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ON THE NAVIER-STOKES EQUATIONS WITH CONSTANT TOTAL TEMPERATURE

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ABSTRACT

For various applications in fluid dynamics, one can assume that the total temperature is constant. Therefore, the energy equation can be replaced by an algebraic relation. The resulting set of equations in the inviscid case is analyzed in this paper. It is shown that the system is strictly hyperbolic and well-posed for the initial value problem. Boundary conditions are described such that the linearized system is well posed. The Hopscotch method is investigated and numerical results are presented.

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1. INTRODUCTION

The Navier-Stokes equations in two space dimensions contain four differential equations: the momentum equations, the continuity equation and the energy equation. For certain applications, the energy equation can be substituted by the assumption that the total temperature is constant, without much loss of accuracy, see [8]. The resulting system for two space dimensions is then in non-dimensionalized form:

$$u_t + uu_x + vu_y + \frac{1}{\rho}p_x = \frac{1}{3\rho Re}\,[4(\mu u_x)_x - 2(\mu v_y)_x + 3(\mu(v_x+u_y))_y]$$

$$v_t + uv_x + vv_y + \frac{1}{\rho}p_y = \frac{1}{3\rho Re}[4(\mu v_y)_y - 2(\mu u_x)_y + 3(\mu(v_x+u_y))_x]$$

$$(1.1)\quad \rho_t + \rho u_x + u\rho_x + \rho v_y + v\rho_y = 0$$

$$1 = T + u^2 + v^2 \quad \text{(total temperature constant)}$$

$$p = R\rho T$$

$$\mu = \frac{c_1 T^{3/2}}{T+C_2}$$

R, C_1, C_2 and the Reynolds number Re are given constants.

In order to obtain good numerical solutions to the initial-boundary-value problem for the system when the Reynolds number is large, we must require that the inviscid equations ($Re \to \infty$) are well posed. In this paper an analysis of the linearized version of the system is presented. The characteristic speeds are no longer the same as for the complete inviscid Navier-Stokes equations, but the system is well posed for the pure initial value problem. The analysis of the mixed initial-boundary-value problem shows that great care must be taken to obtain a well posed problem. If at a subsonic inflow boundary, v and one of the variables u and ρ are specified, the equations are not well posed near the transonic point; v and a combination of u, v, ρ corresponding to an ingoing characteristic must in such a case be specified.

Several numerical experiments have been done for this system by Rudy et al. [8]. One method used by them was the Hopscotch scheme, see [3], [4]. The application of this scheme to the viscous terms is simplified if the function values at the middle point in the approximation of u_{xx}, v_{xx} are taken at time level n in both sweeps. It is shown in section 4 that this simplification introduces a stability limit on Δt. However, for high Reynolds numbers, it is more dissipative than the original method.

2. THE PURE INITIAL-VALUE PROBLEM

In this section we will first show that the linearized inviscid equations are strictly hyperbolic. After elimination of the variables p and T, the linearized system can be written in the form

$$w_t + Aw_x + Bw_y = 0,$$

where

$$w = (\hat{U},\hat{V},\hat{R})^T ,$$

$$A = \begin{bmatrix} (1-2R)u & -2rv & c^2/\rho \\ 0 & u & 0 \\ \rho & 0 & u \end{bmatrix}$$

$$B = \begin{bmatrix} v & 0 & 0 \\ -2Rv & (1-2R)v & c^2/\rho \\ 0 & \rho & v \end{bmatrix}$$

$$c^2 = R(1-u^2-v^2),$$

u, v, ρ now considered to be known functions. To prove strict hyperbolicity we must show that the eigenvalues of $A\omega_1 + A\omega_2$ are real and distinct for all real ω_1, ω_2 with $\omega_1^2 + \omega_2^2 = 1$. An easy calculation shows that these eigenvalues are given by

$$\lambda_1 = u\omega_1 + v\omega_2$$

$$\lambda_{2,3} = (1-R)(u\omega_1+v\omega_2)\pm\sqrt{R^2(u\omega_1+v\omega_2)^2+c^2(\omega_1^2+\omega_2^2)} \;.$$

All eigenvalues are obviously real, and since $R \neq 0$, $c \neq 0$, they are also distinct for $\omega_1^2 + \omega_2^2 = 1$.

The complete inviscid Navier-Stokes equations are symmetric hyperbolic, i.e., the corresponding matrices A and B can be symmetrized by the same similarity transformation, see [9]. This is not the case for the system considered here. With the notation, $a_\pm = Ru\pm\sqrt{R^2u^2+c^2}$, the eigenvectors of A are the column vectors of

$$S = \begin{bmatrix} a_+ & a_- & 0 \\ 0 & 0 & c^2/R \\ -\rho & -\rho & 2v\rho \end{bmatrix}$$

If A and B can be symmetrized by the same similarity transformation, then we can diagonalize A by an orthogonal transformation, and therefore B stays symmetric.

Therefore $S^{-1}AS$ is diagonal, and

$$S^{-1}BS = vI - \frac{1}{a_+ - Ru} \begin{bmatrix} Rva_+ & -\frac{Rva_-^3}{c^2} & -\frac{c^2 a_-}{2R} \\ \frac{Rva_+^3}{c^2} & -Rva_- & \frac{c^2 a_+}{2R} \\ \frac{Ra_+^2(a_+ - Ru)}{c^2} & \frac{Ra_-^2(a_+ - Ru)}{c^2} & 0 \end{bmatrix}$$

which is not symmetric. The eigenvectors in S can be permuted, but that does not effect the symmetry of $S^{-1}BS$. Furthermore, each eigenvector can be scaled by different factors. This corresponds to a similarity transformation of $S^{-1}BS$ by a diagonal matrix. It is easily shown that it is impossible to symmetrize $S^{-1}BS$ by such a transformation, and therefore A and B cannot be symmetrized simultaneously.

The variable c defined by $c^2 = RT$ would for the non-dimensionalized system correspond to the local speed of sound. However, the eigenvalues of A are u, $(1-R)u \pm c\sqrt{1+Ru^2/T}$. These are not the same as the eigenvalues of the corresponding A for the complete Navier-Stokes equations, since the latter are u, $u \pm c$. We note that these are obtained if terms of order R and smaller are neglected. (R is 1/7 for air.) Looking at the velocity component in the x-direction, a supersonic state is most naturally defined as a state for which all the eigenvalues of A have the same sign. Therefore, we will define a state as subsonic if the condition

$$(2.1a) \qquad (1-R)|u| < \sqrt{R^2u^2+c^2}$$

is fulfilled, and supersonic if the condition

$$(2.1b) \qquad (1-R)|u| > \sqrt{R^2u^2+c^2}$$

is fulfilled. (2.1a) can also be written $(1-2R)u^2 < c^2$, or equivalently

$$(2.2) \qquad \frac{1-R}{R}\, u^2+v^2 < 1.$$

3. WELL POSEDNESS OF THE MIXED INITIAL-BOUNDARY VALUE PROBLEM

In this section we will investigate the effect of the boundary conditions, and we assume the problem to be defined on the domain $0 \le x < \infty$, $-\infty < y < \infty$, $0 \le t$. We begin with a brief discussion of the one-dimensional case.

