i=\dim A/\mathfrak{p} \\ \mathfrak{b}_{\mathfrak{p}} & \text{for } i=m\,, \\ 0 & \text{otherwise} \end{cases}$$

and for $\dim A/\mathfrak{p}\geqq m$ we have

$$H_i((L.)_{\mathfrak{p}}) = \begin{cases} \mathfrak{b}_{\mathfrak{p}} & \text{for } i=m\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For all $\mathfrak{p}$ in $\operatorname{Min} A$ and all $A_{\mathfrak{p}}$-modules X let $l_{\mathfrak{p}}(X)$ denote the length of X, so $l_{\mathfrak{p}}(X)<\infty$ if X is finitely generated over $A_{\mathfrak{p}}$. For short we write

$$l(\mathfrak{p}) = l_{\mathfrak{p}}(A_{\mathfrak{p}}) = l_{\mathfrak{p}}(E_{A_{\mathfrak{p}}}(k(\mathfrak{p}))) \quad \text{and} \quad n(\mathfrak{p}) = l_{\mathfrak{p}}(\mathfrak{b}_{\mathfrak{p}})/l(\mathfrak{p})$$

(so $0 \leqq n(\mathfrak{p}) \leqq 1$).

Let χ denote the Euler-characteristic of the complex $L.$, that is,

$$\chi = \sum_{0 \leqq i \leqq m} (-1)^i \mu^i(A) .$$

We have (for $\mathfrak{p}$ in $\operatorname{Min} A$)

$$\chi l(\mathfrak{p}) = \sum_i (-1)^i l_{\mathfrak{p}}((L_i)_{\mathfrak{p}}) = \sum_i (-1)^i l_{\mathfrak{p}}(H_i(L.)_{\mathfrak{p}}) ,$$

so

$$\chi l(\mathfrak{p}) = \begin{cases} (-1)^e l(\mathfrak{p}) + (-1)^m n(\mathfrak{p}) l(\mathfrak{p}) & \text{if } e = \dim A/\mathfrak{p} < m , \\ n(\mathfrak{p}) l(\mathfrak{p}) & \text{if } \dim A/\mathfrak{p} \geqq m . \end{cases}$$

Whence

$$\chi = \begin{cases} (-1)^e + (-1)^m n(\mathfrak{p}) & \text{if } e = \dim A/\mathfrak{p} < m , \\ n(\mathfrak{p}) & \text{if } \dim A/\mathfrak{p} \geqq m , \end{cases}$$

so $n(\mathfrak{p})$ is an integer, and thus $n(\mathfrak{p})=0$ or 1.

Now divide the search for a contradiction into two cases.

1°. $m>d$ (and hence $m>\dim A/\mathfrak{p}$ for all $\mathfrak{p}$ in $\operatorname{Min} A$). In this case there are only five possibilities for χ (namely $-2,-1,0,1,2,$). If χ is odd then $n(\mathfrak{p})=0$, and thereby $\mathfrak{b}_{\mathfrak{p}}=0$, for all $\mathfrak{p}$ in $\operatorname{Min} A = \operatorname{Ass} A$, so $\mathfrak{b}=0$, that is the image of $F_{m+1} \to F_m$ is zero. This is a contradiction by Remark (3.11). If on the other hand χ is even then $n(\mathfrak{p})=1$ for all $\mathfrak{p}$ in $\operatorname{Min} A$, that is $(A/\mathfrak{b})_{\mathfrak{p}}=0$ for all $\mathfrak{p}$ in $\operatorname{Min} A = \operatorname{Ass} A$, but $A/\mathfrak{b}$ is a submodule of the free module F_{m-1}, so $A/\mathfrak{b}=0$. Again we have obtained a contradiction: the image of $F_m \to F_{m-1}$ is zero.

Now that we have outruled the case $m>d$ we turn to the case $m=d$, and if A is Cohen–Macaulay we are done.

2°. $m=d$. We will assume $t=\operatorname{depth} A<d$ and are again looking for a contradiction. If $\mathfrak{a}$ denotes the image of $F_{d+1} \to F_d$ $(=A)$ we have a short exact sequence

$$0 \to \mathfrak{a} \to \mathfrak{b} \to \Omega_A^0 \to 0 .$$

The contradiction will be obtained by showing $\mathfrak{a}_{\mathfrak{p}}=0$ for all $\mathfrak{p}$ in $\operatorname{Min} A$.

