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## ON THE MULTIPLIERS OF HANKEL TRANSFORM

Dedicated to Professor Gen-Ichirô Sunouchi on his 60th birthday

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The Jacobi polynomial of degree *n*, order  $(\alpha, \beta)$ ,  $\alpha, \beta > -1$ , is defined by

$$(1-x)^{lpha}(1+x)^{eta}P_{n}^{(lpha,eta)}(x)=rac{(-1)^{n}}{2^{n}n!}rac{d^{n}}{dx^{n}}[(1-x)^{n+lpha}(1+x)^{n+eta}]\;.$$

 $\{P_n^{(\alpha,\beta)}(\cos\theta)\}_{n=0}^{\infty}$  is an orthogonal system on  $(0, \pi)$  with respect to the measure  $(\sin\theta/2)^{2\alpha+1}(\cos\theta/2)^{2\beta+1}d\theta$ .

For a function  $f(\theta)$  integrable on  $(0, \pi)$  with respect to such a measure define

$$\widehat{f}(n) = \int_{0}^{\pi} f(\theta) P_{n}^{(\alpha,\beta)}(\cos \theta) \Big(\sin \frac{\theta}{2}\Big)^{2\alpha+1} \Big(\cos \frac{\theta}{2}\Big)^{2\beta+1} d\theta$$
.

Put

$$rac{1}{h_n^{(lpha,eta)}} = \int_{\scriptscriptstyle 0}^{\scriptscriptstyle \pi} [P_n^{\scriptscriptstyle (lpha,eta)}(\cos heta)]^2 \Bigl(\sinrac{ heta}{2}\Bigr)^{^{2lpha+1}} \Bigl(\cosrac{ heta}{2}\Bigr)^{^{2eta+1}} d heta \; .$$

Then we have formally

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$$f( heta) = \sum_{n=0}^{\infty} \widehat{f}(n) h_n^{(lpha,eta)} P_n^{(lpha,eta)}(\cos heta)$$
 .

For a sequence  $\phi(n)$  on the non negative integers define a transformation  $T_{\phi}$  by

$$T_{\phi}f( heta) = \sum_{n=0}^{\infty} \phi(n) \widehat{f}(n) h_n^{(lpha,eta)} P_n^{(lpha,eta)}(\cos heta)$$
 .

For  $p \ge 1$  and the function f on  $(0, \pi)$  we define a norm

$$||f||_p = \left(\int_0^\pi |f( heta)|^p \left(\sinrac{ heta}{2}
ight)^{2lpha+1} \left(\cosrac{ heta}{2}
ight)^{2eta+1} d heta
ight)^{1/p}$$

and denote by  $L^p_{(\alpha,\beta)}(0,\pi)$  the set of all measurable functions such that  $||f||_p < \infty$ . The operator norm of  $T_{\phi}$  of  $L^p_{(\alpha,\beta)}(0,\pi)$  to  $L^p_{(\alpha,\beta)}(0,\pi)$  will be denoted by  $||T_{\phi}||_p$  or  $||\phi(n)||_p$ .

Let  $J_{\alpha}(x)$  be the Bessel function of the first kind. For a function g(x) on  $(0, \infty)$  the (modified) Hankel transform of order  $\alpha$  is defined by

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$$\widehat{g}(y) = \int_{_0}^{^\infty}\!\!g(x) rac{J_lpha(xy)}{(xy)^lpha} x^{_{2lpha+1}} dx$$

and the multiplier transformation associated with  $\phi(y)$  is defined formally by

$$U_{_{\phi}}g(x) = \int_{_{0}}^{^{\infty}} \phi(y) \widehat{g}(y) rac{J_{lpha}(xy)}{(xy)^{lpha}} y^{_{2lpha+1}} dy \; .$$

 $L^p_{\alpha}(0,\infty)$  will denote the space of all measurable function g such that

$$\|g\|_p=\left(\int_0^\infty |g(x)|^p\,x^{2lpha+1}dx
ight)^{1/p}<\infty$$
 .