Assume that the system $w_t + Aw_x = 0$ is transformed to diagonal form

$$\phi_t + \Lambda\phi_x = 0, \ \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \ .$$

Then it is well known that the problem is well posed if the boundary conditions can be written on the form

$$\phi^I(0,t) = L\phi^{II}(0,t) + g(t) \ ,$$

where ϕ^I contains those variables $\phi^{(i)}$ which correspond to positive λ_i, and ϕ^{II} contains the remaining ones, see e.g. [6]. Using the same notation as in Section 2, we have in our case $\lambda_1 = u$, $\lambda_2 = u - a_-$, $\lambda_3 = u - a_+$, and

$$\phi = \begin{bmatrix} \hat{V} \\ -a_-\rho\hat{U} - 2Rv\rho\hat{V} + c^2\hat{R} \\ a_+\rho\hat{U} + 2RV\rho\hat{V} - c^2\hat{R} \end{bmatrix}$$

$$w = \begin{bmatrix} \hat{U} \\ \hat{V} \\ \hat{R} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2\rho(a_+ - Ru)} (\phi^{(2)} + \phi^{(3)}) \\ \phi^{(1)} \\ \dfrac{2v\rho}{c^2}\phi^{(1)} - \dfrac{1}{2(a_+ - Ru)}(\dfrac{1}{a_-}\phi^{(2)} + \dfrac{1}{a_+}\phi^{(3)}) \end{bmatrix}$$

For supersonic flow (defined by (2.1b)), all three variables $\hat{U}$, $\hat{V}$, $\hat{R}$ must be specified if $u > 0$ (inflow), and no boundary condition should be given if $u < 0$ (outflow). For subsonic inflow, the boundary conditions must be such that $\phi^{(1)}$, $\phi^{(2)}$ can be expressed in terms of $\phi^{(3)}$ and an inhomogeneous term. If we want to specify two of the physical variables, we see that $\hat{V}$ must be one of them. Either one of $\hat{U}$ and $\hat{R}$ can then be chosen as the other specified variable, since for both of them $\phi^{(2)}$ occurs with a nonzero coefficient. For the same reason, either one of $\hat{U}$ and $\hat{R}$ can be specified for subsonic outflow.

The two-dimensional problem is much more difficult to analyze. The energy method does not work, since the system cannot be symmetrized, and therefore we must use the theory by Kreiss [5]. Since part of the calculations are technically complicated, we give here a summary of the wellposedness proofs; the details are given in the Appendix.

The wellposedness is determined by the behavior of the solutions to the system

(3.2) $$A\hat{w}_x + (sI+i\omega B)\,\hat{w} = 0,$$

which is obtained by a Laplace transformation with respect to t and a Fourier transformation with respect to y. The solutions to (3.2) consist of components of the type $e^{\kappa_i x}\psi_i$ where κ_i is a solution of $\mathrm{Det}(C) = 0$, $C(\kappa) \equiv A\kappa + sI+i\omega B$ (or of the type $x^{\beta_i}e^{\kappa_i x}\tilde{\psi}_1$ for multiple roots κ_i).

With the notation $\alpha = s+i\omega v$, the κ_i's are defined by:

$$\kappa_1 = -\alpha/u$$

(3.3) $$\nu\kappa_{2,3}^2 + 2u[(1-R)\alpha-Rvi\omega]\kappa_{2,3} + \alpha^2-2\alpha Rvi\omega+c^2\omega^2=0,$$

where $\nu = (1-2R)u^2-c^2$. The corresponding vectors ψ_i are given by

$$\psi_1 = \begin{bmatrix} ui\omega \\ \alpha \\ 2\rho R(u^2 i\omega+v\alpha)/c^2 \end{bmatrix}, \quad \psi_{2,3} = \begin{bmatrix} \kappa_{2,3}(\alpha+\kappa_{2,3}u) \\ i\omega(\alpha+\kappa_{2,3}u) \\ \rho(\omega^2-\kappa_{2,3}^2) \end{bmatrix}$$

Note that $\alpha = \pm u\omega$ if and only if $\kappa_2 = \mp\omega$ and therefore $\kappa_1 = \kappa_2$. In this case however, ψ_2 is the nullvector, and we have only two linearly independent vectors ψ_1, ψ_3. If $\kappa_2 = \kappa_3$, it is also clear that there are only two linearly independent vectors ψ_1, ψ_2.

Let M be the class of vector functions $\hat{w}$ satisfying (3.2) and with $\hat{w}\varepsilon L_2(0,\infty)$ for Re $s > 0$. The problem is well posed if there is no nontrivial $\hat{w}\varepsilon M$ satisfying the homogeneous boundary conditions for any s with Re $s \geq 0$. (If there is such a nontrivial solution for a purely imaginary $s = s_o$, then s_o is called a generalized eigenvalue.) We will now investigate this condition for the different cases.

Supersonic Inflow

In this case there are three linearly independent solutions belonging to $L_2(0,\infty)$, and we begin with

Case 1: $\kappa_1 \neq \kappa_2 \neq \kappa_3 \neq \kappa_1$

The general solution is

$$\hat{w}(x) = \xi e^{\kappa_1 x} \psi_1 + \eta e^{\kappa_2 x} \psi_2 + \zeta e^{\kappa_3 x} \psi_3 ,$$

where ξ, η, ζ are scalars. The condition for having nontrivial solutions satisfying $\hat{w}(0) = 0$ is

$$\mathrm{Det}(\psi_1, \psi_2, \psi_3) = 0.$$

A lengthy calculation (see the Appendix) shows that

$$(3.5) \qquad \mathrm{Det}\ (\psi_1, \psi_2, \psi_3) = \nu(\kappa_2 - \kappa_3)\ c^2 (u^2\omega^2 - \alpha^2)^2,$$

and it never vanishes because $\kappa_2 \neq \kappa_3$, and moreover, $\alpha = \pm\ u\omega$ only if $\kappa_1 = \kappa_2$.

Case 2: $\kappa_1 = \kappa_2 \neq \kappa_3$

The general solution is

$$\hat{w}(x) = (\tilde{\psi}_1 + \xi x \psi_1) e^{\kappa_1 x} + \eta \psi_3 e^{\kappa_3 x} ,$$

where

$$(3.6) \qquad C(\kappa_1)\tilde{\psi}_1 = -\ A\xi\psi_1 \quad .$$

The condition $\kappa_1=\kappa_2$ implies $\alpha = \pm\, u\omega$ and $\kappa_1=\pm\omega$. $\hat{w}(0) = 0$ implies $\tilde{\psi}_1=-\eta\psi_3$. When substituting this into (3.6), we obtain from the first and the last equation

$$\text{(3.7a)} \qquad \eta(\kappa_3\pm\omega)^2=\rho i\xi$$

$$\text{(3.7b)} \qquad \eta(\kappa_3\pm\omega)^2 = [-\rho i-2R\rho u(ui+v)/c^2]\xi,$$

which contradict each other if $\xi\neq 0$, $\eta\neq 0$.

