Commun. math. Phys. 20, 167—192 (1971)

On the Nature of Turbulence

DAVID RUELLE and FLORIS TAKENS*
I.H.E.S., Bures-sur-Yvette, France

Received October 5, 1970

Abstract. A mechanism for the generation of turbulence and related phenomena in dissipative systems is proposed.

§ 1. Introduction

If a physical system consisting of a viscous fluid (and rigid bodies) is not subjected to any external action, it will tend to a state of rest (equilibrium). We submit now the system to a steady action (pumping, heating, etc.) measured by a parameter μ[1]. When $\mu = 0$ the fluid is at rest. For $\mu > 0$ we obtain first a *steady state*, i.e., the physical parameters describing the fluid at any point (velocity, temperature, etc.) are constant in time, but the fluid is no longer in equilibrium. This steady situation prevails for small values of μ. When μ is increased various new phenomena occur; (a) the fluid motion may remain steady but change its symmetry pattern; (b) the fluid motion may become periodic in time; (c) for sufficiently large μ, the fluid motion becomes very complicated, irregular and chaotic, we have *turbulence*.

The physical phenomenon of turbulent fluid motion has received various mathematical interpretations. It has been argued by Leray [9] that it leads to a breakdown of the validity of the equations (Navier-Stokes) used to describe the system. While such a breakdown may happen we think that it does not necessarily accompany turbulence. Landau and Lifschitz [8] propose that the physical parameters x describing a fluid in turbulent motion are quasi-periodic functions of time:

$$x(t) = f(\omega_1 t, \ldots, \omega_k t)$$

where f has period 1 in each of its arguments separately and the frequences $\omega_1, \ldots, \omega_k$ are not rationally related[2]. It is expected that k becomes large for large μ, and that this leads to the complicated and irregular behaviour

* The research was supported by the Netherlands Organisation for the Advancement of Pure Research (Z.W.O.).

[1] Depending upon the situation, μ will be the Reynolds number, Rayleigh number, etc.

[2] This behaviour is actually found and discussed by E. Hopf in a model of turbulence [A mathematical example displaying features of turbulence. Commun. Pure Appl. Math. 1, 303–322 (1948)].

characteristic of turbulent motion. We shall see however that a dissipative system like a viscous fluid will not in general have quasi-periodic motions[3]. The idea of Landau and Lifschitz must therefore be modified.

Consider for definiteness a viscous incompressible fluid occupying a region D of $\mathbb{R}^3$. If thermal effects can be ignored, the fluid is described by its velocity at every point of D. Let H be the space of velocity fields v over D; H is an infinite dimensional vector space. The time evolution of a velocity field is given by the Navier-Stokes equations

$$\frac{dv}{dt} = X_\mu(v) \tag{1}$$

where X_μ is a vector field over H. For our present purposes it is not necessary to specify further H or X_μ[4].

In what follows we shall investigate the nature of the solutions of (1), making only assumptions of a very general nature on X_μ. It will turn out that the fluid motion is *expected* to become chaotic when μ increases. This gives a justification for turbulence and some insight into its meaning. To study (1) we shall replace H by a finite-dimensional manifold[5] and use the qualitative theory of differential equations.

For $\mu = 0$, every solution $v(\cdot)$ of (1) tends to the solution $v_0 = 0$ as the time tends to $+\infty$. For $\mu > 0$ we know very little about the vector field X_μ. Therefore it is reasonable to study *generic* deformations from the situation at $\mu = 0$. In other words we shall ignore possibilities of deformation which are in some sense exceptional. This point of view could lead to serious error if, by some law of nature which we have overlooked, X_μ happens to be in a special class with exceptional properties[6]. It appears however that a three-dimensional viscous fluid conforms to the pattern of generic behaviour which we discuss below. Our discussion should in fact apply to very general dissipative systems[7].

The present paper is divided into two chapters. Chapter I is oriented towards physics and is relatively untechnical. In Section 2 we review

[3] Quasi-periodic motions occur for other systems, see Moser [10].

[4] A general existence and uniqueness theorem has not been proved for solutions of the Navier-Stokes equations. We assume however that we have existence and uniqueness locally, i.e., in a neighbourhood of some $v_0 \in H$ and of some time t_0.

[5] This replacement can in several cases be justified, see § 5.

[6] For instance the differential equations describing a Hamiltonian (conservative) system, have very special properties. The properties of a conservative system are indeed very different from the properties of a dissipative system (like a viscous fluid). If a viscous fluid is observed in an experimental setup which has a certain symmetry, it is important to take into account the invariance of X_μ under the corresponding symmetry group. This problem will be considered elsewhere.

[7] In the discussion of more specific properties, the behaviour of a viscous fluid may turn out to be nongeneric, due for instance to the local nature of the differential operator in the Navier-Stokes equations.

some results on differential equations; in Section 3–4 we apply these results to the study of the solutions of (1). Chapter II contains the proofs of several theorems used in Chapter I. In Section 5, center-manifold theory is used to replace H by a finite-dimensional manifold. In Sections 6–8 the theory of Hopf bifurcation is presented both for vector fields and for diffeomorphisms. In Section 9 an example of "turbulent" attractor is presented.

Acknowledgements. The authors take pleasure in thanking R. Thom for valuable discussion, in particular introducing one of us (F. T.) to the Hopf bifurcation. Some inspiration for the present paper was derived from Thom's forthcoming book [12].

Chapter I

§ 2. Qualitative Theory of Differential Equations

Let $B=\{x:|x|<R\}$ be an open ball in the finite dimensional euclidean space H[8]. Let X be a vector field with continuous derivatives up to order r on $\bar{B}=\{x:|x|\leqq R\}$, r fixed $\geqq 1$. These vector fields form a Banach space $\mathscr{B}$ with the norm

$$\|X\| = \sup_{1\leqq i\leqq \nu}\ \sup_{|\varrho|\leqq r}\ \sup_{x\in \bar{B}} \left|\frac{\partial^{|\varrho|}}{\partial x^{\varrho}} X^i(x)\right|$$

where

$$\frac{\partial^{|\varrho|}}{\partial x^{\varrho}} = \left(\frac{\partial}{\partial x^1}\right)^{\varrho_1} \cdots \left(\frac{\partial}{\partial x^{\nu}}\right)^{\varrho_\nu}$$

and $|\varrho|=\varrho_1+\cdots+\varrho_\nu$. A subset E of $\mathscr{B}$ is called *residual* if it contains a countable intersection of open sets which are dense in $\mathscr{B}$. Baire's theorem implies that a residual set is again dense in $\mathscr{B}$; therefore a residual set E may be considered in some sense as a "large" subset of $\mathscr{B}$. A property of a vector field $X\in\mathscr{B}$ which holds on a residual set of $\mathscr{B}$ is called *generic*.

The *integral curve* $x(\cdot)$ through $x_0\in B$ satisfies $x(0)=x_0$ and $dx(t)/dt = X(x(t))$; it is defined at least for sufficiently small $|t|$. The dependence of $x(\cdot)$ on x_0 is expressed by writing $x(t)=\mathscr{D}_{X,t}(x_0)$; $\mathscr{D}_{X,\cdot}$ is called *integral* of the vector field X; $\mathscr{D}_{X,1}$ is the time one integral. If $x(t)\equiv x_0$, i.e. $X(x_0)=0$, we have a *fixed point* of X. If $x(\tau)=x_0$ and $x(t)\neq x_0$ for $0<t<\tau$ we have a closed orbit of period τ. A natural generalization of the idea of *closed orbit* is that of *quasi-periodic* motion:

$$x(t)=f(\omega_1 t,\ldots,\omega_k t)$$

where f is periodic of period 1 in each of its arguments separately and the frequencies $\omega_1,\ldots,\omega_k$ are not rationally related. We assume that f is

[8] More generally we could use a manifold H of class C^r.

a C^k-function and its image a k-dimensional torus T^k imbedded in B. Then however we find that a quasi-periodic motion is non-generic. In particular for $k=2$, Peixoto's theorem[9] shows that quasi-periodic motions on a torus are in the complement of a dense open subset Σ of the Banach space of C^r vector fields on the torus: Σ consists of vector fields for which the non wandering set Ω[10] is composed of a finite number of fixed points and closed orbits only.

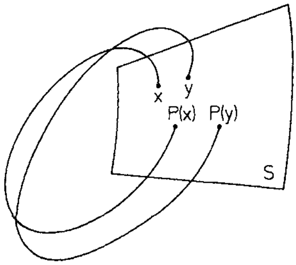


Fig. 1

As $t \to +\infty$, an integral curve $x(t)$ of the vector field X may be attracted by a fixed point or a closed orbit of the vector field, or by a more general attractor[11]. It will probably not be attracted by a quasi-periodic motion because these are rare. It is however possible that the orbit be attracted by a set which is not a manifold. To visualize such a situation in n dimensions, imagine that the integral curves of the vector field go roughly parallel and intersect transversally some piece of $n-1$-dimensional surface S (Fig. 1). We let $P(x)$ be the first intersection of the integral curve through x with S (P is a Poincaré map).

Take now $n-1=3$, and assume that P maps the solid torus Π_0 into itself as shown in Fig. 2,

$$P\Pi_0 = \Pi_1 \subset \Pi_0 .$$

The set $\bigcap_{n>0} P^n \Pi_0$ is an attractor; it is locally the product of a Cantor set and a line interval (see Smale [11], Section I.9). Going back to the vector field X, we have thus a "strange" attractor which is locally the product of a Cantor set and a piece of two-dimensional manifold. Notice that we

[9] See Abraham [1].

