Geophys. J. R. astr. Soc. (1974) **38**, 335–348.

On the 'Nearly Diurnal Wobble' of the Earth

Alar Toomre

(Received 1974 April 3)*

Summary

Serious confusion is evident in the original literature and several review articles from the past decade which have claimed that a second free wobble of the Earth, stemming from gyroscopic coupling with our core fluid, has been detected marginally from latitude or time data. The confusion is betrayed even by the recent labelling of that mode as a nearly diurnal wobble. In fact such a conceivable free mode, as implied for instance by the analysis of Molodenskii, consists fundamentally of a 460-day retrograde nutation *in space* of the supposedly misaligned angular momenta of the core and the mantle. This slow spatial coning of even the mantle axis has to have an amplitude roughly 460 times greater than that of the associated body-related wobble; a motion as large as that would have been noticed long ago if the recent claims of detection of the wobble were sound.

Besides helping to rectify that oversight, the present article (i) comments on the somewhat incomplete theories of such a fluid mode in a realistic Earth, (ii) determines that its combined viscous and magnetic time of damping can hardly exceed 10^3 years, and (iii) estimates the effects of such friction also upon our forced, astronomical nutations. In particular, it appears that the kinematic viscosity of the uppermost core fluid cannot exceed 10^5 stokes without affecting excessively our observed 18·6-year principal nutation. Furthermore, the tidal deceleration of the spin rate of the Earth must have carried us through an exact annual resonance with the free nutation roughly 200 My ago; however, even such resonant coning would probably have had an amplitude only comparable to that of the above principal nutation.

1. Introduction

As recognized already by Hopkins (1839), Hough (1895) and Sludskii (1896), a fluid core within even a rigid Earth makes it possible in principle for such a combined rotating system to exhibit, besides the Eulerian 'nutation' that has deservedly become known as the Chandler wobble, a second and quite different oscillation. Unlike the Chandler wobble, however, this second possible mode has never attracted wide attention. Indeed until Popov (1963a,b), it seems that no one even claimed to have detected it.

It was from latitude data that Popov judged the semi-amplitude of this 'nearly diurnal wobble', or rapid motion relative to surface geography, of the Earth's instantaneous axis of rotation to be roughly 0″.016. Several subsequent analyses (summarized by Yatskiv (1972), and also by Rochester, Jensen & Smylie (1974) in the paper following) of latitude or time data have tended either to agree or to indicate yet slightly smaller values.

*Received in original form 1974 February 15

Curiously, none of the data analysts themselves—nor even reviewers like Vicente & Jeffreys (1964) and lately Rochester (1968, 1970a), who stressed that the detection of such small displacements could at best be considered marginal—seems to have noted one fundamental property of this second mode: at least in the approximation where the Earth's mantle is considered rigid, the rotation axis of that solid exterior must *in space* travel slowly retrograde with an amplitude (and a period) roughly 400 times greater than its nearly diurnal coning relative to geography. This means that the usual labelling of the mode as a diurnal wobble is simply not fitting. It also means that granting it even the tiny geographic amplitude of the order claimed by Popov and others would imply a mysterious spatial nutation that would almost rival in amplitude the 18·6-year principal nutation caused by the regression of the orbital plane of the Moon; the latter nutation, one should remember, was discovered by Bradley before 1750.

As usual, it is difficult to trace the exact origin of this oversight. Certainly it stems from no error in the terse classical analysis of fluid-filled (and other) ellipsoids by Poincaré (1910), reproduced as 'very elegant' on the often-quoted but in the present context surprisingly obscure pages 724–727 of Lamb (1932). Also not at fault, except possibly for a similar lack of explicit emphasis, are the modern theoretical estimates by Jeffreys & Vicente (1957a,b) and Molodenskii (1961) of the frequency and certain other properties of this mode, made with special concern for the known elastic properties of the mantle. Rather, much of the recent confusion seems to have had but one source: in the best tradition of other analyses of Earth tides and/or the Chandler wobble, all those studies were conducted using rotating body-fixed axes or, more exactly, axes chosen to remain about as co-moving with their elastically yielding subject as possible. Unfortunately, though such axes may be best even for analysing the 'nearly diurnal wobble' in a realistic Earth, they are less than optimal for comprehending it intuitively—as that forgotten large spatial motion seems to testify. By contrast, the basic nature of this second mode is much clearer when viewed directly from inertial space; it is clearest if at first one totally excludes all motion of the container, as we now review briefly.

2. Ideal fluid in a fixed spheroidal cavity

Imagine a rigid spheroidal container,

$$(x^2 + y^2)/a^2 + z^2/c^2 = 1 , \tag{1}$$

filled completely with an incompressible, inviscid fluid whose velocity field $\mathbf{u} = (u_1, u_2, u_3)$ relative to inertial space at some instant t is such that its vorticity

$$\boldsymbol{\xi} = \text{curl } \mathbf{u} \tag{2}$$

is independent of x, y, or z. Then, as was known already to Kelvin (1876), regardless of any linear or angular accelerations of this container, the vorticity $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ will at all times *remain* spatially uniform, though in general it will gradually change its magnitude and direction.