If $\mathfrak{p} \in \operatorname{Min} A$ has $\dim A/\mathfrak{p}=d$ (and such exists) then $(\Omega_A^0)_{\mathfrak{p}} \neq 0$, so $\mathfrak{b}_{\mathfrak{p}} \neq 0$, and hence $\chi = n(\mathfrak{p}) = 1$. Since $(\Omega_A^0)_{\mathfrak{p}}$ has length $l(\mathfrak{p})$ we get $\mathfrak{a}_{\mathfrak{p}}=0$.

If $e=\dim A/\mathfrak{p}<d$ then $1=\chi=(-1)^e+n(\mathfrak{p})$ so $n(\mathfrak{p})=0$ (and e is even), and hence $\mathfrak{b}_\mathfrak{p}=0$, that is $\mathfrak{a}_\mathfrak{p}=0$.

As indicated in Remark (1.9) the New Intersection Theorem holds for very short complexes also in mixed characteristics. This is stated below and we give a completely elementary proof of this fact.

LEMMA (4.4). *If* $0 \to H_1 \to F_1 \to F_0 \to H_0 \to 0$ *is an exact sequence of A-modules with* F_0 *and* F_1 *finitely generated free and* $H_0 \neq 0$, *then*

$$\dim A \leqq \max\{\dim H_0+1, \dim H_1\}\ .$$

PROOF. Let $t=\max\{\dim H_0+1, \dim H_1\}$ and assume $\dim A>t$. Choose $\mathfrak{p}$ in $\operatorname{Spec} A$ with $\dim A/\mathfrak{p}>t$. The exact sequence induces an isomorphism $(F_1)_\mathfrak{p} \xrightarrow{\sim} (F_0)_\mathfrak{p}$, so F_1 and F_0 have the same rank, and hence the determinant a of $F_1 \to F_0$ exists and is not a unit in A (since $H_0 \neq 0$). Now $\dim A/(a) \geqq \dim A-1 \geqq t$, so pick $\mathfrak{q}$ in $\operatorname{Spec} A$ such that $a \in \mathfrak{q}$ and $\dim A/\mathfrak{q} \geqq t > \dim H_0$. Since the image of a in $A_\mathfrak{q}$ is not a unit, the homomorphism $(F_1)_\mathfrak{q} \to (F_0)_\mathfrak{q}$ can not be surjective, that is $(H_0)_\mathfrak{q} \neq 0$, a contradiction.

In view of Remark (1.9) it is not surprising that the Conjectures A and B hold for rings of dimensions at most two, but the above lemma gives slightly stronger results.

COROLLARY (4.5). *If* M *is a finitely generated non-zero A-module with* $t=\operatorname{depth} M<\dim A$, *then* $\mu^{t+1}(M)>\mu^t(M)>0$.

COROLLARY (4.6). *Conjecture* B *holds for finitely generated modules* M *with* $\operatorname{depth} M \geqq \dim A-2$.

Conjecture A *holds for rings* A *with* $\operatorname{depth} A \geqq \dim A-2$.

5. The μ^i under localization.

THEOREM (5.1). *Let* $\mathfrak{p}$ *and* $\mathfrak{q}$ *be prime ideals in the ring* A *such that* $\mathfrak{p} \subseteqq \mathfrak{q}$, *and let* l *denote* $\dim(A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q})$. *For each finitely generated A-module* M *and for all integers* i *there is an inequality*:

$$\mu^i(\mathfrak{p}, M) \leqq \mu^{i+l}(\mathfrak{q}, M)\ .$$

PROOF. By [2, (2.4) Corollary] it suffices to prove the following statement:

$\mu^i(\mathfrak{p}, M) \leqq \mu^{i+l}(M)$ for all i and all $\mathfrak{p}$ in $\operatorname{Spec} A$ with $\dim A/\mathfrak{p}=l$.

The proof of this is divided into two steps.