The operator norm of  $U_{\phi}$  of  $L^p_{\alpha}(0, \infty)$  to  $L^p_{\alpha}(0, \infty)$  will be denoted by  $|U_{\phi}|_p$  or  $|\phi(y)|_p$ .

The object of this paper is to study the relation of the multiplier transformations between Jacobi polynomial expansions and Hankel transformations.

THEOREM. Let  $1 \leq p < \infty$  and  $\alpha, \beta > -1$ . Assume that  $\phi$  is a function on  $(0, \infty)$  continuous except on a null set and  $\underline{\lim}_{\varepsilon \to +0} ||\phi(\varepsilon n)||_p$  is finite, then  $|\phi(x)|_p$  is finite and  $|\phi(x)|_p \leq \underline{\lim}_{\varepsilon \to +0} ||\phi(\varepsilon n)||_p$ .

**PROOF.** Let g be an infinitely differentiable function with compact support in a finite interval (0, M) and put  $g_{\lambda}(\theta) = g(\lambda\theta)$  where  $\lambda > 0$  is so large that the support of  $g_{\lambda}(\theta)$  is contained in  $(0, \pi)$ . Then we have by the assumption

(1) 
$$\left\|\sum_{n=0}^{\infty}\phi\left(\frac{n}{\lambda}\right)\widehat{g}(n)h_{n}^{(\alpha,\beta)}P_{n}^{(\alpha,\beta)}(\cos\theta)\right\|_{p} \leq \left\|\phi\left(\frac{n}{\lambda}\right)\right\|_{p}\|g\|_{p}.$$

Changing variable we get

$$\lambda^{(2lpha+2)/p} ||g_{\lambda}||_p = \left(\int_0^M |g( au)|^p \Big(\lambda \sin rac{ au}{2\lambda}\Big)^{2lpha+1} \Big(\cos rac{ au}{2\lambda}\Big)^{2eta+1} d au\Big)^{1/p}$$
 ,

which tends to

$$\left(rac{1}{2^{2lpha+1}}\int_{0}^{\infty}\mid g( au)\mid {}^{p} au^{2lpha+1}d au
ight)^{1/p}$$

as  $\lambda \to \infty$ . Apply the similar argument to the left hand side of (1). Then we get by Fatou's lemma

$$(2) \qquad \left(\frac{1}{2^{2\alpha+1}}\int_{0}^{\infty}\lim_{\lambda\to\infty}\left|\sum_{n=0}^{\infty}\phi\left(\frac{n}{\lambda}\right)\widehat{g}_{\lambda}(n)h_{n}^{(\alpha,\beta)}P_{n}^{(\alpha,\beta)}\left(\cos\frac{\tau}{\lambda}\right)\right|^{p}\tau^{2\alpha+1}d\tau\right)^{1/p}\\ \leq \lim_{\lambda\to\infty}\left\|\phi\left(\frac{n}{\lambda}\right)\right\|_{p}\left(\frac{1}{2^{2\alpha+1}}\int_{0}^{\infty}|g(\tau)|^{p}\tau^{2\alpha+1}d\tau\right)^{1/p}.$$

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Now we proceed to the computation of the left hand side of (2). First we remark that (2) holds for p = 2 and

$$\left\| \phi \left( rac{n}{\lambda} 
ight) 
ight\|_{2} \leq \left\| \phi \left( rac{n}{\lambda} 
ight) 
ight\|_{p}$$
 .

Thus for a sequence  $\lambda_1 < \lambda_2 < \cdots < \lambda_j \rightarrow \infty$ 

$$G(\tau, \lambda) = \sum_{n=0}^{\infty} \phi\left(\frac{n}{\lambda}\right) \hat{g}_{\lambda}(n) h_n^{(lpha,eta)} P_n^{(lpha,eta)}\left(\cos\frac{\tau}{\lambda}\right)$$

converges weakly to a function  $G(\tau)$  in  $L^{2}_{\alpha}(0, K)$  for every K > 0 and  $G(\tau)$  satisfies the inequality