Case 3: $\kappa_1\neq\kappa_2=\kappa_3$.

The general solution is

$$\hat{w}(x) =\xi\psi_1 e^{\kappa_1 x}+(\tilde{\psi}_2+\eta x\psi_2)e^{\kappa_2 x} ,$$

where

$$\text{(3.8)} \qquad C(\kappa_2)\psi_2=-A\eta\psi_2 .$$

In the same way as above, we obtain two equations corresponding to (3.7):

$$\text{(3.9a)} \qquad i\omega u\rho\eta=\alpha\xi$$

$$\text{(3.9b)} \qquad \alpha\rho\eta=-ui\omega\xi .$$

These lead to the case $\alpha^2=u^2\omega^2$, which corresponds to $\kappa_1=\kappa_2$, and this is a contradiction.

Case 4: $\kappa_1=\kappa_2=\kappa_3$

In this case we must have $\alpha=\pm u\omega$, $\kappa=\mp\omega$, and from (3.3) we get $v\kappa=-u[(1-R)\alpha-Rvi\omega]$. The imaginary part of the equation yields $v=0$ and the real part yields $u=0$.

This completes the proof of wellposedness for supersonic inflow.

Supersonic Outflow

This case is trivial, since all κ's have positive real part for Re $s > 0$, and no boundary conditions should be given.

Subsonic Inflow

In this case there are only two linearly independent solutions belonging to $L_2(0,\infty)$. The general solution is

$$(3.10) \qquad \hat{w} = \xi e^{\kappa_1 x}\psi_1+\eta e^{\kappa_2 x}\psi_2 , \qquad \text{if } \kappa_1\neq\kappa_2$$

and

$$(3.11) \qquad \hat{w} = (\tilde{\psi}_1+\xi x\psi_1)e^{\mp\omega x} \qquad \text{if } \kappa_1=\kappa_2$$

where $\mathfrak{V}$ is defined by (3.6).

With the assumption that $\hat{U}$ and $\hat{V}$ are specified, we get immediately the condition for a nontrivial solution

$$(3.12)\qquad (u\omega^2+\alpha\kappa_2)(\alpha+\kappa_2 u) = 0 \qquad\qquad \text{if } \kappa_1\neq\kappa_2 .$$

Since by assumption $\alpha+\kappa_2 u\neq 0$, we obtain the equivalent condition $\alpha=-u\omega^2/\kappa_2$. With this α-value inserted into (3.3), we obtain an equation for $\hat{\kappa}=\kappa_2/i\omega$:

$$(3.13)\qquad (\hat{\kappa}^2+1)\ (\nu\hat{\kappa}^2-2Ru\nu\hat{\kappa}+u^2) = 0 .$$

Since $\hat{\kappa}=\pm i$, $\alpha=\pm u\omega$ corresponds to $\kappa_1=\kappa_2$, the critical $\hat{\kappa}$-values are

$$(3.14)\qquad \hat{\kappa} = u(R\nu\pm\sqrt{R^2\nu^2-\nu})/\nu$$

The subsonic condition is equivalent to $\nu < 0$; therefore, both $\hat{\kappa}$-values are real. The corresponding α-values

$$(3.15)\qquad \alpha=\hat{\alpha}i\omega \equiv \frac{\nu}{R\nu\pm\sqrt{R^2\nu^2-\nu}}\, i\omega$$

are purely imaginary. This means that the only nontrivial solution corresponds to $s_0 = (\alpha-v)i\omega$. With $\alpha=\hat{\alpha}i\omega+\delta$, $\delta>0$, we want to see if the corresponding κ satisfies $\mathrm{Re}\kappa<0$. In that case we have a generalized eigenvalue $s=s_0$, and the problem is not well posed in the sense of Kreiss [5]. In the Appendix it is shown that Re $\kappa < 0$ if

$$(3.16) \qquad (1-R)(u^2+v^2) > R.$$

Unfortunately, this condition can be fulfilled even if the subsonic condition (2.1a) is satisfied, provided that $2R<1$. Therefore, we have proved that the problem is not well posed for example with a transonic boundary, because there is always some part near the transonic point where (3.16) is fulfilled if $v\neq 0$.

It remains to treat the case with double roots $\kappa_1=\kappa_2$, when the solution has the form (3.11). A straightforward calculation (see the Appendix) shows that the condition for a nontrivial solution leads to the trivial case $u=v=0$.

Another natural choice of boundary conditions would be the specificiation of $\hat{V}$ and $\hat{R}$. Assuming $\kappa_1\neq\kappa_2$, the condition for a nontrivial solution is

$$(3.17) \qquad c^2\alpha(\omega^2-\kappa_2^2)-2Ri\omega(u^2 i\omega+v\alpha)(\alpha+u\kappa_2) = 0 .$$

Solving this equation for $\omega^2-\kappa_2^2$ and inserting that into (3.3) gives (since $\alpha+u\kappa_2\neq 0$)

$$(3.18) \qquad (1-2R)u\alpha\kappa_2 = 2Ru^2\omega^2 - \alpha^2 .$$

With the expression for κ_2 defined by this equation inserted into (3.17), we obtain

$$(3.19) \qquad (\alpha^2-u^2\omega^2)[c^2\alpha^2-4R^2u^2vi\omega(1-2R)\alpha-4R^2u^2\omega^2(c^2-(1-2R)u^2)]=0.$$

$\alpha=\pm u\omega$ corresponds to $\kappa_1=\kappa_2$, so the critical α-values are given by the zeros of the second factor:

$$(3.20) \qquad \alpha=2Ru\omega\left(Ruv(1-2R)i\pm\sqrt{-vc^2-(1-2R)^2R^2u^2v^2}\right)/c^2 .$$

From (3.18) it is easily seen that $Re\kappa_2 > 0$ if $Re\alpha>0$ and $|\alpha|^2<2Ru^2\omega^2$. But this inequality follows immediately from (3.19), where the magnitude of the constant term in the second factor equals $c^2|\alpha|^2$. Hence, the critical α-values (giving $Re\kappa_2<0$) must be imaginary, and this is the case when

$$\frac{1-R}{R}u^2+\left(1+\frac{u^2(1-2R)^2R}{c^2}\right)v^2 > 1.$$

A perturbation calculation shows that there is a generalized eigenvalue in a neighborhood of the transonic point. The above analysis shows that we must resort to the specification of the characteristic variables $\phi^{(1)}$ and $\phi^{(2)}$ (see the definition (3.1).) Following the same lines as above, we arrive for $\kappa_1 \neq \kappa_2$ at the final equation (stated in terms of α this time) for a nontrivial solution:

$$(3.21) \qquad (\alpha^2-u^2\omega^2)(2R\nu i\omega(u-a_+)a_+\alpha-\nu a_+^2\omega^2) = 0$$

$\alpha^2=\pm u\omega$ is ruled out by the assumption $\kappa_1 \neq \kappa_2$, and the remaining critical α-value is, therefore,

$$(3.22) \qquad \alpha = -\frac{\nu a_+}{2R\nu(u-a_+)} i\omega = \hat{\alpha} i\omega \ .$$

A perturbation calculation shows that the condition Re $\kappa<0$ for $\alpha=\hat{\alpha}i\omega+\delta, \delta>0$, is equivalent to the supersonic condition.