1°. *A is complete.* Since A (by [4]) is a homomorphic image of a regular local ring we may apply (2.3) and (2.4). Let $I_M^{\cdot}$ be a minimal injective resolution of M and write $F. = \Gamma_{\mathfrak{m}}(I_M^{\cdot})^{\vee}$. Then $F.$ is isomorphic (in the derived category $\mathscr{D}$) to $\operatorname{Hom}(M, D^{\cdot})\,[d]$ by (2.4)), and hence

$$X^{\cdot} = \operatorname{Hom}^{\cdot}(F^{\cdot}, D^{\cdot}) \cong M[-d] \qquad (\text{in } \mathscr{D})$$

(by (2.3)). We have used the convention $F^i = F_{-i}$ and $M[-d]$ is the complex with the module M in degree d and zero in other degrees. We have

$$H^i(X^{\cdot}) = \begin{cases} M & \text{for } i = d\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

This means that we can break the complex $X^{\cdot}$ up into exact sequences as follows:

(III)
$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & & & \\
 & & & & & & \downarrow & & & & \\
0 \to X^0 \to & \dots & \to X^{d-1} & \to & & & B^d \to 0 & & & & \\
 & & & & & & \downarrow & & & & \\
 & & & & & 0 \to & Z^d & \to & X^d & \to & X^{d+1} \to \dots \\
 & & & & & & \downarrow & & & & \\
 & & & & & & M & & & & \\
 & & & & & & \downarrow & & & & \\
 & & & & & & 0 & & & &
\end{array}$$

Before we continue we will look more closely on the modules X^i. Using the fact that $F^p = F_{-p}$ is free of rank $\mu^{-p}(M)$ (cf. Remark (2.7)) and the definition of $X^{\cdot}$ we get for a fixed i:

$$\begin{aligned}
X^i &= \prod_p \operatorname{Hom}(F^p, D^{p+i}) \\
&= \prod_p \operatorname{Hom}\Big(A^{\mu^{-p}(M)}, \coprod^{*p} E(A/\mathfrak{p})\Big) \\
&= \prod_p \Big(\coprod^{*p} E(A/\mathfrak{p})\Big)^{\mu^{-p}(M)} \\
&= \coprod_{\mathfrak{p}} E(A/\mathfrak{p})^{\mu^{u(i,\mathfrak{p})}(M)}
\end{aligned}$$

where (of typographical reasons $\coprod^{*p}$ means the sum over the $\mathfrak{p}$ in $\operatorname{Spec} A$ with $\dim A/\mathfrak{p} = d - p - i$ and where $u(i, \mathfrak{p}) = \dim A/\mathfrak{p} + i - d$.

In other words, X^i is an injective module and

$$\mu^0(\mathfrak{p}, X^i) = \mu^{u(i,\mathfrak{p})}(M) \qquad \text{for all } i\,.$$

Now back to the diagram (III). The top row shows that B^d is injective, so the column splits: $Z^d = B^d \oplus M$. We get also: $X^d = B^d \oplus \tilde{X}^d$, and hence an exact sequence:

$$0 \to M \to \tilde{X}^d \to X^{d+1} \to \dots \to X^{d+i} \to \dots$$

that is, an injective resolution of M (not necessarily minimal). We get (for $\mathfrak{p} \in \operatorname{Spec} A$)

$$\mu^0(\mathfrak{p}, M) \leqq \mu^0(\mathfrak{p}, \tilde{X}^d) \leqq \mu^0(\mathfrak{p}, X^d) = \mu^l(M) \quad \text{and}$$

$$\mu^i(\mathfrak{p}, M) \leqq \mu^0(\mathfrak{p}, X^{i+d}) = \mu^{i+l}(M) \quad \text{for all } i>0\,,$$

where $l = \dim A/\mathfrak{p}$ (so $u(i+d, \mathfrak{p}) = i+l$), and hence we are done with the proof in the complete case.