If in fact this container is held firmly fixed in space, the velocity components of the fluid within are uniquely given by

$$u_1 = \frac{a^2}{a^2 + c^2}\, \xi_2 z - \frac{1}{2}\, \xi_3 y , \quad u_2 = \frac{1}{2}\, \xi_3 x - \frac{a^2}{a^2 + c^2}\, \xi_1 z , \tag{3}$$

$$u_3 = \frac{c^2}{a^2 + c^2}\, (\xi_1 y - \xi_2 x).$$

It follows from the curl of the equation of motion of an inviscid fluid (cf. Lamb (1932) p. 205) that the vorticity components will then change with time at the rates

$$\frac{d\xi_1}{dt} = \frac{1}{2}\frac{a^2-c^2}{a^2+c^2}\,\xi_2\xi_3\,,\quad \frac{d\xi_2}{dt} = -\frac{1}{2}\frac{a^2-c^2}{a^2+c^2}\,\xi_1\xi_3\,,\quad \frac{d\xi_3}{dt} = 0\,. \tag{4}$$

Hence $\xi_3(t) = \text{const.} = 2\omega$, say, and

$$\xi_1(t) = K\sin(\Omega t+\varphi),\quad \xi_2(t) = K\cos(\Omega t+\varphi)\,, \tag{5}$$

where

$$\Omega = \frac{a^2-c^2}{a^2+c^2}\,\omega\,. \tag{6}$$

All this, too, was almost certainly known to Kelvin, and to Greenhill (1880).

For an oblate ($a > c$) spheroid, equations (5) and (6) indicate that whenever the fluid vorticity $\boldsymbol{\xi}$ has for some reason become displaced from exact alignment with the minor axis, it will thereafter simply precess along a space-fixed cone in a direction *opposite* to the intrinsic spin $\omega = \xi_3/2$ of the fluid about that z axis. The angular speed Ω is typically quite small: in a container of the ellipticity $(a - c)/a$ presumed for our core–mantle interface from the Clairaut equation, the period $2\pi/\Omega$ of this regression in space would be approximately 400 'days', and of course inside a sphere it would be infinite.

Besides the vorticity $\boldsymbol{\xi}$, the total angular momentum $\mathbf{L}_f$ of the fluid is well defined; its components work out as

$$(L_1, L_2) = \frac{2}{5}\,M\,\frac{a^2c^2}{a^2+c^2}\,(\xi_1, \xi_2)\,,\qquad L_3 = \frac{1}{5}\,Ma^2\,\xi_3\,, \tag{7}$$

where M denotes the fluid mass. Hence we observe that even $\mathbf{L}_f$ partakes in the retrograde motion, though around a slightly tighter cone than $\boldsymbol{\xi}$, when the container is oblate. That in turn removes any conceivable doubt that the pressure forces at the spheroidal boundary manage to exert a net restoring torque upon the fluid whenever that vector is not aligned exactly with the minor axis (or else with the equator).

This gyroscopic coning or 'tilt-over mode' forms the essence of what is sometimes referred to as Poincaré coupling, or more often nowadays as just inertial coupling, of a rotating fluid to its oblate container. Poincaré (1910) himself would have termed it an example of 'rigidité gyrostatique'. Whatever its name, such a slow nutation of the angular momentum of the contained fluid is dynamically analogous to our likewise retrograde lunisolar precession, and even to the 18·6-year regression of the tilted orbital plane of the Moon caused by restoring torques from the Sun. If yet a third analogy is desired, one can also imagine the spheroid drained of all but one last droplet of fluid. This cohesive but frictionless droplet, like the marble discussed by Toomre (1966), will then slide along a path that is a geodesic of the spheroid. Moreover, if that path almost coincides with the equator, a little calculation shows that the mean orbital plane of also this particle will regress in space—at the rate $\omega(a-c)/c$ which actually matches, for small tilt and ellipticity, the speed Ω from equation (6).

3. Ideal fluid within a free spheroid

Obviously the analysis becomes more involved when the spheroid itself can no longer be imagined fixed. Yet for the purpose of just thinking about these matters, we are fortunate that our terrestrial core has barely one-tenth the angular inertia and momentum of its 'container'.

For one thing, such a relatively minor core can intuitively be expected only to modify the basic rigid-body mode, the Chandler wobble, by perhaps one part in ten—and indeed roughly such a change of frequency is borne out by detailed study

(Hough 1895; Jeffreys 1948) before somewhat greater allowance has to be made for elasticity. More to the point, it also seems self-evident that the fluid tilt-over mode in a shell of such large inertia cannot differ much from the nutation within a truly fixed cavity. Of course, it is now the combined angular momentum **L** of the fluid plus container that must remain invariant in space. But as one might suspect—and as the mathematics again confirms (see Hough or Jeffreys once more)—this requirement is easily met by a slightly revised mode in which *both* the fluid angular momentum $\mathbf{L}_f$ and the angular momentum $\mathbf{L}_m$ of the mantle precess slowly retrograde. Each vector obviously has to tilt a little from the invariant direction **L**, but in opposite senses, and the former by an angle about ten times greater than the latter. That modal geometry is sketched in Fig. 1, with certain features considerably exaggerated; however, notice all vectors shown in this caricature are indeed coplanar, and also that in reality $\alpha \cong \beta/10$.