[10] A point x belongs to Ω (i.e. is non wandering) if for every neighbourhood U of x and every $T>0$ one can find $t>T$ such that $\mathscr{D}_{X,t}(U) \cap U \neq \emptyset$. For a quasi-periodic motion on T^k we have $\Omega = T^k$.

[11] A closed subset A of the non wandering set Ω is an attractor if it has a neighbourhood U such that $\bigcap_{t>0} \mathscr{D}_{X,t}(U) = A$. For more attractors than those described here see Williams [13].

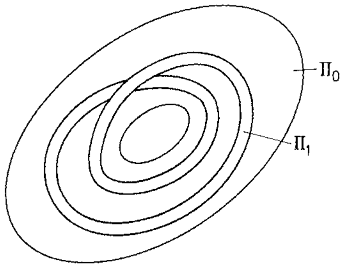


Fig. 2

keep the same picture if X is replaced by a vector field Y which is sufficiently close to X in the appropriate Banach space. An attractor of the type just described can therefore not be thrown away as non-generic pathology.

§ 3. A Mathematical Mechanism for Turbulence

Let X_μ be a vector field depending on a parameter μ[12]. The assumptions are the same as in Section 2, but the interpretation we have in mind is that X_μ is the right-hand side of the Navier-Stokes equations. When μ varies the vector field X_μ may change in a number of manners. Here we shall describe a pattern of changes which is physically acceptable, and show that it leads to something like turbulence.

For $\mu = 0$, the equation

$$\frac{dx}{dt} = X_\mu(x)$$

has the solution $x = 0$. We assume that the eigenvalues of the Jacobian matrix A_k^j defined by

$$A_k^j = \frac{\partial X_0^j}{\partial x^k}(0)$$

have all strictly negative real parts; this corresponds to the fact that the fixed point 0 is attracting. The Jacobian determinant is not zero and therefore there exists (by the implicit function theorem) $\xi(\mu)$ depending continuously on μ and such that

$$X_\mu(\xi(\mu)) = 0\,.$$

In the hydrodynamical picture, $\xi(\mu)$ describes a steady state.

We follow now $\xi(\mu)$ as μ increases. For sufficiently small μ the Jacobian matrix $A_k^j(\mu)$ defined by

$$A_k^j(\mu) = \frac{\partial X_\mu^j}{\partial x^k}(\xi(\mu)) \tag{2}$$

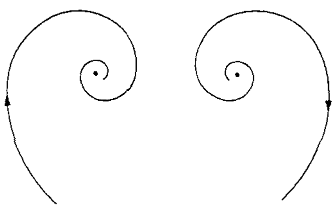
[12] To be definite, let $(x, \mu) \to X_\mu(x)$ be of class C^r.

has only eigenvalues with strictly negative real parts (by continuity). We assume that, as μ increases, successive pairs of complex conjugate eigenvalues of (2) cross the imaginary axis, for $\mu = \mu_1, \mu_2, \mu_3, \ldots$[13]. For $\mu > \mu_1$, the fixed point $\xi(\mu)$ is no longer attracting. It has been shown by Hopf[14] that when a pair of complex conjugate eigenvalues of (2) cross the imaginary axis at μ_i, there is a one-parameter family of closed orbits of the vector field in a neighbourhood of $(\xi(\mu_i), \mu_i)$. More precisely there are continuous functions $y(\omega)$, $\mu(\omega)$ defined for $0 \leq \omega < 1$ such that

(a) $y(0) = \xi(\mu_i)$, $\mu(0) = \mu_i$,

(b) the integral curve of $X_{\mu(\omega)}$ through $y(\omega)$ is a closed orbit for $\omega > 0$.

Generically $\mu(\omega) > \mu_i$ or $\mu(\omega) < \mu_i$ for $\omega \neq 0$. To see how the closed orbits are obtained we look at the two-dimensional situation in a neighbourhood of $\xi(\mu_1)$ for $\mu < \mu_1$ (Fig. 3) and $\mu > \mu_1$ (Fig. 4). Suppose that when μ crosses μ_1 the vector field remains like that of Fig. 3 at large distances of $\xi(\mu)$; we get a closed orbit as shown in Fig. 5. Notice that Fig. 4 corresponds to $\mu > \mu_1$ and that the closed orbit is attracting. Generally we shall assume that the closed orbits appear for $\mu > \mu_i$ so that the vector field at large distances of $\xi(\mu)$ remains attracting in accordance with physics. As μ crosses we have then replacement of an attracting fixed point by an attracting closed orbit. The closed orbit is physically interpreted as a periodic motion, its amplitude increases with μ.



Figs. 3 and 4

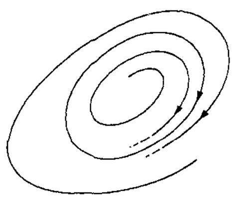


Fig. 5

§ 3a) Study of a Nearly Split Situation

To see what happens when μ crosses the successive μ_i, we let E_i be the two-dimensional linear space associated with the i-th pair of eigenvalues of the Jacobian matrix. In first approximation the vector field X_μ is, near $\xi(\mu)$, of the form

$$\tilde{X}_\mu(x) = \tilde{X}_{\mu 1}(x_1) + \tilde{X}_{\mu 2}(x_2) + \cdots \tag{3}$$

[13] Another less interesting possibility is that a real eigenvalue vanishes. When this happens the fixed point $\xi(\lambda)$ generically coalesces with another fixed point and disappears (this generic behaviour is changed if some symmetry is imposed to the vector field X_μ).

[14] Hopf [6] assumes that X is real-analytic; the differentiable case is treated in Section 6 of the present paper.

where $\tilde{X}_{\mu i}$, x_i are the components of $\tilde{X}_\mu$ and x in E_i. If μ is in the interval (μ_k, μ_{k+1}), the vector field $\tilde{X}_\mu$ leaves invariant a set $\tilde{T}^k$ which is the cartesian product of k attracting closed orbits $\Gamma_1, \ldots, \Gamma_k$ in the spaces $E_1, \ldots, E_k$. By suitable choice of coordinates on $\tilde{T}^k$ we find that the motion defined by the vector field on $\tilde{T}^k$ is quasi-periodic (the frequencies $\tilde{\omega}_1, \ldots, \tilde{\omega}_k$ of the closed orbits in $E_1, \ldots, E_k$ are in general not rationally related).

Replacing $\tilde{X}_\mu$ by X_μ is a perturbation. We assume that this perturbation is small, i.e. we assume that X_μ nearly splits according to (3). In this case there exists a C^r manifold (torus) T^k close to $\tilde{T}^k$ which is invariant for X_μ and attracting[15]. The condition that $X_\mu - \tilde{X}_\mu$ be small depends on how attracting the closed orbits $\Gamma_1, \ldots, \Gamma_k$ are for the vector field $\tilde{X}_{\mu 1}, \ldots, \tilde{X}_{\mu k}$; therefore the condition is violated if μ becomes too close to one of the μ_i.

We consider now the vector field X_μ restricted to T^k. For reasons already discussed, we do not expect that the motion will remain quasi-periodic. If $k=2$, Peixoto's theorem implies that generically the non-wandering set of T^2 consists of a finite number of fixed points and closed orbits. What will happen in the case which we consider is that there will be one (or a few) attracting closed orbits with frequencies ω_1, ω_2 such that ω_1/ω_2 goes continuously through rational values.

Let $k>2$. In that case, the vector fields on T^k for which the non-wandering set consists of a finite number of fixed points and closed orbits are no longer dense in the appropriate Banach space. Other possibilities are realized which correspond to a more complicated orbit structure; "strange" attractors appear like the one presented at the end of Section 2. Taking the case of T^4 and the C^3-topology we shall show in Section 9 that in any neighbourhood of a quasi-periodic $\tilde{X}$ there is an open set of vector fields with a strange attractor.

We propose to say that the motion of a fluid system is turbulent when this motion is described by an integral curve of the vector field X_μ which tends to a set A[16], and A is neither empty nor a fixed point nor a closed orbit. In this definition we disregard nongeneric possibilities (like A having the shape of the figure 8, etc.). This proposal is based on two things:

(a) We have shown that, when μ increases, it is not unlikely that an attractor A will appear which is neither a point nor a closed orbit.

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[15] This follows from Kelley [7], Theorem 4 and Theorem 5, and also from recent work of Pugh (unpublished). That T^k is attracting means that it has a neighbourhood U such that $\bigcap_{t>0} \mathscr{D}_{X,t}(U) = T^k$. We cannot call T^k an attractor because it need not consist of non-wandering points.

[16] More precisely A is the ω^+ limit set of the integral curve $x(\cdot)$, i.e., the set of points ξ such that there exists a sequence (t_n) and $t_n \to \infty$, $x(t_n) \to \xi$.

(b) In the known generic examples where A is not a point or a closed orbit, the structure of the integral curves on or near A is complicated and erratic (see Smale [11] and Williams [13]).

We shall further discuss the above definition of turbulent motion in Section 4.

§ 3b) Bifurcations of a Closed Orbit

We have seen above how an attracting fixed point of X_μ may be replaced by an attracting closed orbit γ_μ when the parameter crosses the value μ_1 (Hopf bifurcation). We consider now in some detail the next bifurcation; we assume that it occurs at the value μ' of the parameter[17] and that $\lim_{\mu \to \mu'} \gamma_\mu$ is a closed orbit $\gamma_{\mu'}$ of $X_{\mu'}$[18].