The multiple root case leads to no new restriction on u, v, and, therefore, we have a well posed problem.

Subsonic Outflow

For this case only one variable should be specified and the solution to (3.2) always has the form $\hat{v} = \xi e^{\kappa_3 x}\psi_3$. Since $\alpha+\kappa_3 u \neq 0$, the only possibility for a nontrivial solution with $\hat{U}$ specified is $\kappa_3 = 0$. This corresponds to the α-value $\alpha=(R\nu\pm\sqrt{c^2+R^2v^2})i\omega=\hat{\alpha}i\omega$. Substituting $\alpha=\hat{\alpha}i\omega+\delta$ into (3.3), we obtain after dropping second order terms in δ and κ

$$\kappa = \frac{\mp\sqrt{R^2v^2+c^2}}{\pm\sqrt{R^2v^2+c^2} - R(Rv\pm\sqrt{R^2v^2+c^2})} \frac{\delta}{u}, \quad u<0 .$$

Since R^2v is always less than $(1-R)\sqrt{R^2v^2+c^2}$ if $2R < 1$, we see that $\kappa>0$ if $\delta>0$, and wellposedness is proved.

For $\hat{R}$ specified, the condition for a nontrivial solution is $\kappa_3=\pm\omega$. It is enough to investigate the case $\kappa_3=-\omega, \omega>0$. From (3.3) we obtain the corresponding α-values $\alpha_1=u\omega, \alpha_2=((1-2R)u+2Rvi)\omega$ which both have negative real parts. This proves the wellposedness.

Let us finally mention that for a subsonic outflow boundary it is often difficult to specify accurate values for the variable u or ρ. In that case one could think of using numerical boundary conditions that approximate the condition $\partial w^r/\partial x^r=0$, $r>0$, i.e., vanishing derivatives of some order is assumed for all the variables. However, it is easily proved that this leads to a non wellposed problem for all $r > 0$. Since the derivative boundary condition applies also to the ingoing characteristic variable, it is sufficient to study the scalar equation $\phi_t+\lambda\phi_x=0$, $\lambda>0$. After transformation the general solution is $\hat{\phi}=\exp(-s/\lambda)\psi$, and the condition for a nontrivial solution becomes

$$(-s/\lambda)^r = 0.$$

Therefore, $s = 0$ is a generalized eigenvalue.

4. THE HOPSCOTCH METHOD

Using the notation $u_j^n = u(j\Delta x, n\Delta t)$, the Hopscotch method (see [3], [4]) is defined by

$$\text{(4.1)} \qquad u_j^{n+1} = u_j^n + Lu_j^n \;, \; j+n \text{ even}$$

$$\text{(4.2)} \qquad u_j^{n+1} = u_j^n + Lu_j^{n+1}, \; j+n \text{ odd}$$

where L is a difference operator. Assuming $j+n$ even and combining equation (4.1) with

$$u_j^n = u_j^{n-1} + Lu_j^n \;,$$

we obtain immediately the two equations

$$\text{(4.3)} \qquad u_j^{n+1} = 2u_j^n - u_j^{n-1}$$

$$j+n \text{ even}$$

$$\text{(4.4)} \qquad u_j^{n+1} = u_j^{n-1} + 2Lu_j^n$$

Similarly for $j+n$ odd, we combine equation (4.2) with

$$u_j^n = u_j^{n-1} + Lu_j^{n-1}$$

to obtain

$$u_j^{n+1} = u_j^{n-1} + L(u_j^{n+1} + u_j^{n-1}) .$$

If L is defined by $Lu_j^n = A\frac{\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n)$, the extrapolation formula (4.3) is valid with u_j^n replaced by Lu_j^n, and we get

$$(4.5) \qquad u_j^{n+1} = u_j^{n-1} + 2Lu_j^n, \quad j+n \text{ odd.}$$

Therefore, the Hopscotch scheme for the model equation $u_t = Au_x$ is equivalent to the Leap-frog scheme at every point, provided that the first time level for the latter one is generated by (4.1) and (4.2). The stability condition is the CFL-condition

$$(4.6) \qquad \frac{\Delta t}{\Delta x}\rho(A) \leq 1 ,$$

where $\rho(A)$ is the spectral radius of A. For the equation $u_t = Au_{xx}$ and with L defined by $Lu_j^n = A\frac{\Delta t}{(\Delta x)^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$, then, as pointed out in [3], the scheme is equivalent to the DuFort-Frankel scheme

$$(4.7) \qquad u_j^{n+1} = u_j^{n-1} + \frac{2\Delta t}{(\Delta x)^2} A(u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n)$$

with u_j^1 defined by (4.1), (4.2). Accordingly it is unconditionally stable. In order to avoid the solving of a system of equations to obtain u^{n+1} at each time step, Au_j^{n+1} can be replaced by Au_j^n. If only the steady state solution is wanted, the inconsistency with the time dependent problem thereby introduced if of no importance. However, we will show that a stability limit is imposed on Δt by this modification.

For the scalar equation $u_t=\sigma u_{xx}$, the modified method is defined by

$$(4.8) \qquad u_j^{n+1}=u_j^n+\lambda(u_{j+1}^n-2u_j^n+u_{j-1}^n), \quad j+n \text{ even}$$

$$(4.9) \qquad u_j^{n+1}=u_j^n+\lambda(u_{j+1}^{n+1}-2u_j^n+u_{j-1}^{n+1}), \quad j+n \text{ odd},$$

where $\lambda = \dfrac{\Delta t\sigma}{(\Delta x)^2}$. Proceeding along the same lines as above, we derive the extrapolation formula corresponding to (4.3):

$$(4.10) \qquad 2(1-\lambda)u_j^n = u_j^{n+1} + (1-2\lambda)u_j^{n-1} \text{ , } j+n \text{ even.}$$

Using this we then obtain an equation containing function values $u_{j\pm1}^n$, $u_j^{n\pm1}$ only:

$$(4.11) \qquad u_j^{n+1}=2\lambda(1-\lambda)(u_{j+1}^n+u_{j-1}^n)+(1-\lambda)^2u_j^{n-1} \text{ , } j+n \text{ even.}$$

A trivial calculation shows that the von Neumann condition for (4.11) is

(4.12) $\lambda \leq 1$,

and the unconditional stability is lost. With $u_{j\pm1}^{n}$, u_j^{n+1} given, the equation (4.10) defines u_j^n assuming $\lambda < 1$.