2°. *The general case.* Let $\mathfrak{p}$ ($\in \operatorname{Spec} A$) be fixed and put $l = \dim A/\mathfrak{p}$. Choose $\mathfrak{q}$ in $\operatorname{Spec} \hat{A}$ such that $\mathfrak{q} \supseteqq \mathfrak{p}\hat{A}$ and $\dim \hat{A}/\mathfrak{q} = l$. This is possible since $\dim A/\mathfrak{p} = \dim \hat{A}/\mathfrak{p}\hat{A}$, and $\mathfrak{q}$ is necessarily minimal in $\operatorname{Supp}_{\hat{A}} (\hat{A}/\mathfrak{p}\hat{A})$, so $C = \hat{A}_\mathfrak{q}/\mathfrak{p}\hat{A}_\mathfrak{q}$ is an artinian ring, and hence $\mu_C^0(\mathfrak{q}C, C) \neq 0$.

Using [8] and Case 1° we get the desired assertion:

$$\begin{aligned}
\mu_A^i(\mathfrak{p}, M) &\leqq \mu_C^0(\mathfrak{q}C, C)\mu_A^i(\mathfrak{p}, M) \\
&\leqq \sum_{p+q=i} \mu_C^p(\mathfrak{q}C, C)\mu_A^q(\mathfrak{p}, M) \\
&= \mu_{\hat{A}}^i(\mathfrak{q}, \hat{M}) \\
&\leqq \mu_{\hat{A}}^{i+l}(\hat{M}) = \mu_A^{i+l}(M)\,.
\end{aligned}$$

6. Injective dimension of modules of depth zero.

THEOREM (6.1). *If A is essentially equicharacteristic and if M is an A-module with* depth $M = 0$ *then*

$$\dim A \leqq \operatorname{id} M + \dim (M^{\vee})\,.$$

Note that the module M is not supposed to be finitely generated, but if M is finitely generated then the result is known (since it is the Bass' conjecture, see [21] and [15], but also Remark (3.6)).

PROOF. One of the standard duality formulas [3, chapter VI] gives $\operatorname{Tor}_i (L, M^{\vee}) = \operatorname{Ext}^i (L, M)^{\vee}$ for all i and all finitely generated modules L, and hence $k \otimes M^{\vee} \neq 0$ and $\operatorname{id} M = \operatorname{fd} M^{\vee}$. This shows that it suffices to prove the dual version (6.2) of (6.1) stated just below.

THEOREM (6.2). *If A is essentially equicharacteristic and if the A-module N is such that* $\mathfrak{m}N \neq N$ *then*

$$\dim A \leqq \operatorname{fd} N + \dim N\,.$$

In the proof of (6.2) we need the existence of a big maximal Cohen–Macaulay module satisfying the conditions of the following Proposition in which the (countably generated) big maximal Cohen–Macaulay module C has been constructed by Griffith (see [10]) (beginning with Hochster's big maximal Cohen–Macaulay module).

PROPOSITION (6.3). *Let A be a complete essentially equicharacteristic local ring. Then there exists a (countably generated) maximal Cohen–Macaulay A-module C such that*

$$\operatorname{Tor}_i^A(C,N) = 0$$

for all $i>0$ and all modules N of finite flat dimension.

PROOF. Pick $\mathfrak{p}$ in $\operatorname{Spec} A$ such that $B=A/\mathfrak{p}$ is equicharacteristic and of dimension $d=\dim A$. This complete domain B is a module-finite extension of a regular local subring R (by [4]). In this situation Griffith has constructed a (countably generated) B-module C which is free as an R-module (cf. [10, Theorem 3.1]). This module C is a maximal Cohen–Macaulay module both as an R-module, as a B-module, and as an A-module, since if $(a_1,\ldots,a_d)$ is a system of parameters for A such that the images $(\bar{a}_1,\ldots,\bar{a}_d)$ in B lie in R and form a system of parameters for R, then $a_1,\ldots,a_d$ is a C-regular sequence. It is this module C that works in the Proposition.

To see this let $\mathscr{C}_h$ (for $0\leqq h\leqq d$) denote the class of non-zero B-modules D such that D is free over $R/(r_1,\ldots,r_h)$ for a suitable R-regular sequence $r_1,\ldots,r_h$ of length h in $\operatorname{Ann}_R D$. Note that C belongs to $\mathscr{C}_0$ and hence $\mathscr{C}_h\neq\varnothing$ for all h, $0\leqq h\leqq d$.