Let Φ_μ be the Poincaré map associated with a piece of hypersurface S transversal to γ_μ, for $\mu \in (\mu_1, \mu']$. Since γ_μ is attracting, $p_\mu = S \cap \gamma_\mu$ is an attracting fixed point of Φ_μ for $\mu \in (\mu_1, \mu')$. The derivative $d\Phi_\mu(p_\mu)$ of Φ_μ at the point p_μ is a linear map of the tangent hyperplane to S at p_μ to itself.

We assume that the spectrum of $d\Phi_{\mu'}(p_{\mu'})$ consists of a finite number of isolated eigenvalues of absolute value 1, and a part which is contained in the open unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$[19]. According to § 5, Remark (5.6), we may assume that S is finite dimensional. With this assumption one can say rather precisely what kind of generic bifurcations are possible for $\mu = \mu'$. We shall describe these bifurcations by indicating what kind of attracting subsets for X_μ (or Φ_μ) there are near $\gamma_{\mu'}$ (or $p_{\mu'}$) when $\mu > \mu'$.

Generically, the set E of eigenvalues of $d\Phi_{\mu'}(p_{\mu'})$, with absolute value 1, is of one of the following types:

1. $E = \{+1\}$,
2. $E = \{-1\}$,
3. $E = \{\alpha, \bar{\alpha}\}$ where α, $\bar{\alpha}$ are distinct.

For the cases 1 and 2 we can refer to Brunovsky [3]. In fact in case 1 the attracting closed orbit disappears (together with a hyperbolic closed orbit); for $\mu > \mu'$ there is no attractor of X_μ near $\gamma_{\mu'}$. In case 2 there is for $\mu > \mu'$ (or $\mu < \mu'$) an attracting (resp. hyperbolic) closed orbit near $\gamma_{\mu'}$, but the period is doubled.

If we have case 3 then Φ_μ has also for μ slightly bigger than μ' a fixed point p_μ; generically the conditions (a)', ..., (e) in Theorem (7.2) are

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[17] In general μ' will differ from the value μ_2 introduced in § 3a).

[18] There are also other possibilities: If γ_μ tends to a point we have a Hopf bifurcation with parameter reversed. The cases where $\lim_{\mu \to \mu'} \gamma_\mu$ is not compact or where the period of γ_μ tends to ∞ are not well understood; they may or may not give rise to turbulence.

[19] If the spectrum of $d\Phi_{\mu'}(p_{\mu'})$ is discrete, this is a reasonable assumption, because for $\mu_1 < \mu < \mu'$ the spectrum is contained in the open unit disc.

satisfied. One then concludes that when $\gamma_{\mu'}$ is a "vague attractor" (i.e. when the condition (f) is satisfied) then, for $\mu > \mu'$, there is an attracting circle for Φ_μ; this amounts to the existence of an invariant and attracting torus T^2 for X_μ. If $\gamma_{\mu'}$ is not a "vague attractor" then, generically, X_μ has no attracting set near $\gamma_{\mu'}$ for $\mu > \mu'$.

§ 4. Some Remarks on the Definition of Turbulence

We conclude this discussion by a number of remarks:

1. The concept of genericity based on residual sets may not be the appropriate one from the physical view point. In fact the complement of a residual set of the μ-axis need not have Lebesgue measure zero. In particular the quasi-periodic motions which we had eliminated may in fact occupy a part of the μ-axis with non vanishing Lebesgue measure[20]. These quasi-periodic motions would be considered turbulent by our definition, but the "turbulence" would be weak for small k. There are arguments to define the quasi-periodic motions, along with the periodic ones, as non turbulent (see (4) below).

2. By our definition, a periodic motion (= closed orbit of X_μ) is not turbulent. It may however be very complicated and appear turbulent (think of a periodic motion closely approximating a quasi-periodic one, see § 3b) second footnote).

3. We have shown that, under suitable conditions, there is an attracting torus T^k for X_μ if μ is between μ_k and μ_{k+1}. We assumed in the proof that μ was not too close to μ_k or μ_{k+1}. In fact the transition from T^1 to T^2 is described in Section 3b, but the transition from T^k to T^{k+1} appears to be a complicated affair when $k > 1$. In general, one gets the impression that the situations not covered by our description are more complicated, hard to describe, and probably turbulent.

4. An interesting situation arises when statistical properties of the motion can be obtained, via the pointwise ergodic theorem, from an ergodic measure m supported by the attracting set A. An observable quantity for the physical system at a time t is given by a function x_t on H, and its expectation value is $m(x_t) = m(x_0)$. If m is "mixing" the time correlation functions $m(x_t y_0) - m(x_0)\, m(y_0)$ tend to zero as $t \to \infty$. This situation appears to prevail in turbulence, and "pseudo random" variables with correlation functions tending to zero at infinity have been studied by Bass[21]. With respect to this property of time correlation functions the quasi-periodic motions should be classified as non turbulent.

[20] On the torus T^2, the rotation number ω is a continuous function of μ. Suppose one could prove that, on some μ-interval, ω is non constant and is absolutely continuous with respect to Lebesgue measure; then ω would take irrational values on a set of non zero Lebesgue measure.

[21] See for instance [2].

5. In the above analysis the detailed structure of the equations describing a viscous fluid has been totally disregarded. Of course something is known of this structure, and also of the experimental conditions under which turbulence develops, and a theory should be obtained in which these things are taken into account.

6. Besides viscous fluids, other dissipative systems may exhibit time-periodicity and possibly more complicated time dependence; this appears to be the case for some chemical systems[22].

Chapter II

§ 5. Reduction to Two Dimensions

Definition (5.1). Let $\Phi: H \to H$ be a C^1 map with fixed point $p \in H$, where H is a Hilbert space. The spectrum of Φ at p is the spectrum of the induced map $(d\Phi)_p: T_p(H) \to T_p(H)$.

Let X be a C^1 vectorfield on H which is zero in $p \in H$. For each t we then have $d(\mathscr{D}_{X,t})_p: T_p(H) \to T_p(H)$, induced by the time t integral of X. Let $L(X): T_p(H) \to T_p(H)$ be the unique continuous linear map such that $d(\mathscr{D}_{X,t})_p = e^{t \cdot L(X)}$.

We define the spectrum of X at p to be the spectrum of $L(X)$, (note that $L(X)$ also can be obtained by linearizing X).

Proposition (5.2). *Let X_μ be a one-parameter family of C^k vectorfields on a Hilbert space H such that also X, defined by $X(h, \mu) = (X_\mu(h), 0)$, on $H \times \mathbb{R}$ is C^k. Suppose:*

(a) *X_μ is zero in the origin of H.*

(b) *For $\mu < 0$ the spectrum of X_μ in the origin is contained in $\{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$.*

(c) *For $\mu = 0$, resp. $\mu > 0$, the spectrum of X_μ at the origin has two isolated eigenvalues $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ with multiplicity one and $\mathrm{Re}(\lambda(\mu)) = 0$, resp. $\mathrm{Re}(\lambda(\mu)) > 0$. The remaining part of the spectrum is contained in $\{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$.*

Then there is a (small) 3-dimensional C^k-manifold $\tilde{V}^c$ of $H \times \mathbb{R}$ containing $(0, 0)$ such that:

1. *$\tilde{V}^c$ is locally invariant under the action of the vectorfield X (X is defined by $X(h, \mu) = (X_\mu(h), 0)$); locally invariant means that there is a neighbourhood U of $(0, 0)$ such that for $|t| \leqq 1$, $\tilde{V}^c \cap U = \mathscr{D}_{X,t}(\tilde{V}^c) \cap U$.*

2. *There is a neighbourhood U' of $(0, 0)$ such that if $p \in U'$, is recurrent, and has the property that $\mathscr{D}_{X,t}(p) \in U'$ for all t, then $p \in \tilde{V}^c$*

3. *in $(0, 0)$ $\tilde{V}^c$ is tangent to the μ axis and to the eigenspace of $\lambda(0)$, $\overline{\lambda(0)}$.*

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[22] See Pye, K., Chance, B.: Sustained sinusoidal oscillations of reduced pyridine nucleotide in a cell-free extract of Saccharomyces carlbergensis. Proc. Nat. Acad. Sci. U.S.A. **55**, 888–894 (1966).

Proof. We construct the following splitting $T_{(0,0)}(H\times\mathbb{R})=V^c\oplus V^s$: V^c is tangent to the μ axis and contains the eigenspace of $\lambda(\mu)$, $\overline{\lambda(\mu)}$; V^s is the eigenspace corresponding to the remaining (compact) part of the spectrum of $L(X)$. Because this remaining part is compact there is a $\delta>0$ such that it is contained in $\{z\in\mathbb{C}\mid \mathrm{Re}(z)<-\delta\}$. We can now apply the centermanifold theorem [5], the proof of which generalizes to the case of a Hilbert space, to obtain $\tilde{V}^c$ as the centermanifold of X at $(0,0)$ [by assumption X is C^k, so $\tilde{V}^c$ is C^k; if we would assume only that, for each μ, X_μ is C^k (and X only C^1), then $\tilde{V}^c$ would be C^1 but, for each μ_0, $\tilde{V}^c\cap\{\mu=\mu_0\}$ would be C^k].