We also want to investigate the dissipative properties for small λ-values, which correspond to large Reynolds numbers in (1.1). The eigenvalues of the amplification matrix for the original Hopscotch method are denoted by z, and for the modified version by $\hat{z}$. They satisfy the equations

$$(1+2\lambda)z^2 - 4\lambda\alpha z - 1+2\lambda = 0$$

$$\hat{z}^2-4\lambda(1-\lambda)\alpha\hat{z}-(1-2\lambda)^2 = 0,$$

where $\alpha = \cos(\omega\Delta x)$. The solutions to these equations are, after expanding the square roots and dropping $0(\lambda^3)$ terms,

$$z=2\lambda\alpha(1-\lambda) \pm (1-2\lambda+2\lambda^2(1+\alpha^2))$$

$$\hat{z}=2\lambda\alpha(1-2\lambda)+2\lambda^2\alpha\pm (1-2\lambda+2\lambda^2(1+\alpha^2)-2\lambda^2) .$$

It is clear that if $\alpha\neq1$, $\alpha\neq-1$, then $|\hat{z}| < |z|$. Accordingly, the dissipation is larger for the modified scheme, except for the lowest and the highest frequency, where both schemes have no damping at all.

There is also another way to look at this difference. From (4.8) and (4.9) we can for even j+n derive an equation on the form

$$u_j^{n+1}=u_j^{n-1}+2\lambda(u_{j+1}^n-u_j^{n+1}-u_j^{n-1}+u_{j-1}^n)+2\lambda(2u_j^{n+1}-u_j^n).$$

The last term represents the deviation from the original Hopscotch scheme, and it is an approximation to $2\lambda\Delta t u_t$. Accordingly, the equation

$$u_t = \frac{\sigma}{1-\lambda} u_{xx}$$

is approximated by the modified scheme, and the dissipation coefficient is obviously larger for $\lambda < 1$.

5. NUMERICAL EXPERIMENTS

The system (1.1) has been solved by Rudy et al. [8] using several numerical methods. For the high Reynolds numbers used there, we consider the system as a singular perturbation of the inviscid hyperbolic system. The choice of boundary conditions should therefore be made based on the analysis in section 3.

When the experiments in [8] were made, only the one-dimensional analysis had been performed. Therefore, all the experiments were made with u and v specified at subsonic inflow boundaries. As we have seen in Section 3, this gives rise to a non wellposed two-dimensional problem for transonic speeds at the boundary. However, for the actual boundary data, the condition (3.16) for a non wellposed problem was never satisfied at any grid point. In this section we will present results for another set of data, where condition (3.16) is fulfilled for the whole subsonic boundary.

Figure 1 shows the computational domain and boundary data initially. In the neighborhood of the transonic point B the data were chosen in a way such that they were smooth on the whole line AC.

The boundary conditions on the line BC were

(5.1) $$v_{oj}^{n+1} = f_j$$

(5.2) $$-a_{oj}^{n}\rho_{oj}^{n}u_{oj}^{n+1} - 2RV_{oj}^{n}\rho_{oj}^{n}v_{oj}^{n+1} + (c^2)_{oj}^{n}\rho_{oj}^{n+1} = g_j \ .$$

The outgoing characteristic variable $\phi^{(3)}$ was defined at the boundary by using linear extrapolation. At the upper boundary the analogous formulas were used. On AB every variable was specified, on EF linear extrapolation was used, and on the symmetry line AF we used the conditions $u_y = \rho_y = 0, v = 0$. This set of boundary conditions will be denoted by B.C.1.

The scheme was also run with the subsonic inflow boundary condition (5.2) replaced by the condition $u_{oj}^{n+1} = h_j$

and where ρ_{oj}^{n+1} was defined by extrapolation. This set will be denoted by B.C.2. In both cases the conditions

$$|u^{n+1}-u^{n}| \leq 10^{-2}\Delta t|u^{n}|$$

(5.3) $$|v^{n+1}-v^{n}| \leq 10^{-2}\Delta t|u^{n}|$$

$$|\rho^{n+1}-\rho^{n}| \leq 10^{-2}\Delta t|\rho^{n}|$$

were checked. For the Hopscotch scheme an artificial viscosity term approximating $0.1\Delta t((\Delta x)^2 y_{xx}+(\Delta y)^2 y_{yy})$ was included in the equation for ρ. A 20x60 grid was used, and $\Delta t=.019$, Re=80,625. Figure 2 shows the pressure p after 500 time steps when conditions (5.3) were first fulfilled for B.C.1. It is seen that the pressure is far from being a constant in both cases; B.C.2 produces very large and oscillating p-values. There are no oscillations for B.C.1. Figure 3 shows the pressure after 1300 steps. B.C.1 gives a smooth and almost constant pressure profile. Figure 4 shows the results from B.C.2 after 1700 steps when the conditions (5.3) were first satisfied, and also after 2800 steps. Oscillations are still present in the whole subsonic region.