CLAIM 1. The classes $\mathscr{C}_h$ have the following property: For any D in $\mathscr{C}_h$ and any $\mathfrak{q}$ in $\operatorname{Supp}_A D$ with $\operatorname{ht}_A \mathfrak{q}>h$ there exists a D-regular element x in $\mathfrak{q}$ such that D/xD belongs to $\mathscr{C}_{h+1}$.

PROOF OF CLAIM 1. Let D be in $\mathscr{C}_h$, say D is free over $\bar{R}=R/(r_1,\ldots,r_h)$ where $r_1,\ldots,r_h$ is an R-regular sequence. Since A is complete, it is catenary, so all saturated chains of prime ideals between $\mathfrak{p}$ and $\mathfrak{m}$ have the same length, namely d ($=\dim A$). From this follows that the prime ideal $\mathfrak{q}'=\mathfrak{q}/\mathfrak{p}$ in B has height $>h$, so the prime ideal $\mathfrak{r}=\mathfrak{q}'\cap R$ in R has also height $>h$ (since B is module-finite over R). Therefore, and since R is a regular local ring, the R-regular sequence $r_1,\ldots,r_h$ ($\in \mathfrak{r}$) can be extended to an R-regular sequence $r_1,\ldots,r_h,r_{h+1}$ in $\mathfrak{r}$ of length $h+1$. Pick x in A such that its image in B is r_{h+1}. Then D/xD is free over $\bar{R}/r_{h+1}\bar{R}=R/(r_1,\ldots,r_h,r_{h+1})$ and x is D-regular (since r_{h+1} is R-regular and D is R-free).

The Proposition follows from the next Claim.

CLAIM 2. $\mathrm{Tor}_i(D,N)=0$ for $i>h$ if $D\in\mathscr{C}_h$ and $\mathrm{fd}\,N<\infty$.

Proof of Claim 2 by descending induction on h. Note first of all that $\mathrm{fd}\,N\leqq d$ if $\mathrm{fd}\,N<\infty$ (cf. [1, Theorem 2.4]) so the case $h=d$ is trivial.

The inductive step. Assume $i>h$ and $\mathrm{Tor}_i(D,N)\neq 0$, and pick $\mathfrak{q}$ in $\mathrm{Ass}\,\mathrm{Tor}_i(D,N)$. If $\mathrm{ht}\,\mathfrak{q}>h$ then there exists (by Claim 1) a D-regular element x in $\mathfrak{q}$ such that $\bar{D}=D/xD\in\mathscr{C}_{h+1}$. The short exact sequence $0\to D\xrightarrow{x} D\to\bar{D}\to 0$ induces an exact sequence:

$$\mathrm{Tor}_{i+1}(\bar{D},N)\to\mathrm{Tor}_i(D,N)\xrightarrow{x}\mathrm{Tor}_i(D,N)\,.$$

Here $\mathrm{Tor}_{i+1}(\bar{D},N)=0$ by the inductive hypothesis, and hence x is regular on $\mathrm{Tor}_i(D,N)$ contrary to the assumptions $x\in\mathfrak{q}$ and $\mathfrak{q}\in\mathrm{Ass}\,\mathrm{Tor}_i(D,N)$.

Therefore necessarily $\mathrm{ht}\,\mathfrak{q}\leqq h$, but this leads also to a contradiction:

$$\mathrm{Tor}_i^{A_\mathfrak{q}}(D_\mathfrak{q},N_\mathfrak{q}) = \mathrm{Tor}_i^A(D,N)_\mathfrak{q} \neq 0\,,$$

so $i\leqq\mathrm{fd}_{A_\mathfrak{q}}N_\mathfrak{q}\leqq\dim A_\mathfrak{q}=\mathrm{ht}\,\mathfrak{q}\leqq h$ (cf. again [1, Theorem 2.4]).

PROOF OF (6.2). The proof is divided into 3 steps, and throughout the proof we assume (of course) $\mathrm{fd}\,N<\infty$.

STEP 1. It suffices to assume that A is complete, since

$$\dim\hat{A} = \dim A,\qquad \mathrm{fd}_{\hat{A}}(N\otimes_A\hat{A}) = \mathrm{fd}_A N\,,$$

and

$$\dim_{\hat{A}}(N\otimes_A\hat{A}) = \dim_A N\,.$$

The equality of the flat dimensions follows since $\hat{A}$ is faithfully flat over A, and $\dim_{\hat{A}}(N\otimes_A\hat{A})=\dim_A N$, since this holds if N is finitely generated, and in general N is the direct union of finitely generated submodules and tensoring with $\hat{A}$ respects direct unions.