For positive t, $d(\mathscr{D}_{X,t})_{0,0}$ induces a contraction on V^s (the spectrum is contained in $\{z\in\mathbb{C}\mid |z|<e^{-\delta t}\}$). Hence there is a neighbourhood U' of $(0,0)$ such that

$$U'\cap\left[\bigcap_{t=1}^{\infty}\mathscr{D}_{X,t}(U')\right]\subset(U'\cap\tilde{V}^c).$$

Now suppose that $p\in U'$ is recurrent and that $\mathscr{D}_{X,t}(p)\in U'$ for all t. Then given $\varepsilon>0$ and $N>0$ there is a $t>N$ such that the distance between p and $\mathscr{D}_{X,t}(p)$ is $<\varepsilon$. It then follows that $p\in(U'\cap\tilde{V}^c)\subset\tilde{V}^c$ for U' small enough. This proves the proposition.

Remark (5.3). The analogous proposition for a one parameter set of diffeomorphisms Φ_μ is proved in the same way. The assumptions are then:

(a)′ The origin is a fixed point of Φ_μ.

(b)′ For $\mu<0$ the spectrum of Φ_μ at the origin is contained in $\{z\in\mathbb{C}\mid |z|<1\}$.

(c)′ For $\mu=0$ resp. $\mu>0$ the spectrum of Φ_μ at the origin has two isolated eigenvalues $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ with multiplicity one and $|\lambda(\mu)|=1$ resp. $|\lambda(\mu)|>1$. The remaining part of the spectrum is contained in $\{z\in\mathbb{C}\mid |z|<1\}$.

One obtains just as in Proposition (5.2) a 3-dimensional center manifold which contains all the local recurrence.

Remark (5.4). If we restrict the vectorfield X, or the diffeomorphism Φ [defined by $\Phi(h,\mu)=(\Phi_\mu(h),\mu)$], to the 3-dimensional manifold $\tilde{V}^c$ we have locally the same as in the assumptions (a), (b), (c), or (a)′, (b)′, (c)′ where now the Hilbert space has dimension 2. So if we want to prove a property of the local recurrent points for a one parameter family of vectorfield, or diffeomorphisms, satisfying (a) (b) and (c), or (a)′, (b)′ and (c)′, it is enough to prove it for the case where $\dim(H)=2$.

Remark (5.5). Everything in this section holds also if we replace our Hilbert space by a Banach space with C^k-norm; a Banach space B has C^k-norm if the map $x\to\|x\|$, $x\in B$ is C^k except at the origin. This C^k-norm is needed in the proof of the center manifold theorem.

Remark (5.6). The Propositions (5.2) and (5.3) remain true if

1. we drop the assumptions on the spectrum of X_μ resp. Φ_μ for $\mu > 0$.
2. we allow the spectrum of X_ϱ resp. Φ_ϱ to have an arbitrary but finite number of isolated eigenvalues on the real axis resp. the unit circle. The dimension of the invariant manifold $\tilde{V}^c$ is then equal to that number of eigenvalues plus one.

§ 6. The Hopf Bifurcation

We consider a one parameter family X_μ of C^k-vectorfields on $\mathbb{R}^2$, $k \geqq 5$, as in the assumption of proposition (5.2) (with $\mathbb{R}^2$ instead of H); $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ are the eigenvalues of X_μ in (0, 0). Notice that with a suitable change of coordinates we can achieve $X_\mu = (\operatorname{Re}\lambda(\mu)x_1 + \operatorname{Im}\lambda(\mu)x_2)\frac{\partial}{\partial x_1} + (-\operatorname{Im}\lambda(\mu)x_1 + \operatorname{Re}\lambda(\mu)x_2)\frac{\partial}{\partial x_2}$ + terms of higher order.

Theorem (6.1). (Hopf [6]). *If* $\left(\frac{d(\lambda(\mu))}{d\mu}\right)_{\mu=0}$ *has a positive real part, and if* $\lambda(0) \neq 0$, *then there is a one-parameter family of closed orbits of* $X(=(X_\mu, 0))$ *on* $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}^1$ *near* (0, 0, 0) *with period near* $\frac{2\pi}{|\lambda(0)|}$; *there is a neighbourhood U of* (0, 0, 0) *in* $\mathbb{R}^3$ *such that each closed orbit of X, which is contained in U, is a member of the above family.*

If (0, 0) *is a "vague attractor" (to be defined later) for* X_0, *then this one-parameter family is contained in* $\{\mu > 0\}$ *and the orbits are of attracting type.*

Proof. We first have to state and prove a lemma on polar-coordinates:

Lemma (6.2). *Let X be a* C^k *vectorfield on* $\mathbb{R}^2$ *and let* $X(0,0) = 0$. *Define polar coordinates by the map* $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$, *with* $\Psi(r, \varphi) = (r\cos\varphi, r\sin\varphi)$. *Then there is a unique* C^{k-2}*-vectorfield* $\tilde{X}$ *on* $\mathbb{R}^2$, *such that* $\Psi_*(\tilde{X}) = X$ *(i.e. for each* (r, φ) $d\Psi(\tilde{X}(r,\varphi)) = X(r\cos\varphi, r\sin\varphi)$*).*

Proof of Lemma (6.2). We can write

$$\begin{aligned} X &= X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2} \\ &= \frac{x_1 X_1 + x_2 X_2}{\sqrt{x_1^2 + x_2^2}} \left(\frac{1}{\sqrt{x_1^2 + x_2^2}} \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) \right) \\ &\quad + \frac{(-x_2 X_1 + x_1 X_2)}{(x_1^2 + x_2^2)} \left(-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} \right) \\ &= \frac{f_r(x_1, x_2)}{r} \cdot \Psi_*(\tilde{Z}_r) + \frac{f_\varphi(x_1, x_2)}{r^2} \Psi_*(\tilde{Z}_\varphi). \end{aligned}$$

Where $\tilde{Z}_r\left(=\frac{\partial}{\partial r}\right)$ and $\tilde{Z}_\varphi\left(=\frac{\partial}{\partial \varphi}\right)$ are the "coordinate vectorfields" with respect to (r,φ) and $r=\pm\sqrt{x_1^2+x_2^2}$. (Note that r and $\Psi_*(\tilde{Z}_r)$ are bi-valued.)

Now we consider the functions $\Psi^*(f_r)=f_r\circ\Psi$ and $\Psi^*(f_\varphi)$. They are zero along $\{r=0\}$; this also holds for $\frac{\partial}{\partial r}(\Psi^*(f_r))$ and $\frac{\partial}{\partial r}(\Psi^*(f_\varphi))$. By the division theorem $\frac{\Psi^*(f_r)}{r}$, resp. $\frac{\Psi^*(f_\varphi)}{r^2}$, are C^{k-1} resp. C^{k-2}.

We can now take $\tilde{X}=\frac{\Psi^*(f_r)}{r}\tilde{Z}_r+\frac{\Psi^*(f_\varphi)}{r^2}\tilde{Z}_\varphi$; the uniqueness is evident.

Definition (6.3). We define a Poincaré map P_X for a vectorfield X as in the assumptions of Theorem (6.1):

P_X is a map from $\{(x_1,x_2,\mu)\,|\,|x_1|<\varepsilon,\ x_2=0,\ |\mu|\leqq\mu_0\}$ to the (x_1,μ) plane; μ_0 is such that $\mathrm{Im}(\lambda(\mu))\neq 0$ for $|\mu|\leqq\mu_0$; ε is sufficiently small. P_X maps (x_1,x_2,μ) to the first intersection point of $\mathscr{D}_{X,t}(x_1,x_2,\mu)$, $t>0$, with the (x_1,μ) plane, for which the sign of x_1 and the x_1 coordinate of $\mathscr{D}_{X,t}(x_1,x_2,\mu)$ are the same.

Remark (6.4). P_X preserves the μ coordinate. In a plane $\mu=$ constant the map P_X is illustrated in the following figure $\mathrm{Im}(\lambda(u))\neq 0$ means that

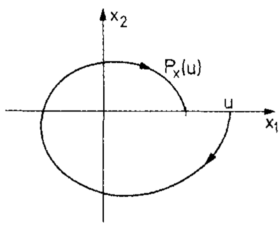

Fig. 6. Integral curve of X at $\mu=$ constant

X has a "non vanishing rotation"; it is then clear that P_X is defined for ε small enough.

Remark (6.5). It follows easily from Lemma (6.2) that P_X is C^{k-2}. We define a *displacement function* $V(x_1,\mu)$ on the domain of P_X as follows:

$$P_X(x_1,0,\mu)=(x_1+V(x_1,\mu),0,\mu);\qquad V \text{ is } C^{k-2}.$$

This displacement function has the following properties:

(i) V is zero on $\{x_1=0\}$; the other zeroes of V occur in pairs (of opposite sign), each pair corresponds to a closed orbit of X. If a closed orbit γ of X is contained in a sufficiently small neighbourhood of $(0,0)$, and intersects $\{x_1=0\}$ only twice then V has a corresponding pair of zeroes (namely the two points $\gamma\cap$(domain of P_X)).

(ii) For $\mu<0$ and $x_1=0$, $\frac{\partial V}{\partial x_1}<0$; for $\mu>0$ and $x_1=0$, $\frac{\partial V}{\partial x_1}>0$ and for $\mu=0$ and $x=0$, $\frac{\partial^2 V}{\partial\mu\,\partial x_1}>0$. This follows from the assumptions on $\lambda(\mu)$. Hence, again by the division theorem, $\tilde{V}=\frac{V}{x_1}$ is C^{k-3}. $\tilde{V}(0,0)$ is zero, $\frac{\partial\tilde{V}}{\partial\mu}>0$, so there is locally a unique C^{k-3}-curve l of zeroes of $\tilde{V}$ passing through (0, 0). Locally the set of zeroes of V is the union of l and $\{x_1=0\}$. l induces the one-parameter family of closed orbits.