(3) 
$$\left( \int_{0}^{\infty} |G(\tau)|^{p} \tau^{2\alpha+1} d\tau \right)^{1/p} \\ \leq \lim_{\lambda \to \infty} \left\| \phi\left(\frac{n}{\lambda}\right) \right\|_{p} \left( \int_{0}^{\infty} |g(\tau)|^{p} \tau^{2\alpha+1} d\tau \right)^{1/p} .$$

To show that  $G(\tau)$  is the Hankel transform of  $\phi \hat{g}$  put

$$egin{aligned} G( au,\lambda) &= \Big(\sum\limits_{n=0}^{N[\lambda]}+\sum\limits_{n=N[\lambda]+1}^{\infty}\Big)\phi\Big(rac{n}{\lambda}\Big)\widehat{g}_{\lambda}(n)h_n^{(lpha,eta)}P_n^{(lpha,eta)}\Big( ext{cos}rac{ au}{\lambda}\Big) \ &= G^N( au,\lambda)\,+\,H^N( au,\lambda), ext{ say,} \end{aligned}$$

for  $N = 1, 2, \cdots$ .

Since

$$\frac{d}{d\theta} \left[ \left( \sin \frac{\theta}{2} \right)^{2\alpha+2} \left( \cos \frac{\theta}{2} \right)^{2\beta+2} P_{n-1}^{(\alpha+1,\beta+1)}(\cos \theta) \right] \\ = n \left( \sin \frac{\theta}{2} \right)^{2\alpha+1} \left( \cos \frac{\theta}{2} \right)^{2\beta+1} P_n^{(\alpha,\beta)}(\cos \theta)$$

(cf. [5, p. 97]), integrating by parts we get

$$\hat{g}_{\lambda}(n) = -\frac{\lambda}{n} \int_{0}^{\pi} \frac{g'(\lambda\theta)}{\sin \theta/2 \cos \theta/2} P_{n-1}^{(\alpha+1,\beta+1)}(\cos \theta) \left(\sin \frac{\theta}{2}\right)^{2\alpha+3} \left(\cos \frac{\theta}{2}\right)^{2\beta+3} d\theta .$$

This, if K > 0 is any fixed number and  $\pi \lambda > K$ , then

$$egin{aligned} &\int_0^\kappa |\, H^{\scriptscriptstyle N}( au,\,\lambda)\,|^2 \Bigl(\lambda\,\sinrac{ au}{2\lambda}\Bigr)^{2lpha+1} \Bigl(\cosrac{ au}{2\lambda}\Bigr)^{2eta+1}d au\ &\leq \int_0^\pi |\, H^{\scriptscriptstyle N}( au,\,\lambda)\,|^2 \Bigl(\lambda\,\sinrac{ au}{2\lambda}\Bigr)^{2lpha+1} \Bigl(\cosrac{ au}{2\lambda}\Bigr)^{2eta+1}d au\ &= \lambda^{2lpha+2} \int_0^\pi |\, H^{\scriptscriptstyle N}(\lambda au,\,\lambda)\,|^2 \Bigl(\sinrac{ au}{2}\Bigr)^{2lpha+1} \Bigl(\cosrac{ au}{2}\Bigr)^{2eta+1}d au\ . \end{aligned}$$

By Parseval's relation the last term equals

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$$\lambda^{2lpha+2}\sum_{n=N[\lambda]+1}^{\infty}\left|\phi\left(rac{n}{\lambda}
ight)
ight|^{2}|\,\widehat{g}_{\lambda}(n)\,|^{2}h_{n}^{\scriptscriptstyle(lpha,eta)}$$
 .