From now on A is complete and C is the module from Proposition (6.3).

STEP 2. For all N with $\mathrm{fd}\,N<\infty$ and all i with $0\leqq i\leqq d$ we have an isomorphism

$$H_\mathfrak{m}^{d-i}(C\otimes N) = \mathrm{Tor}_i(H_\mathfrak{m}^d(C),N)\,.$$

PROOF. Let T denote the functor $H_\mathfrak{m}^d(C\otimes -)$ from the category $\mathscr{F}$ of modules of finite flat dimension to the category of modules. T is a right exact functor on $\mathscr{F}$ and T commutes with direct limits, so there is an isomorphism of functors on $\mathscr{F}$:

$$T\cong H_\mathfrak{m}^d(C)\otimes -\,.$$

Now the result of Step 2 follows, because the ith left derived functor $L_i T$ of T is $H_{\mathfrak{m}}^{d-i}(C\otimes -)$, since if P is a projective module, then $\operatorname{depth} C\otimes P=d$, so $H_{\mathfrak{m}}^{i}(C\otimes P)=0$ for $i\neq d$, and since $C\otimes -$ is exact by Proposition (6.3).

The last step completes the proof of (6.2).

STEP 3. $\operatorname{fd} N \geqq d-\dim N$.

PROOF. Since $(C\otimes_A N)\otimes_A k=(C\otimes_A k)\otimes_k (N\otimes_A k)\neq 0$ by assumption, Corollary (2.2) gives that

$$t = \operatorname{depth}(C\otimes N) \leqq \dim(C\otimes N) \leqq \dim N ,$$

and hence

$$\operatorname{Tor}_{d-t}(H^d(C),N) \cong H^t(C\otimes N) \neq 0$$

by Step 2 and Lemma (2.1.2).

REMARK (6.4). For M finitely generated Theorem (6.2) follows immediately from the New Intersection Theorem. We get even $\dim B \leqq \operatorname{pd} N+\dim(B\otimes N)$ for all finitely generated modules B and N. So it might be that (6.2) also in the general case follows directly from an appropriate generalization of the New Intersection Theorem to complexes of flat modules.

COROLLARY (6.5). *If there exists a non-zero A-module N such that* $\operatorname{Supp} N =\{\mathfrak{m}\}$, $\mathfrak{m}N\neq N$, *and* $\operatorname{fd} N<\infty$, *then A is Cohen–Macaulay* (*and* $\operatorname{pd} N=\operatorname{fd} N =\dim A=\operatorname{depth} A$) *provided A is essentially equicharacteristic.*

PROOF. From (6.2) it follows that $\dim A \leqq \operatorname{fd} N$, and hence A is Cohen–Macaulay by [1, Theorem 2.4]. From [12, Corollaire (3.2.7)] (see also [9, Corollary 3.4]) it follows that $\operatorname{pd} N \leqq \dim A$.

EXAMPLE (6.6). If A is a two-dimensional Cohen–Macaulay ring with an A-regular sequence x,y, then $C=A\oplus E_A(A/(y))$ is a big maximal Cohen–Macaulay module (x,y is a C-regular sequence (but y,x is not)) and $\operatorname{fd}(A/(y)) =1<\infty$. However,

$$\operatorname{Tor}_1(C,A/(y)) = \{c\in C \mid yc=0\} \neq 0 .$$

This shows that not any big maximal Cohen–Macaulay module C works in Proposition (6.3).

ADDED IN PROOF. Szpiro has informed me that the New Intersection Theorem holds in general for complexes of length two (that is, $s=2$ in Theorem (1.2)). This is a consequence of [21, Chap. II Théorème (1.3)], and it shows that the bound $\geqq \dim A-2$ in both paragraphs of Corollary (4.6) can be replaced by $\geqq \dim A-3$.