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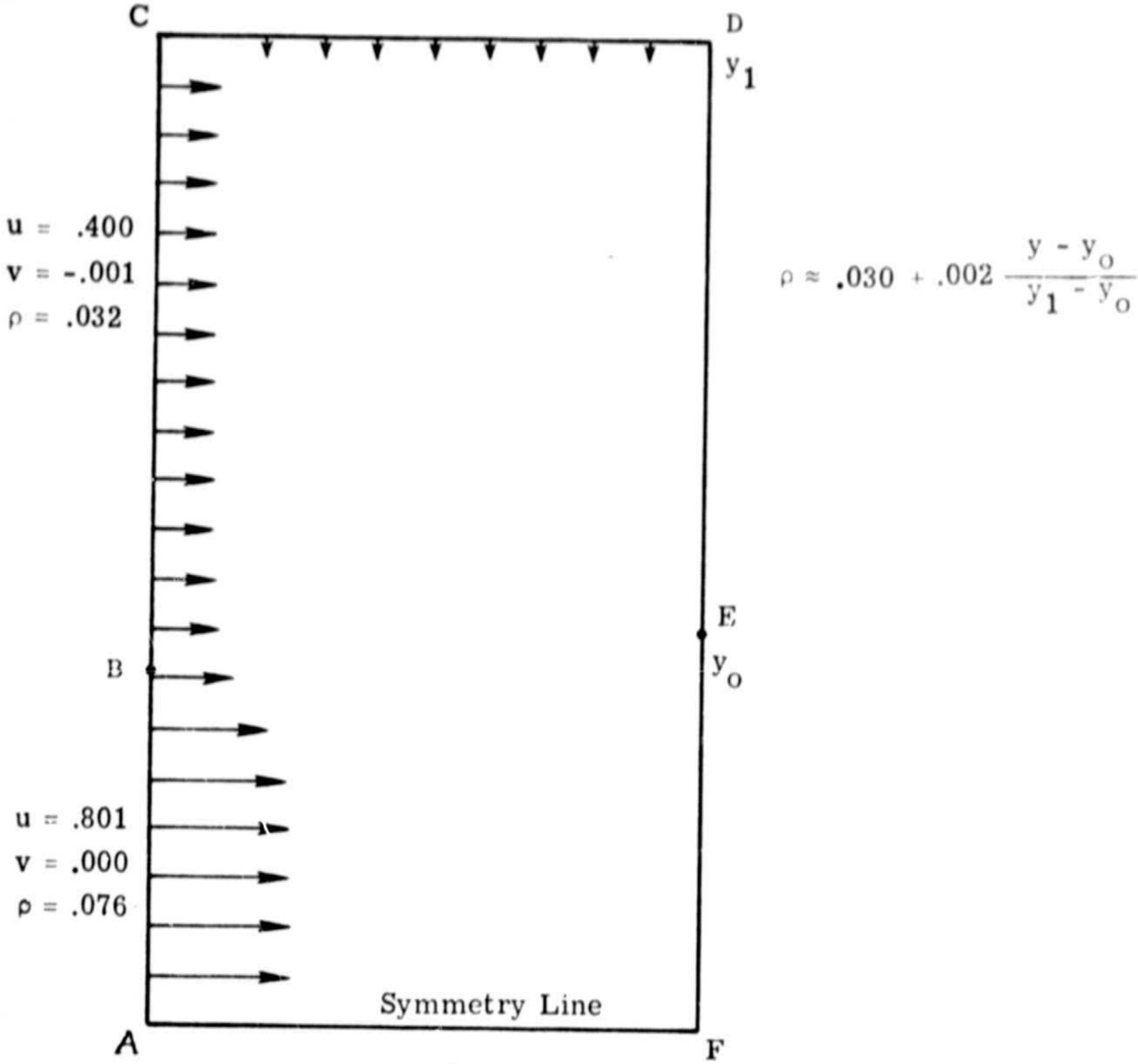

Fig. 1. Computational domain and boundary data.

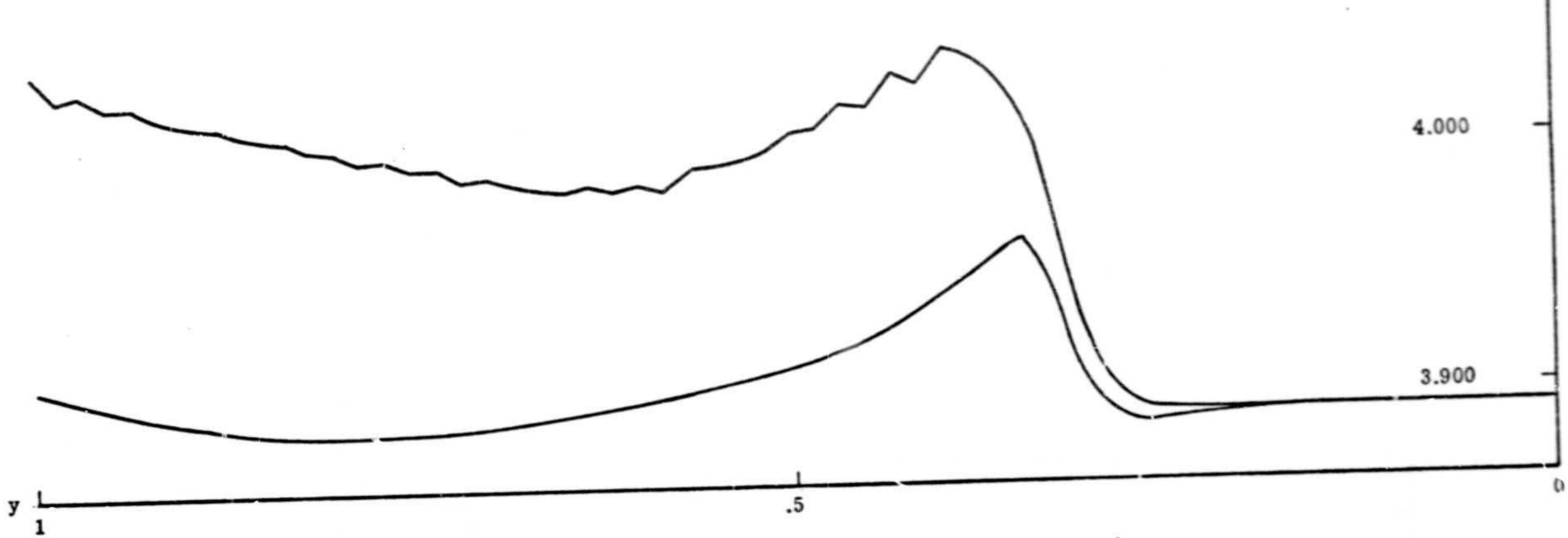

Fig. 2. Pressure profile after 500 steps for B.C.1 (lower curve) and B. C.2 (upper curve)