(iii) Let us say that (0, 0) is a "vague attractor" for X_0 if $V(x_1,0)=-Ax_1^3+$ terms of order >3 with $A>0$. This means that the 3rd order terms of X_0 make the flow attract to (0, 0). In that case $\tilde{V}=\alpha_1\mu-Ax_1^2$ + terms of higher order, with α_1 and $A>0$, so $\tilde{V}(x_1,\mu)$ vanishes only if $x_1=0$ or $\mu>0$. This proves that the one-parameter family is contained in $\{\mu>0\}$.

(iv) The following holds in a neighbourhood of (0, 0, 0) where $\frac{\partial V}{\partial x_1}>-1$.

If $V(x_1,\mu)=0$ and $\left(\frac{\partial V}{\partial x_1}\right)_{(x_1,\mu)}<0$, then the closed orbit which cuts the domain of P_X in (x_1,μ) is an attractor of X_μ. This follows from the fact that (x_1,μ) is a fixed point of P_X and the fact that the derivative of P_X in (x_1,μ), restricted to this μ level, is smaller than 1 (in absolute value).

Combining (iii) and (iv) it follows easily that, if (0, 0) is a vague attractor, the closed orbits of our one parameter family are, near (0, 0), of the attracting type.

Finally we have to show that, for some neighbourhood U of (0, 0), every closed orbit of X, which is contained in U, is a member of our family of closed orbits. We can make U so small that every closed orbit γ of X, which is contained in U, intersects the domain of P_X.

Let $p=(x_1(\gamma),0,\mu(\gamma))$ be an intersection point of a closed orbit γ with the domain of P_X. We may also assume that U is so small that $P_X[U\cap(\text{domain of }P_X)]\subset(\text{domain of }P_X)$. Then $P_X(p)$ is in the domain of P_X but also $P_X(p)\subset U$ so $(P_X)^2(p)$ is defined etc.; so $P_X^i(p)$ is defined.

Restricted to $\{\mu=\mu(\gamma)\}$, P_X is a local diffeomorphism of a segment of the half line ($x_1\geqq 0$ or $x_1\leqq 0$, $x_2=0$, $\mu=\mu(\gamma)$) into that half line.

If the x_1 coordinate of $P_X^i(p)$ is < (resp. >) than $x_1(\gamma)$ then the x_1 coordinate of $P_X^{i+1}(p)$ is < (resp. >) than the x_1 coordinate of $P_X^i(p)$, so p does not lie on a closed orbit. Hence we must assume that the x_1 coordinate of $P_X(p)$ is $x_1(\gamma)$, hence p is a fixed point of P_X, hence p is a zero of V, so, by property (ii), γ is a member of our one parameter family of closed orbits.

§ 7. Hopf Bifurcation for Diffeomorphisms*

We consider now a one parameter family $\Phi_\mu : \mathbb{R}^2 \to \mathbb{R}^2$ of diffeomorphisms satisfying (a)', (b)' and (c)' (Remark (5.3)) and such that:

(d) $\frac{d}{d\mu}(|\lambda(\mu)|)_{\mu=0} > 0.$

Such a diffeomorphism can for example occur as the time one integral of a vectorfield X_μ as we studied in Section 2. In this diffeomorphism case we shall of course not find any closed (circular) orbit (the orbits are not continuous) but nevertheless we shall, under rather general conditions, find, near (0, 0) and for μ small, a one parameter family of invariant circles.

We first bring Φ_μ, by coordinate transformations, into a simple form:

We change the μ coordinate in order to obtain

(d)' $|\lambda(\mu)| = 1 + \mu$.

After an appropriate (μ dependent) coordinate change of $\mathbb{R}^2$ we then have $\Phi(r, \varphi, \mu) = ((1+\mu)r, \varphi + f(\mu), \mu) +$ terms of order r^2, where $x_1 = r\cos\varphi$ and $x_2 = r\sin\varphi$; "$\Phi = \Phi' +$ terms of order r^l" means that the derivatives of Φ and Φ' up to order $l-1$ with respect to (x_1, x_2) agree for $(x_1, x_2) = (0, 0)$.

We now put in one extra condition:

(e) $f(0) \neq \frac{k}{l} \cdot 2\pi$ for all $k, l \leqq 5$.

Proposition (7.1). *Suppose* Φ_μ *satisfies* (a)', (b)', (c)', (d)' *and* (e) *and is* C^k, $k \geqq 5$. *Then for* μ *near* 0, *by a* μ *dependent coordinate change in* $\mathbb{R}^2$, *one can bring* Φ_μ *in the following form:*

$$\Phi_\mu(r, \varphi) = ((1+\mu)r - f_1(\mu)\cdot r^3,\ \varphi + f_2(\mu) + f_3(\mu)\cdot r^2) + \text{terms of order } r^5.$$

For each μ, *the coordinate transformation of* $\mathbb{R}^2$ *is* C^∞; *the induced coordinate transformation on* $\mathbb{R}^2 \times \mathbb{R}$ *is only* C^{k-4}.

The next paragraph is devoted to the proof of this proposition**. Our last condition on Φ_μ is:

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* *Note added in proof.* J. Moser kindly informed us that the Hopf bifurcation for diffeomorphisms had been worked out by Neumark (reference not available) and R. Sacker (Thesis, unpublished). An example of "decay" (loss of differentiability) of T^2 under perturbations has been studied by N. Levinson [a second order differentiable equation with singular solutions. Ann. of Math. **50**, 127–153 (1949)].

** *Note added in proof.* The desired normal form can also be obtained from § 21 of C. L. Siegel, Vorlesungen über Himmelsmechanik, Springer, Berlin, 1956 (we thank R. Jost for emphasizing this point).

(f) $f_1(0) \neq 0$. We assume even that $f_1(0) > 0$ (this corresponds to the case of a vague attractor for $\mu = 0$, see Section 6); the case $f_1(0) < 0$ can be treated in the same way (by considering $\Phi_{-\mu}^{-1}$ instead of Φ_μ).

Notation. We shall use $N\Phi_\mu$ to denote the map

$$(r, \varphi) \to ((1+\mu)r - f_1(\mu) \cdot r^3, \varphi + f_2(\mu) + f_3(\mu) \cdot r^2)$$

and call this "the simplified Φ_μ".

Theorem (7.2). *Suppose Φ_μ is at least C^5 and satisfies* (a)′, (b)′, (c)′, (d)′ *and* (e) *and $N\Phi_\mu$, the simplified Φ_μ, satisfies* (f). *Then there is a continuous one parameter family of invariant attracting circles of Φ_μ, one for each $\mu \in (0, \varepsilon)$, for ε small enough.*

Proof. The idea of the proof is as follows: the set $\Sigma = \{\mu = f_1(\mu) \cdot r^2\}$ in (r, q, μ)-space is invariant under $N\Phi$; $N\Phi$ even "attracts to this set". This attraction makes Σ stable in the following sense: $\{\Phi^n(\Sigma)\}_{n=0}^{\infty}$ is a sequence of manifolds which converges (for μ small) to an invariant manifold (this is actually what we have to prove); the method of the proof is similar to the methods used in [4, 5].

First we define $U_\delta = \left\{(r, \varphi, \mu) \mid r \neq 0 \text{ and } \frac{\mu}{r^2} \in [f_1(\mu) - \delta, f_1(\mu) + \delta]\right\}$, $\delta \ll f_1(\mu)$, and show that $N\Phi(U_\delta) \subset U_\delta$ and also, in a neighbourhood of $(0, 0, 0)$, $\Phi(U_\delta) \subset U_\delta$. This goes as follows:

If $p \in \partial U_\delta$, and $r(p)$ is the r-coordinate of p, then the r-coordinate of $N\Phi(p)$ is $r(p) \pm \delta \cdot (r(p))^3$ and p goes towards the interior of U_δ. Because Φ equals $N\Phi$, modulo terms of order r^5, also, locally, $\Phi(U_\delta) \subset U_\delta$. From this it follows that, for ε small enough and all $n \geqq 0$ $\Phi^n(\Sigma_\varepsilon) \subset U_\delta$; $\Sigma_\varepsilon = \Sigma \cap \{0 < \mu < \varepsilon\}$.

Next we define, for vectors tangent to a μ level of U_δ, the slope by the following formula: for X tangent to $U_\delta \cap \{\mu = \mu_0\}$ and $X = X_r \frac{\partial}{\partial r} + X_\varphi \frac{\partial}{\partial \varphi}$ the slope of X is $\left|\frac{X_r}{\mu_0 \cdot X_\varphi}\right|$; for $X_\varphi = 0$ the slope is not defined.

By direct calculations it follows that if X is a tangent vector of $U_\delta \cap \{\mu = \mu_0\}$ with slope $\leqq 1$, and μ_0 is small enough, then the slope of $d(N\Phi)(X)$ is $\leqq (1 - K\mu_0)$ for some positive K. Using this, the fact that $\frac{\mu}{r^2} \sim$ constant on U_δ and the fact that Φ and $N\Phi$ only differ by terms of order r^5 one can verify that for ε small enough and X a tangent vector of $U_\delta \cap \{\mu = \mu_0\}$, $\mu_0 \leqq \varepsilon$, with slope $\leqq 1$, $d\Phi(X)$ has slope < 1.