Since  $h_n^{(\alpha,\beta)} = 2n + O(1)$  as  $n \to \infty$  and  $\phi$  is uniformly bounded, the above is dominated by

$$A\lambda^{2lpha+2}\left(rac{\lambda}{N\lambda}
ight)^2\sum_{n=N[\lambda]+1}^{\infty}\left|rac{n}{\lambda}\widehat{g}_{\lambda}(n)
ight|^2h_{n-1}^{(lpha+1,\,eta+1)}$$
 ,

where A is a constant independent on  $\lambda$  and N. By Bessel's inequality this is bounded by

$$egin{aligned} &Arac{\lambda^{2lpha+2}}{N^2} \int_{_0}^{\pi} \Big|rac{g'(\lambda heta)}{\sin heta/2\cos heta/2}\Big|^2 \Big(\sinrac{ heta}{2}\Big)^{2lpha+3} \Big(\cosrac{ heta}{2}\Big)^{2eta+3}d heta \ &=rac{A}{N^2} \int_{_0}^{_M} |\,g'( heta)\,|^2 \Big(\lambda\sinrac{ heta}{2\lambda}\Big)^{2lpha+1} \Big(\cosrac{ heta}{2\lambda}\Big)^{2eta+1}d heta \ &=O\Big(rac{1}{N^2}\Big) \end{aligned}$$

uniformly in  $\lambda$ .

Thus we get

$$\int_{_0}^{_\pi} |H^{\scriptscriptstyle N}( au,\,\lambda)|^2\, au^{_{2lpha+1}}d au\,=\,O\Bigl(rac{1}{N^2}\Bigr)$$

uniformly in  $\lambda$ .

Thus by the diagonal argument there exists a subsequence  $\{\lambda_{k_j}\}$  of  $\{\lambda_j\}$  such that  $H^N(\tau, \lambda_{k_j})$  converges weakly to a function  $H^N(\tau)$  in  $L^2_{\alpha}(0, K)$  for every  $N = 1, 2, \cdots$  and

$$\int_{_{0}}^{^{K}} \mid H^{_{N}}( au) \mid^{_{2}} au^{^{2lpha+1}} d au \, = \, O\!\!\left(rac{1}{N^{^{2}}}
ight)$$
 .

For a subsequence  $\{N_j\}$ ,  $H^{N_j}(\tau)$  converges to zero almost everywhere. Since

$$G^{\scriptscriptstyle N}(\tau,\,\lambda) = G(\tau,\,\lambda) - H^{\scriptscriptstyle N}(\tau,\,\lambda)$$
,

 $G^{\scriptscriptstyle N}(\tau, \lambda_{k_j})$  converges weakly in  $L^{\scriptscriptstyle 2}_{\alpha}(0, K)$  to a limit  $G^{\scriptscriptstyle N}(\tau)$  as  $j \to \infty$  and  $G(\tau) = G^{\scriptscriptstyle N}(\tau) + H^{\scriptscriptstyle N}(\tau)$  for  $N = 1, 2, \cdots$ . Thus  $G^{\scriptscriptstyle N_j}(\tau)$  converges to  $G(\tau)$  almost everywhere.

We prove that  $G^{\scriptscriptstyle N}(\tau, \lambda)$  converges pointwise to a function as  $\lambda \to \infty$ . Then the limit function coincides with  $G^{\scriptscriptstyle N}(\tau)$ .

First we note that

$$\left(\sin\frac{\theta}{2}\right)^{\alpha} \left(\cos\frac{\theta}{2}\right)^{\beta} P_{n}^{(\alpha,\beta)}(\cos\theta)$$

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$$=\widetilde{n}^{-lpha}rac{arGamma(n+lpha+1)}{n^{lpha}}\Bigl(rac{ heta}{\sin heta}\Bigr)^{^{1/2}}J_{lpha}(\widetilde{n} heta)+egin{cases} O( heta^{^{1/2}}n^{^{-3/2}}) & ext{for } Cn^{^{-1}}\leq heta\leq\pi-arepsilon\ O( heta^{^{lpha+2}}n^{lpha}) & ext{for } 0< heta\leq Cn^{^{-1}} \ ,$$

where  $\tilde{n} = n + (\alpha + \beta + 1)/2$ , and  $\varepsilon$  and C are fixed positive numbers ([5, p. 197]).