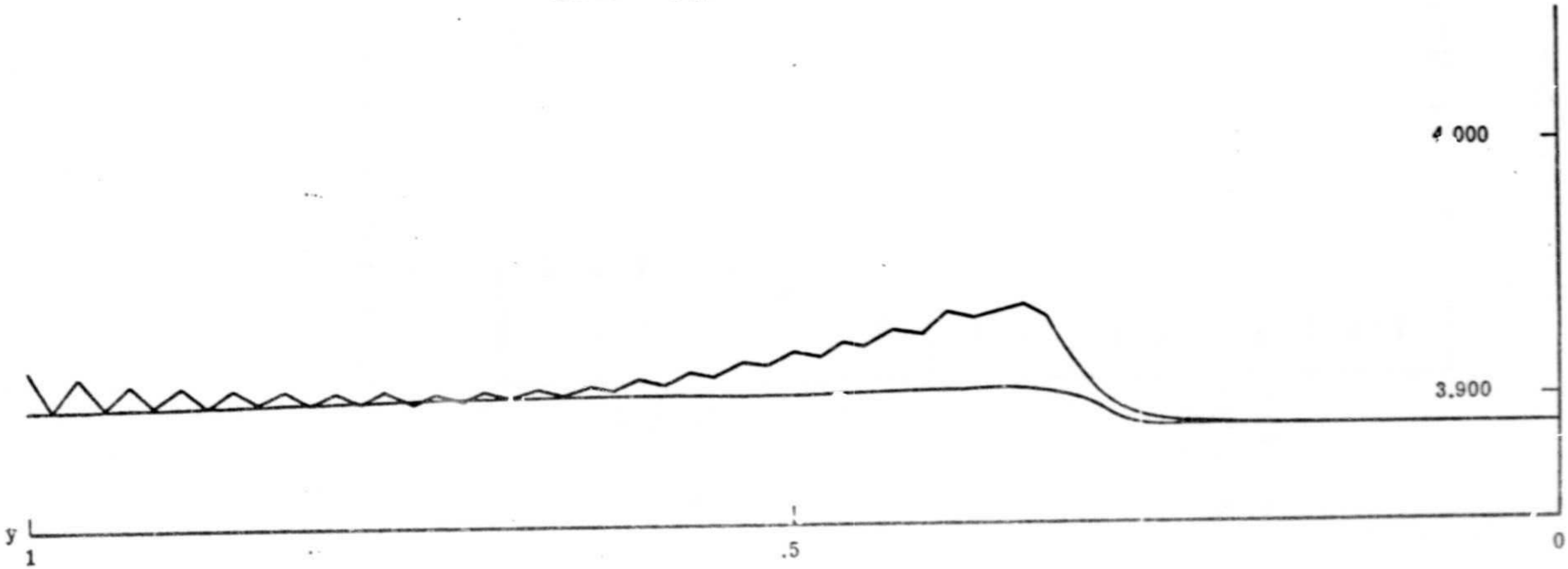

Fig. 3. Pressure profile after 1300 steps for B.C.1 (lower curve) and B.C.2 (upper curve)

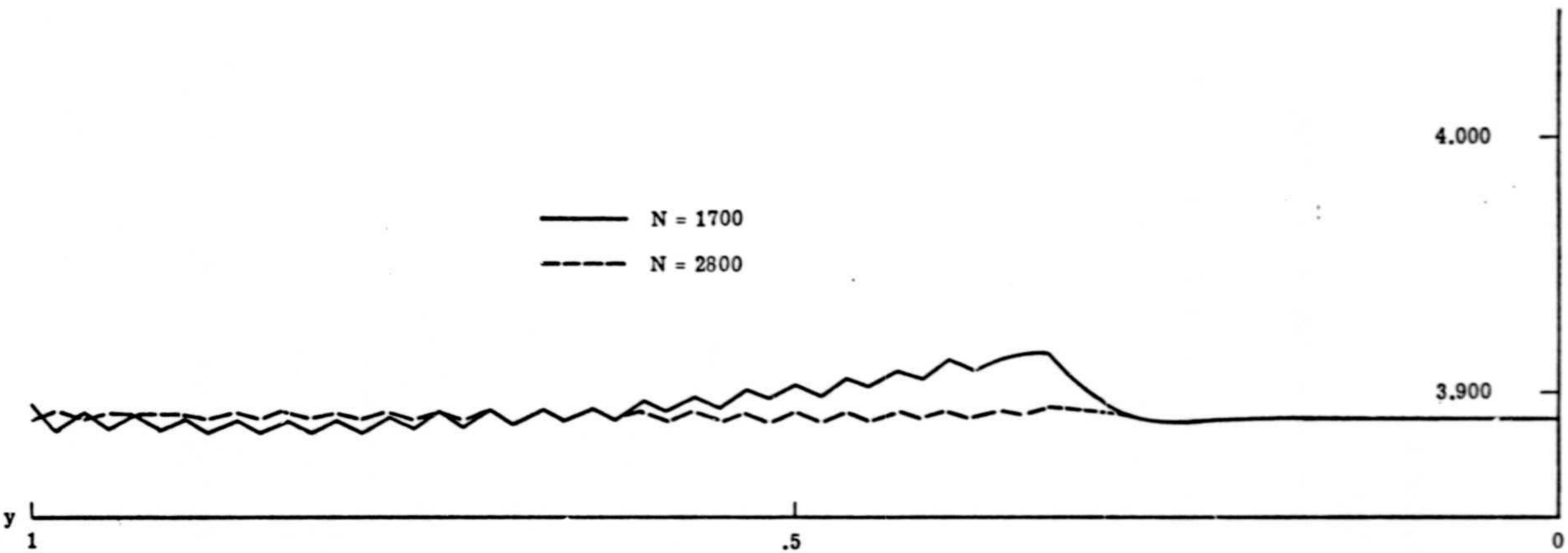

Fig. 4. Pressure profile for B.C.1 after 1700 steps and after 2800 steps

APPENDIX

Proof of (3.4)

The eigenvector ψ_1 is found directly by applying

$$C(-\frac{\alpha}{u})\psi_1 = 0.$$

However, since we don't have the explicit form of κ_2 and κ_3, the derivation of ψ_2 and ψ_3 will be more involved. Let

$$\psi_2 = (\delta_1, \delta_2, \delta_3)^T$$

and substitute it into

$$C(\kappa_2)\psi_2 = 0$$

to get with $\kappa=\kappa_2$

$$[\alpha+(1-2R)\kappa u]\delta_1 - 2Rv\kappa\delta_2 + \frac{c^2}{\rho}\kappa\delta_3 = 0$$

and

$$-2Rui\omega\delta_1 + (\alpha+\kappa u-2Rvi\omega)\delta_2 + \frac{c^2}{\rho}\, i\omega\ \delta_3 = 0$$

Eliminating δ_2 from the last two equations yields

$$\{(\alpha+(1-2R)+u)(\alpha+\kappa u-2Rvi\omega)-4R^2uvi\omega\kappa\}\delta_1$$

$$+\ \{\frac{c^2}{\rho}\kappa(\alpha+\kappa u-2Rvi\omega) + \frac{c^2}{\rho}2Rv\kappa i\omega\}\delta_3 = 0.$$

We use now (3.3) to get the expression

$$c^2(\kappa^2-\omega^2)\delta_1 = \frac{c^2}{\rho}\ \kappa(\alpha+\kappa u)\delta_3$$

and therefore,

$$\delta_1 = \frac{\kappa}{\rho}(\alpha+\kappa u),\ \delta_2 = \frac{i\omega}{\rho}(\kappa u+\alpha)\ ,\ \delta_3 = -\ (\kappa^2-\omega^2).$$

Derivation of Equation (3.5)

We start with

$\det(\psi_1,\psi_2,\psi_2) =$

$$\begin{vmatrix} i\omega u & \frac{\kappa_2}{\rho}(\alpha+\kappa_2 u) & \frac{\kappa_3}{\rho}(\alpha+\kappa_3 u) \\ \alpha & \frac{i\omega}{\rho}(\alpha+\kappa_2 u) & \frac{i\omega}{\rho}(\alpha+\kappa_3 u) \\ \frac{2\rho R}{c^2}(u^2 i\omega+v\alpha) & \omega^2-\kappa_2^2 & \omega^2-\kappa_3^2 \end{vmatrix}$$

Expanding the determinant around the first column yields

$$\det(\psi_1,\psi_2,\psi_3) = (\kappa_2-\kappa_3)\{(\kappa_3+\kappa_2)[-2c^2\omega^2\alpha u+2R\alpha u(vi\alpha\omega-u^2\omega^2)]$$

$$\text{(A1)} \qquad + (\kappa_3-\kappa_2)[-c^2\omega^2u^2-\alpha^2c^2+2Ru^2(vi\alpha\omega-u^2\omega^2)]$$

$$- c^2u^2\omega^4-\alpha^2c^2\omega^2+2R\alpha^2(vi\alpha\omega-u^2\omega^2)\}.$$