From this it follows that for ε small enough and any $n \geqq 0$,

1. $\Phi^n(\Sigma_\varepsilon) \subset U_\delta$ and
2. the tangent vectors of $\Phi^n(\Sigma_\varepsilon) \cap \{\mu = \mu_0\}$, for $\mu_0 \leqq \varepsilon$ have slope < 1.

This means that for any $\mu_0 \leqq \varepsilon$ and $n \geqq 0$

$$\Phi^n(\Sigma_\varepsilon) \cap \{\mu = \mu_0\} = \{(f_{n,\mu_0}(\varphi), \varphi, \mu_0)\}$$

where f_{n,μ_0} is a unique smooth function satisfying:

1'. $f_{n,\mu_0}(\varphi) \in \left[\sqrt{\frac{\mu_0}{f_1(\mu_0)+\delta}}, \sqrt{\frac{\mu_0}{f_1(\mu_0)-\delta}}\right]$ for all φ

2'. $\frac{d}{d\varphi}(f_{n,\mu_0}(\varphi)) \leqq \mu_0$ for all φ.

We now have to show that, for μ_0 small enough, $\{f_{n,\mu_0}\}_{n=0}^{\infty}$ converges. We first fix a φ_0 and define

$$p_1 = (f_n(\varphi_0), \varphi_0, \mu_0), \qquad p_1' = \Phi(p_1) = (r_1', \varphi_1', \mu),$$

$$p_2 = (f_{n+1}(\varphi_0), \varphi_0, \mu_0), \qquad p_2' = \Phi(p_2) = (r_2', \varphi_2', \mu).$$

Using again the fact that $(f_{n,\mu_0}(\varphi))^2/\mu_0 \sim$ constant (independent of μ_0), one obtains:

$$|r_1' - r_2'| \leqq (1 - K_1\mu_0)\,|f_{n,\mu_0}(\varphi_0) - f_{n+1,\mu_0}(\varphi_0)|$$

and

$$|\varphi_1' - \varphi_2'| \leqq K_2\sqrt{\mu_0} \cdot |f_{n,\mu_0}(\varphi_0) - f_{n+1,\mu_0}(\varphi_0)| \quad \text{where} \quad K_1, K_2 > 0$$

and independent of μ_0.

By definition we have $f_{n+1,\mu_0}(\varphi_1') = r_1'$ and $f_{n+2,\mu_0}(\varphi_2') = r_2'$. We want however to get an estimate for the difference between $f_{n+1,\mu_0}(\varphi_1')$ and $f_{n+2,\mu_0}(\varphi_1')$. Because

$$\frac{d}{d\varphi}(f_{n+2,\mu_0}(\varphi)) \leqq \mu_0,$$

$$|f_{n+2,\mu_0}(\varphi_2') - f_{n+2,\mu_0}(\varphi_1')| \leqq \mu_0 |\varphi_2' - \varphi_1'| \leqq K_2 \cdot \mu_0^{\frac{3}{2}} |f_{n,\mu_0}(\varphi_0) - f_{n+1,\mu_0}(\varphi_0)|.$$

We have seen that

$$|f_{n+1,\mu_0}(\varphi_1') - f_{n+2,\mu_0}(\varphi_2')| = |r_1' - r_2'|$$
$$\leqq (1 - K_1\mu_0)\,|f_{n,\mu_0}(\varphi_0) - f_{n+1,\mu_0}(\varphi_0)|.$$

So

$$|f_{n+1,\mu_0}(\varphi_1') - f_{n+2,\mu_0}(\varphi_1')| \leqq (1 + K_2\mu_0^{\frac{3}{2}} - K_1\mu_0)\,|f_{n,\mu_0}(\varphi_0) - f_{n+1,\mu_0}(\varphi_0)|.$$

We shall now assume that μ_0 is so small that $(1 + K_2\mu_0^{\frac{3}{2}} - K_1\mu_0) = K_3(\mu_0) < 1$, and write $\varrho(f_{n,\mu_0}, f_{n+1,\mu_0}) = \max_\varphi(|f_{n,\mu_0}(\varphi) - f_{n+1,\mu_0}(\varphi)|)$.

It follows that

$$\varrho(f_{m,\mu_0}, f_{m+1,\mu_0}) \leqq (K_3(\mu_0))^m \cdot \varrho(f_{0,\mu_0}, f_{1,\mu_0}).$$

This proves convergence, and gives for each small $\mu_0 > 0$ an invariant and attracting circle. This family of circles is continuous because the limit functions f_{∞,μ_0} depend continuously on μ_0, because of uniform convergence.

Remark (7.3). For a given μ_0, f_{∞,μ_0} is not only continuous but even Lipschitz, because it is the limit of functions with derivative $\leqq \mu_0$. Now we can apply the results on invariant manifolds in [4, 5] and obtain the following:

If Φ_μ is C^r for each μ then there is an $\varepsilon_r > 0$ such that the circles of our family which are in $\{0 < \mu < \varepsilon_r\}$ are C^r. This comes from the fact that near $\mu = 0$ in U_δ the contraction in the r-direction dominates sufficiently the maximal possible contraction in the φ-direction.

§ 8. Normal Forms (the Proof of Proposition (7.1))

First we have to give some definitions. Let $\underline{V}_r$ be the vectorspace of r-jets of vectorfields on $\mathbb{R}^2$ in 0, whose $(r-1)$-jet is zero (i.e. the elements of $\underline{V}_r$ can be uniquely represented by a vectorfield whose component functions are homogeneous polynomials of degree r). V_r is the set of r-jets of diffeomorphisms $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, whose $(r-1)$-jet is "the identity". $\mathrm{Exp}: \underline{V}_r \to V_r$ is defined by: for $\alpha \in \underline{V}_r$, $\mathrm{Exp}(\alpha)$ is the (r-jet of) the diffeomorphism obtained by integrating α over time 1.

Remark (8.1). For $r \geqq 2$, Exp is a diffeomorphism onto and $\mathrm{Exp}(\alpha) \circ \mathrm{Exp}(\beta) = \mathrm{Exp}(\alpha + \beta)$. The proof is straightforward and left to the reader.

Let now $A: (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a linear map. The induced transformations $A_r: \underline{V}_r \to \underline{V}_r$ are defined by $A_r(\alpha) = A_* \alpha$, or, equivalently, $\mathrm{Exp}(A_r(\alpha)) = A \circ \mathrm{Exp}\,\alpha \circ A^{-1}$.

Remark (8.2). If $[\Psi]_r$ is the r-jet of $\Psi: (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $d\Psi$ is A, then, for every $\alpha \in \underline{V}_r$, the r-jets $[\Psi]_r \circ \mathrm{Exp}(\alpha)$ and $\mathrm{Exp}(A_r \alpha) \circ [\Psi]_r$ are equal. The proof is left to the reader.

A splitting $\underline{V}_r = \underline{V}_r' \oplus \underline{V}_r''$ of $\underline{V}_r$ is called an *A-splitting*, $A: (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ linear, if

1. $\underline{V}_r'$ and $\underline{V}_r''$ are invariant under the action of A_r.
2. $A_r \mid \underline{V}_r''$ has no eigenvalue one.

Example (8.3). We take A with eigenvalues λ, $\bar{\lambda}$ and such that $|\lambda| \neq 1$ or such that $|\lambda| = 1$ but $\lambda \neq e^{k/l\, 2\pi i}$ with $k, l \leqq 5$. We may assume that A

is of the form

$$|\lambda|\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}.$$

For $2 \leqq i \leqq 4$ we can obtain a A-splitting of $\underline{V}_i$ as follows:

$\underline{V}_i'$ is the set of those (i-jets of) vectorfields which are, in polar coordinates of the form $\alpha_1 r^i \frac{\partial}{\partial r} + \alpha_2 r^{i-1} \frac{\partial}{\partial \varphi}$. More precisely $\underline{V}_2' = 0$, $\underline{V}_3'$ is generated by $r^3 \frac{\partial}{\partial r}$ and $r^2 \frac{\partial}{\partial \varphi}$ and $\underline{V}_4' = 0$ (the other cases give rise to vectorfields which are not differentiable, in ordinary coordinates).

$\underline{V}_i''$ is the set of (i-jets of) vectorfields of the form

$$g_1(\varphi)\, r^i \frac{\partial}{\partial r} + g_2(\varphi)\, r^{i-1} \frac{\partial}{\partial \varphi} \quad \text{with} \quad \int_0^{2\pi} g_1(\varphi) = \int_0^{2\pi} g_2(\varphi) = 0\,.$$

$g_1(\varphi)$ and $g_2(\varphi)$ have to be linear combinations of $\sin(j\cdot\varphi)$ and $\cos(j\cdot\varphi)$, $j \leqq 5$, because otherwise the vectorfield will not be differentiable in ordinary coordinates (not all these linear combinations are possible).

Proposition (8.4). *For a given diffeomorphism* $\Phi: (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *with* $(d\Phi)_0 = A$ *and a given* A*-splitting* $\underline{V}_i = \underline{V}_i' \oplus \underline{V}_i''$ *for* $2 \leqq i \leqq i_0$, *there is a coordinate transformation* $\varkappa: (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *such that:*

1. $(d\varkappa)_0 =$ *identity.*

2. *For each* $z \leqq i \leqq i_0$ *the i-jet of* $\Phi' = \varkappa \circ \Phi \circ \varkappa^{-1}$ *is related to its* $(i-1)$*-jet as follows: Let* $[\Phi']_{i-1}$ *be the polynomial map of degree* $\leqq i-1$ *which has the same* $(i-1)$*-jet. The i-jet of* Φ' *is related to its* $(i-1)$*-jet if there is an element* $\alpha \in \underline{V}_i'$ *such that* $\operatorname{Exp}\alpha \circ [\Phi']_{i-1}$ *has the same i-jet as* Φ'.