Let K be a fixed number and  $0 < \tau \leq K$ . For  $n, 0 \leq n \leq N[\lambda]$ , we have

$$egin{aligned} &rac{h_n^{(lpha,eta)}}{\lambda^lpha} P_n^{(lpha,eta)} \Big( \cosrac{ au}{\lambda} \Big) \ &= h_n^{(lpha,eta)} \widetilde{n}^{-lpha} rac{arLambda(n+lpha+1)}{n^lpha} \Big( rac{ au/\lambda}{\sin au/\lambda} \Big)^{1/2} J_lpha \Big( rac{ ilde n}{\lambda} au \Big) rac{1}{(\lambda \sin au/2\lambda)^lpha (\cos au/2\lambda)^eta} + O \Big( rac{n^{lpha+1}}{\lambda^{lpha+2}} \Big) \ &= h_n^{(lpha,eta)} J_lpha \Big( rac{ ilde n}{\lambda} au \Big) \Big( rac{1}{2} \sum r \Big)^lpha + o(n) \ &= 2n J_lpha \Big( rac{n}{\lambda} au \Big) \Big( rac{2}{ au} \Big)^lpha + o(n) \ . \end{aligned}$$

On the other hand

$$egin{aligned} \lambda^lpha \widehat{g}_{\lambda}(n) &= rac{1}{\lambda^{lpha+2}} \int_{0}^{M} g( heta) P_n^{(lpha,eta)} \Big( \cosrac{ heta}{\lambda} \Big) \Big( \sinrac{ heta}{2\lambda} \Big)^{2lpha+1} \Big( \cosrac{ heta}{2\lambda} \Big)^{2eta+1} d heta \ &= rac{1}{\lambda^2} \int_{0}^{M} g( heta) \widetilde{n} rac{arLambda(n+\alpha+1)}{lpha} \Big( rac{ heta/\lambda}{\sin heta/\lambda} \Big)^{1/2} J_{lpha} \Big( rac{ ilde n}{\lambda} heta \Big) \Big( \lambda \sinrac{ heta}{2\lambda} \Big)^{lpha+1} \Big( \cosrac{ heta}{2\lambda} \Big)^{eta+1} d heta \ &+ o\Big(rac{1}{\lambda^2}\Big) \ &= rac{1}{\lambda^2} rac{1}{2^{lpha+1}} \int_{0}^{\infty} g( heta) J_{lpha} \Big( rac{ heta}{\lambda} heta \Big) heta^{lpha+1} d heta \ &+ o\Big(rac{1}{\lambda^2}\Big) \;. \end{aligned}$$

Thus

$$egin{aligned} &\lim_{\lambda o\infty}\sum_{n=0}^{N[\lambda]} \phiigg(rac{n}{\lambda}ig) \widehat{g}_{\lambda}(n) h_n^{(lpha,eta)} P_n^{(lpha,eta)} \Big( \cosrac{ au}{\lambda} \Big) \ &= \lim_{\lambda o\infty} igg\{\sum_{n=0}^{N[\lambda]} \phiigg(rac{n}{\lambda}igg) \int_0^\infty g( heta) J_lpha igg(rac{n}{\lambda} heta \Big) heta^{lpha+1} d heta J_lpha igg(rac{n}{\lambda} au igg) rac{1}{ au^lpha} rac{n}{\lambda} rac{1}{\lambda} + o(1)rac{n}{\lambda^2} igg\} \ &= \int_0^N \phi(v) \widehat{g}(v) rac{J_lpha(v au)}{(v au)^lpha} v^{2lpha+1} dv \;. \end{aligned}$$

Thus we get

(4) 
$$G(\tau) = \int_0^\infty \phi(v) \hat{g}(v) \frac{J_\alpha(v\tau)}{(v\tau)^\alpha} v^{2\alpha+1} dv \quad \text{a.e.}$$

From (3) it follows that

$$|\phi(x)|_p \leq \lim_{\lambda \to \infty} \left\| \phi\left(\frac{n}{\lambda}\right) \right\|_p$$
,

which proves the theorem.

Our theorem proves the mean convergence, mean Cesàro summability, the multiplier theorems of Marcinkiewicz' type and decomposition theorem for Hankel transform by the theorems in [4], [1] and [2].

We remark that our theorem is reduced to a theorem in [3] when  $\alpha = \beta = -1/2$ .

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