BIBLIOGRAPHY

1. M. Auslander and D. Buschbaum, *Homological dimension in Noetherian rings* II, Trans. Amer. Math. Soc. 88 (1958), 194–206.
2. H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. 82 (1963), 8–28.
3. H. Cartan and S. Eilenberg, *Homological algebra* (Princeton Math. Series 19) Princeton University Press, Princeton, 1956.
4. I. S. Cohen, *On the structure and ideal theory of complete local rings*, Trans. Amer. Math. Soc. 59 (1946), 54–106.
5. R. Fossum, H.-B. Foxby, P. Griffith, and I. Reiten, *Minimal injective resolutions with applications to dualizing modules and Gorenstein modules*, Publ. Math. Inst. Hautes Études Sci. 45 (1976), 193–215.
6. H.-B. Foxby, *On the μ^i in a minimal injective resolution*, Math. Scand. 29 (1971), 175–186.
7. H.-B. Foxby, *Gorenstein modules and related modules*, Math. Scand. 31 (1972), 267–284.
8. H.-B. Foxby and A. Thorup, *Minimal injective resolutions under flat base change*, Proc. Amer. Math. Soc., to appear.
9. H.-B. Foxby, *Isomorphisms between complexes with applications to the homological theory of modules*, Math. Scand. 40 (1977), 5–19.
10. P. Griffith, *A representation theorem for complete local rings*, J. Pure Appl. Algebra 7 (1976), 303–315.
11. A. Grothendieck, *Local cohomology* (Notes by R. Hartshorne) (Lecture Notes Math. 41), Springer-Verlag, Berlin · Heidelberg · New York, 1967.
12. L. Gruson and M. Raynaud, *Critères de platitude*, Invent. Math. 13 (1971), 1–89.
13. R. Hartshorne, *Residues and duality* (Lecture Notes Math. 20), Springer-Verlag, Berlin Heidelberg · New York, 1966.
14. J. Herzog and E. Kunz, *Der kanonische Modul eines Cohen–Macaulay-Rings* (Lecture Notes Math. 238), Springer-Verlag, Berlin · Heidelberg · New York, 1971.
15. M. Hochster, *Topics in the homological theory of modules over commutative rings* (C.B.M.S. Regional Conference Ser. Math. 24), Amer. Math. Soc., Providence, R.I., 1976.
16. M. Hochster, *Big Cohen–Macaulay modules and algebras and embeddability in rings of Witt vectors*, Queen's Papers Pure Appl. 42 (1975), 106–195.
17. M. Hochster, *Cohen–Macaulay modules*, in *Conference on Commutative Algebra*, ed. J. W. Brewer and E. A. Rutter (Lecture Notes Math. 311) (pp. 120–152), Springer-Verlag, Berlin · Heidelberg · New York, 1973.
18. D. Lazard, *Sur les modules plats*, C.R. Acad. Sci. Paris Sér. A 258 (1964), 6313–6316.
19. I. G. Macdonald and R. Y. Sharp, *An elementary proof of the non-vanishing of certain local cohomology modules*, Quart. J. Math. (Oxford) (2) 23 (1972), 197–204.
20. M. Nagata, *Local rings* (Interscience Tracts Pure Appl. Math. 13), Interscience Publ., New York, London, 1962.
21. C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale*, Publ. Math. Inst. Hautes Études Sci. 42 (1973), 49–119.

22. C. Peskine and L. Szpiro, *Syzygies et multiplicités*, C.R. Acad. Sci. Paris Sér. A 278 (1974), 1421–1424.
23. R. Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. École Norm. Sup. (4) 9 (1976), 103–106.
24. R. Y. Sharp, *Gorensteins modules*, Math. Z. 115 (1970), 117–139.
25. R. Y. Sharp, *Finitely generated modules of finite injective dimension over certain Cohen–Macaulay rings*, Proc. London Math. Soc. 25 (1972), 303–328.
26. R. Y. Sharp, *Local cohomology theory in commutative algebra*, Quart. J. Math. (Oxford) (2) 21 (1970), 425–434.
27. R. Y. Sharp, *Dualizing complexes for commutative Noetherian rings*, Math. Proc. Cambridge Philos. Soc. 78 (1975), 369–386.
28. W. V. Vasconcelos, *Divisor theory in module categories* (Math. Studies 14), North Holland Publ. Co. Amsterdam, 1975.

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