Proof. We use induction: Suppose we have a map $\varkappa$ such that 1 and 2 hold for $i < i_1 \leqq i_0$. Consider the i_1 jet of $\varkappa \circ \Phi \circ \varkappa^{-1}$. We now replace $\varkappa$ by $\operatorname{Exp}\alpha \circ \varkappa$ for some $\alpha \in \underline{V}_{i_1}''$. $\varkappa \circ \Phi \circ \varkappa^{-1}$ is then replaced by $\operatorname{Exp}(\alpha) \circ \varkappa \circ \Phi \circ \varkappa^{-1} \circ \operatorname{Exp}(-\alpha)$, according to remark (8.2) this equal to $\operatorname{Exp}(-A_{i_1}\alpha) \circ \operatorname{Exp}(\alpha) \circ \varkappa \circ \Phi \circ \varkappa^{-1} = \operatorname{Exp}(\alpha - A_{i_1}\alpha) \circ \varkappa \circ \Phi \circ \varkappa^{-1}$.

$A_{i_1} \mid \underline{V}_{i_1}''$ has no eigenvalue one, so for each $\beta \in \underline{V}_{i_1}''$ there is a unique $\alpha \in \underline{V}_{i_1}''$ such that if we replace $\varkappa$ by $\operatorname{Exp}\alpha \circ \varkappa$, $\varkappa \circ \Phi \circ \varkappa^{-1}$ is replaced by $\operatorname{Exp}\beta \circ \varkappa \circ \Phi \circ \varkappa^{-1}$. It now follows easily that there is a unique $\alpha \in \underline{V}_{i_1}''$ such that $\operatorname{Exp}\alpha \circ \varkappa$ satisfies condition 2 for $i \leqq i_1$. This proves the proposition.

Proof of Proposition (7.1). For μ near 0, $d\Phi_\mu$ is a linear map of the type we considered in example (8.3). So the splitting given there is a $d\Phi_\mu$-splitting of $\underline{V}_i$, $i = 2, 3, 4$, for μ near zero. We now apply Proposition (8.4) for each μ and obtain a coordinate transformation $\varkappa_\mu$ for each μ which brings Φ_μ in the required form. The induction step then becomes:

Given $\varkappa_\mu$, satisfying 1 and 2 for $i < i_1$ there is for each μ a unique $\alpha_\mu \in \underline{V}_{i_1}''$ such that $\operatorname{Exp}\alpha_\mu \circ \varkappa_\mu$ satisfies 1 and 2 for $i \leqq i_1$. α_μ depends then C^r on μ if the i_1-jet of Φ depends C^r on μ; this gives the loss of differentiability in the μ direction.

§ 9. Some Examples

In this section we show how a small perturbation of a quasi-periodic flow on a torus gives flows with strange attractors (Proposition (9.2)) and, more generally, flows which are not Morse-Smale (Proposition (9.1)).

Proposition (9.1). *Let ω be a constant vector field on $T^k = (\mathbb{R}/\mathbb{Z})^k$, $k \geqq 3$. In every C^{k-1}-small neighbourhood of ω there exists an open set of vector fields which are not Morse-Smale.*

We consider the case $k = 3$. We let $\omega = (\omega_1, \omega_2, \omega_3)$ and we may suppose $0 \leqq \omega_1 \leqq \omega_2 \leqq \omega_3$. Given $\varepsilon > 0$ we may choose a constant vector field ω' such that

$$\|\omega' - \omega\|_2 = \|\omega' - \omega\|_0 < \varepsilon/2,$$

$$\omega_3' > 0, \qquad 0 < \frac{\omega_1'}{\omega_3'} = \frac{p_1}{q_1} < 1, \qquad 0 < \frac{\omega_2'}{\omega_3'} = \frac{p_2}{q_2} < 1,$$

where p_1, p_2, q_1, q_2 are integers, and $p_1 q_2$ and $p_2 q_1$ have no common divisor. We shall also need that q_1, q_2 are sufficiently large and satisfy

$$\tfrac{1}{2} < q_1/q_2 < 2.$$

All these properties can be satisfied with $q_1 = 2^{m_1}$, $q_2 = 3^{m_2}$.

Let $I = \{x \in \mathbb{R} : 0 \leqq x \leqq 1\}$ and define $g, h : I^3 \to T^3$ by

$$g(x_1, x_2, x_3) = (x_1 (\operatorname{mod} 1), x_2 (\operatorname{mod} 1), x_3 (\operatorname{mod} 1))$$

$$h(x_1, x_2, x_3) = (q_1^{-1} x_1 + p_1 q_2 x_3 (\operatorname{mod} 1), q_2^{-1} x_2 + p_2 q_1 x_3 (\operatorname{mod} 1), q_1 q_2 x_3 (\operatorname{mod} 1)).$$

We have $gI^3 = hI^3 = T^3$ and g (resp. h) has a unique inverse on points gx (resp. hx) with $x \in \mathring{I}^3$.

We consider the map f of a disc into itself (see [11] Section I.5, Fig. 7) used by Smale to define the horseshoe diffeomorphism. Imbedding Δ in T^2:

$$\Delta \subset \{(x_1, x_2) : \tfrac{1}{3} < x_1 < \tfrac{2}{3}, \tfrac{1}{3} < x_2 < \tfrac{2}{3}\} \subset T^2$$

we can arrange that f appears as Poincaré map in $T^3 = T^2 \times T^1$. More precisely, it is easy to define a vector field $X = (\tilde{X}, 1)$ on $T^2 \times T^1$ such that if $\xi \in \Delta$, we have

$$(f(\xi), 0) = \mathscr{D}_{X,1}(\xi, 0)$$

where $\mathscr{D}_{X,1}$ is the time one integral of X (see Fig. 7). Finally we choose the restriction of X to a neighbourhood of $g(\partial I^2 \times I)$ to be $(0, 1)$ (i.e. $\tilde{X} = 0$).

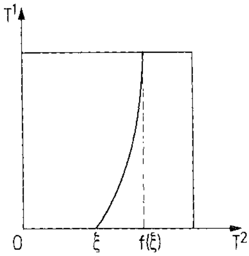

Fig. 7

If $x \in g\mathring{I}^3$, then $\Phi x = h \circ g^{-1}$ is uniquely defined and the tangent mapping to Φ applied to X gives a vector field Y:

$$Y(\Phi(x)) = [d\Phi(x)]\, X(x)$$

where

$$[d\Phi(x)] = \begin{pmatrix} q_1^{-1} & & p_1 q_2 \\ & q_2^{-1} & p_2 q_1 \\ & & q_1 q_2 \end{pmatrix}.$$

Y has a unique smooth extension to T^3, again called Y. Let now $Z = (q_1 q_2)^{-1} \omega'_3 Y$. We want to estimate

$$\|Z - \omega'\|_r = \sup_{\varrho;\, |\varrho| \leq r} N^\varrho$$

where

$$N^\varrho = \sup_{y \in T^3} \sup_{i=1,2,3} |D^\varrho Z_i(y) - D^\varrho \omega'_i| \tag{*}$$

and D^ϱ denotes a partial differentiation of order $|\varrho|$. Notice that it suffices to take the first supremum in (*) over $y \in h\mathring{I}^3$, i.e. $y = \Phi x$ where $x \in g\mathring{I}^3$. We have

$$\frac{\partial}{\partial y} = \begin{pmatrix} q_1 & & \\ & q_2 & \\ -p_1 & -p_2 & (q_1 q_2)^{-1} \end{pmatrix} \frac{\partial}{\partial x}$$

so that

$$\sup_i \left| \frac{\partial}{\partial y_i} \right| < (q_1 + q_2) \sup_i \left| \frac{\partial}{\partial x_i} \right|.$$

Notice also that

$$Z_i(y) - \omega'_i = (q_1 q_2)^{-1} \omega'_3 \begin{pmatrix} q_1^{-1} X_1 + p_1 q_2 \\ q_2^{-1} X_2 + p_2 q_1 \\ q_1 q_2 \end{pmatrix} - \omega'_3 \begin{pmatrix} p_1 q_1^{-1} \\ p_2 q_2^{-1} \\ 1 \end{pmatrix}$$

$$= (q_1 q_2)^{-1} \omega'_3 \begin{pmatrix} q_1^{-1} X_1 \\ q_2^{-1} X_2 \\ 0 \end{pmatrix}.$$

Therefore

$$N^\varrho \leqq (q_1 q_2)^{-1} \omega'_3 (q_1 + q_2)^{|\varrho|} \Big(\sup_{i=1,2} q_i^{-1} \Big) \sup_{i=1,2} \|X_i\|_{|\varrho|}$$

$$\|Z - \omega'\|_r \leqq (q_1 q_2)^{-2} (q_1 + q_2)^{r+1} [\omega'_3 \|\tilde{X}\|_r] .$$

If we have chosen q_1, q_2 sufficiently large, we have

$$\|Z - \omega'\|_2 < \varepsilon/2$$

and therefore

$$\|Z - \omega\|_2 < \varepsilon .$$

Consider the Poincaré map $P: T^2 \to T^2$ defined by the vector field Z on $T^3 = T^2 \times T^1$. By construction the non wandering set of P contains a Cantor set, and the same is true if Z is replaced by a sufficiently close vector field Z'. This concludes the proof for $k = 3$.