We use now the characteristic equation (3.3) in order to get

$$\text{(A2)} \qquad \kappa_2+\kappa_3 = -2u[(1-R)\alpha-2Rvi\omega]/\nu$$

and

$$\text{(A3)} \qquad \kappa_2\kappa_3 = (\alpha^2+c^2\omega^2-2Rvi\omega\alpha)/\nu .$$

After using (A2) and (A3) in (A1) we get

$$\det(\psi_1,\psi_2,\psi_3) = \nu(\alpha^2-\omega^2u^2)^2c^2(\kappa_2-\kappa_3)$$

Derivation of inequality (3.16)

With $\hat{\kappa}$ and $\hat{\alpha}$ defined by (3.14) and (3.15) respectively, we insert $\alpha = \hat{\alpha}i\omega + \delta$ and $\kappa = \hat{\kappa}i\omega + \varepsilon$ into (3.3). Dropping terms of order δ^2, $\delta\varepsilon$ and ε^2, we obtain

$$\varepsilon = \frac{\theta}{-\theta+c^2+R(u^2+\hat{\alpha}^2)} \cdot \frac{\delta}{u} ,$$

where $\theta = u^2(1-R) + \hat{\alpha}^2 - R\nu\hat{\alpha}$. Since $u > 0$, ε is negative for $\delta > 0$ if and only if one of the inequalities $\theta < 0$ or

(A4) $\qquad \theta > c^2+R(u^2+\hat{\alpha}^2)$

hold. With $z = -\nu/(R\nu)^2$, the inequality $0 < \theta$ is equivalent to

$$\nu\left(\frac{z}{(1-\sqrt{1+z})^2} + \frac{1}{1-\sqrt{1+z}}\right) > (1-R)u^2,\ 0 < z < \infty.$$

But since $\nu < 0$ and $z > \sqrt{1+z} - 1$, this inequality can never be fulfilled.

The inequality (A4) is equivalent to

$$(1-R)\hat{\alpha}^2 - R\nu\hat{\alpha} + \nu > 0 ,$$

which by the definition of $\hat{\alpha}$ and z and after division by $-\nu$ (>0) can be written in the form

$$\frac{z(1-R)}{(1\pm\sqrt{1+z})^2} + \frac{1}{1\pm\sqrt{1+z}} - 1 > 0.$$

After multiplying by $(1\pm\sqrt{1+z})^2$ we obtain $Rz + 1 < \sqrt{1+z}$, which gives the inequality

$$z < \frac{1-2R}{R^2} .$$

Using the definition of z, ν and c^2 we get the inequality (3.16).

<u>The case</u> $\kappa_1=\kappa_2$ <u>for subsonic inflow,</u> $\hat{U}$, $\hat{V}$ <u>specified</u>.

Denoting the elements of $\mathcal{V}_1$ by a_1, a_2, a_3, the last two equations of (3.6) are

(A5) $$-2Ruia_1-2Rvia_2+\frac{c^2}{\rho}\, ia_3 = \mp u^2\xi$$

(A6) $$\mp a_1+ia_2= -\frac{2Ru^2}{c^2}(ui\pm v)\xi$$

For $a_1=a_2=0$, (A6) implies $\xi=0$ for $u\neq0$, and from (A5) we then get $a_3=0$, and there is no nontrivial solution.

The case with $\hat{V}$ and $\hat{R}$ specified

With $\alpha=\hat{\alpha}i\omega+\delta$ and $\kappa=\hat{\kappa}i\omega+\varepsilon$, we get from (3.3) dropping δ^2, $\delta\ \varepsilon$ and ε^2 order terms

$$\text{(A7)} \qquad \varepsilon = \frac{u(1-R)\hat{\kappa}+\hat{\alpha}-Rv}{-\nu\frac{\hat{\kappa}}{u} - (1-R)\hat{\alpha}+Rv} \; \frac{\delta}{u} \, .$$

From (3.20) we have

$$\hat{\alpha} = 2Ru(Ruv(1-2R)\pm\sqrt{(1-2R)^2R^2u^2v^2+\nu c^2})/c^2$$

and the corresponding $\hat{\kappa}$ from (3.18)

$$(1-2R)u\hat{\kappa}\hat{\alpha} = -\, 2Ru^2 - \hat{\alpha}^2.$$

Since $|\alpha| < Rv$ and sign $\hat{\kappa}$ = - sign $\hat{\alpha}$, the numerator in (A7) is negative if $v > 0$ and $\hat{\alpha} > 0$. Therefore $\varepsilon < 0$ at the transonic point $\nu = 0$, and by continuity also in a neighborhood.

The case with Characteristic Variables Specified

With $\hat{\alpha}$ defined by (3.22), the corresponding $\hat{\kappa}$ is given by

$$\text{(A8)} \qquad \hat{\kappa}=\kappa_2/i\omega = \frac{\nu a_+}{2Rv(u-a_+)^2} + \frac{2Ruv^2}{\nu} \, .$$

With $\alpha=\delta+\hat{\alpha}i\omega$ and $\kappa=\varepsilon+\hat{\kappa}i\omega$ we have (A7). Since sign $\hat{\kappa}$ sign $\hat{\alpha}$ = - sign v. it is sufficient to investigate the case $v < 0$. For wellposedness we want to prove $\varepsilon>0$ which is equivalent to

$$-\nu\hat{\kappa}/u - (1-R)\hat{\alpha} + Rv > 0.$$

Using the $\hat{\alpha}$ - and $\hat{\kappa}$ - values defined by (3.22) and (A7) we get

$$\text{(A9)} \qquad (1-R)u|u-a_+| < |\nu| + \frac{2R^2uv^2(u-a_+)^2}{a_+|\nu|} \quad .$$

But it is easily seen that the subsonic condition is equivalent to the condition $(1-R)u|u-a_+| < |\nu|$, and therefore (A9) is always satisfied.

For $\kappa_1=\kappa_2$ and with the notation $\tilde{\psi}=(a_1,a_2,a_3)T$, we get from the condition $\hat{V}=\phi^{(1)}=0$ on the boundary, that $a_2=0$ and

$$a_1 = \pm \frac{2Ru^2}{c^2} (ui\pm v)\xi$$

$$c^2a_3 = \pm (u^2\rho i + \frac{4Ru^3\rho}{c^2}(ui\pm v))\xi \ .$$

The condition for a nontrivial solution satisfying $\phi^{(2)}=0$ on the boundary, becomes

$$\frac{2Rv}{c^2}(-a_-+2Ru) \pm (1- \frac{2Ra_-u}{c^2} + \frac{4R^2u^2}{c^2})\, \imath = 0 \ ,$$

and this cannot be satisfied since all terms of the imaginary part have the same sign.