In the general case $k \geqq 3$ we approximate again ω by ω' rational and let

$$0 < \frac{\omega'_i}{\omega'_k} = \frac{p_i}{q_i} < 1 \quad \text{for} \quad i = 1, \ldots, k-1 .$$

We assume that the integers $p_1 \prod_{i \neq 1} q_i, \ldots, p_{k-1} \prod_{i \neq k-i} q_i$ have no common divisor. Furthermore $q_1, \ldots, q_{k-1}$ are chosen sufficiently large and such that

$$\big(\max_i q_i\big) / \big(\min_i q_i\big) < C$$

where C is a constant depending on k only.

The rest of the proof goes as for $k = 2$, with the horseshoe diffeomorphism replaced by a suitable $k-1$-diffeomorphism. In particular, using the diffeomorphism of Fig. 2 (end of § 2) we obtain the following result.

Proposition (9.2). *Let ω be a constant vector field on T^k, $k \geqq 4$. In every C^{k-1}-small neighbourhood of ω there exists an open set of vector fields with a strange attractor.*

Appendix

Bifurcation of Stationary Solutions of Hydrodynamical Equations

In this appendix we present a bifurcation theorem for fixed points of a non linear map in a Banach space. Our result is of a known type[23], but has the special interest that the fixed points are shown to depend differentiably on the bifurcation parameter. The theorem may be used to study the bifurcation of stationary solutions in the Taylor and Bénard[24] problems for instance. By reference to Brunovský (cf. § 3b) we see that the bifurcation discussed below is *nongeneric*. The bifurcation of stationary solutions in the Taylor and Bénard problems is indeed nongeneric, due to the presence of an invariance group.

Theorem. *Let H be a Banach space with C^k norm, $1 \leqq k < \infty$, and $\Phi_\mu : H \to H$ a differentiable map such that $\Phi_\mu(0) = 0$ and $(x, \mu) \to \Phi_\mu x$ is C^k from $H \times \mathbb{R}$ to H. Let*

$$L_\mu = [d\Phi_\mu]_0, \qquad N_\mu = \Phi_\mu - L_\mu . \tag{1}$$

We assume that L_μ has a real simple isolated eigenvalue $\lambda(\mu)$ depending continuously on μ such that $\lambda(0) = 1$ and $(d\lambda/d\mu)(0) > 0$; we assume that the rest of the spectrum is in $\{\mathfrak{z} \in \mathbb{C} : |\mathfrak{z}| < 1\}$.

(a) *There is a one parameter family (a C^{k-1} curve l) of fixed points of $\Phi : (x, \mu) \to (\Phi_\mu x, \mu)$ near $(0, 0) \in H \times \mathbb{R}$. These points and the points $(0, \mu)$ are the only fixed points of Φ in some neighbourhood of* $(0, 0)$.

(b) *Let $\mathfrak{z}$ (resp. $\mathfrak{z}^*$) be an eigenvector of L_0 (resp. its adjoint L_0^* in the dual H^* of H) to the eigenvalue* 1, *such that $(\mathfrak{z}^*, \mathfrak{z}) = 1$. Suppose that for all $\alpha \in \mathbb{R}$*

$$(\mathfrak{z}^*, N_0 \alpha \mathfrak{z}) = 0 . \tag{2}$$

Then the curve l of (a) *is tangent to* $(\mathfrak{z}, 0)$ *at* $(0, 0)$.

From the center-manifold theorem of Hirsch, Pugh, and Shub[25] it follows that there is a 2-dimensional C^k-manifold V^c, tangent to the vectors $(\mathfrak{z}, 0)$ and $(0, 1)$ at $(0, 0) \in H \times \mathbb{R}$ and locally invariant under Φ. Furthermore there is a neighbourhood U of $(0, 0)$ such that every fixed point of Φ in U is contained in $V^c \cap U$[26].

We choose coordinates (α, μ) on V^c so that

$$\Phi(\alpha, \mu) = (f(\alpha, \mu), \mu)$$

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[23] See for instance [16] and [14].

[24] The bifurcation of the Taylor problem has been studied by Velte [18] and Yudovich [19]. For the Bénard problem see Rabinowitz [17], Fife and Joseph [15].

[25] Usually the center manifold theorem is only formulated for diffeomorphisms; C. C. Pugh pointed out to us that his methods in [5], giving the center manifold, also work for differentiable maps which are not diffeomorphisms.

[26] See § 5.

with $f(0, \mu) = 0$ and $\frac{\partial f}{\partial \alpha}(0, \mu) = \lambda(\mu)$. The fixed points of Φ in V^c are given by $\alpha = f(\alpha, \mu)$, they consist of points $(0, \mu)$ and of solutions of

$$g(\alpha, \mu) = 0$$

where, by the division theorem, $g(\alpha, \mu) = \frac{f(\alpha, \mu)}{\alpha} - 1$ is C^{k-1}. Since $\frac{\partial g}{\partial \mu}(0, 0) = \frac{d\lambda}{d\mu}(0) > 0$, the implicit function theorem gives (a).

Let $(x, 0) \in V^c$. We may write

$$x = \alpha \mathfrak{z} + Z \tag{3}$$

where $(\mathfrak{z}^*, Z) = 0$, $Z = 0(\alpha^2)$. Since $(\mathfrak{z}^*, \mathfrak{z}) = 1$ we have

$$\begin{aligned} f(\alpha, 0) &= (\mathfrak{z}^*, \Phi_0(\alpha \mathfrak{z} + Z)) \\ &= \alpha + (\mathfrak{z}^*, L_0 Z + N_0(\alpha \mathfrak{z} + Z)) \\ &= \alpha + (\mathfrak{z}^*, L_0 Z + N_0 \alpha \mathfrak{z}) + 0(\alpha^3) . \end{aligned} \tag{4}$$

Notice that

$$(\mathfrak{z}^*, L_0 Z) = (L_0^* \mathfrak{z}^*, Z) = (\mathfrak{z}^*, Z) = 0 .$$

We assume also that (2) holds:

$$(\mathfrak{z}^*, N_0 \alpha \mathfrak{z}) = 0 .$$

Then

$$\begin{aligned} f(\alpha, 0) &= \alpha + 0(\alpha^3) , \\ f(\alpha, \mu) &= \alpha(\lambda(\mu) + 0^2) , \end{aligned} \tag{5}$$

where 0^2 represents terms of order 2 and higher in α and μ. The curve l of fixed points of Φ introduced in (a) is given by

$$\lambda(\mu) - 1 + 0^2 = 0 \tag{6}$$

and (b) follows from $\lambda(0) = 0$, $\frac{d\lambda}{d\mu}(0) > 0$.

Remark 1. From (2) and the local invariance of V^c we have

$$\Phi_0(\alpha \mathfrak{z} + Z) = \alpha \mathfrak{z} + Z + 0(\alpha^3)$$

hence

$$Z = L_0 Z + N_0 \alpha \mathfrak{z} + 0(\alpha^3) ,$$

$$Z = (1 - L_0)^{-1} N_0 \alpha \mathfrak{z} + 0(\alpha^3) ,$$

and (5) can be replaced by the more precise

$$f(\alpha, 0) = \alpha + (\mathfrak{z}^*, N_0[\alpha \mathfrak{z} + (1 - L_0)^{-1} N \alpha \mathfrak{z}]) + 0(\alpha^4) \tag{7}$$

from which one can compute the coefficient A of α^3 in $f(\alpha, 0)$. Then (6) is, up to higher order terms

$$\left[\frac{d\lambda}{d\mu}(0)\right]\mu + A\alpha^2 = 0 .$$

Depending on whether $A < 0$ or $A > 0$, this curve lies in the region $\mu > 0$ or $\mu < 0$, and consists of attracting or non-attracting fixed points. This is seen by discussing the sign of $f(\alpha, \mu) - \alpha$ (see Fig. 8b, c); the fixed points $(0, \mu)$ are always attracting for $\mu < 0$, non attracting for $\mu > 0$.

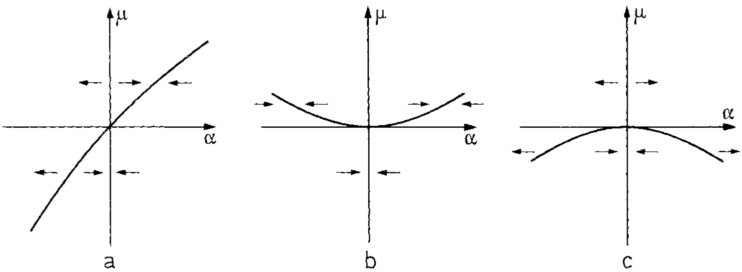

Fig. 8

Remark 2. If the curve l of fixed points is not tangent to $(\mathfrak{z}, 0)$ at $(0, 0)$, then the points of l are attracting for $\mu > 0$, non-attracting for $\mu < 0$ (see Fig. 8a).

Remark 3. If it is assumed that L_μ has the real simple isolated eigenvalue $\lambda(\mu)$ as in the theorem and that the rest of the spectrum lies in $\{\mathfrak{z} \in \mathbb{C} : |\mathfrak{z}| \neq 1\}$ (rather than $\{\mathfrak{z} \in \mathbb{C} : |\mathfrak{z}| < 1\}$, the theorem continues to hold but the results on the attractive character of fixed points are lost.

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D. Ruelle
The Institute for Advanced Study
Princeton, New Jersey 08540, USA

F. Takens
Universiteit van Amsterdam
Roetersstr. 15
Amsterdam, The Netherlands