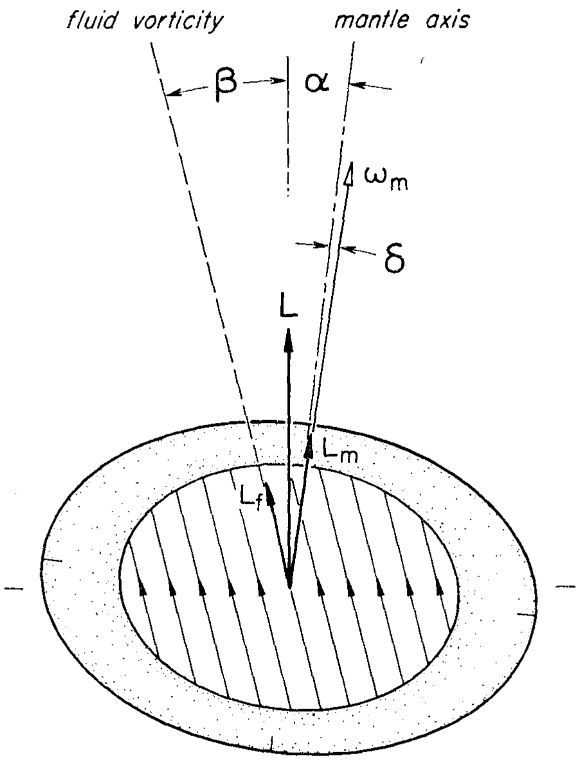

FIG. 1. Geometry of the free core nutation.

In the present context it hardly matters that such 'unlocking' of the massive container *increases* the speed Ω of the regression by about one-tenth (essentially because the torque exerted on the fluid is nearly proportional to the total angle $\alpha+\beta$ rather than just β). Nor is it vital to recognize here that Ω in turn decreases by about one-quarter—to yield a period of about 460 days, as estimated by Jeffreys & Vicente (1957a, though not in 1957b) and by Molodenskii (1961)—when our Earth is further endowed with realistic elasticity, and it thereby manages to exert less torque for any given $\alpha + \beta$.

Instead, the real key to the oversight which prompted this paper is the small angle δ in Fig. 1. It denotes the discrepancy between the axis of symmetry and the instantaneous angular velocity ω_m of that sketched rigid container. (For a flexible container, δ might have to be reinterpreted as, say, the angle between the geographic north pole and the instantaneous surface point of no motion. However, apart from the added complication of elastic tiltings of the various observatories, the basic reasoning should remain the same).

As the well-known Poinsot representation involving body- and space-fixed cones (cf. Rochester *et al.* 1974) reminds us, some such small angular difference is *kinematically* inevitable, if the mantle axis of symmetry is to pursue its sympathetic 460-day motion in space. Hence relative to that container, its spatially likewise regressing angular velocity vector ω_m will indeed execute a nearly diurnal wobble around a cone of semi-angle δ centred on the symmetry axis. This rapid angular travel will itself appear to be retrograde, and its period will equal one sidereal day minus one part in about 460.

Yet as any skeptical reader should now verify for himself, the same kinematics demands also that

$$\alpha \cong 460\delta. \tag{8}$$

This is the simple but critical point that was overlooked in much of the recent literature, even though technically it is closely analogous to the amplitudes of various astronomical nutations *vis-à-vis* their own small 'Oppolzer terms' (cf. Takagi & Murakami 1968). Hence, though it is true that *some* related body-fixed wobble cannot be avoided, the recent alias of 'nearly diurnal wobble' clearly does injustice to a mode which already Hough (1895) knew to be one where 'the oscillations of the shell are similar in character to the 'forced' nutations of the Earth produced by the action of the Sun and Moon'. It seems much less confusing to refer to that mode instead as the 'core nutation'—perhaps prefixing the word 'principal', to be fussy—regardless of whether it can actually be detected.

4. Some reservations

The above review should not be mistaken to imply that the role of a realistic fluid core has been fully elucidated with these classical uniform-vorticity solutions. It certainly has not been.

Of greatest concern here is not the complication that the real fluid contains magnetic fields and has a poorly known viscosity; at least the damping from those causes can be roughly estimated, as we shall see below. Nor is it that even an 'ideal' rotating fluid within a rigid spheroid can in principle exhibit many other inertial oscillations (cf. already Hough 1895 for some hints of those, or Greenspan 1968 for a modern account; also see Aldridge & Toomre 1969 for related axisymmetric experiments) besides the simple tilt-over mode that leaves the vorticity uniform: for infinitesimal amplitudes, at any rate, it can be shown that only the tilt-over mode exerts a net pressure torque on its container, and hence it alone can in that sense be 'felt'.

Rather, the one really awkward complication is the variable density of our core fluid; this fact probably invalidates any claimed tilt-over mode of uniform vorticity. To be sure, it seems very likely that some analogous mode of internally variable tilt angle exists for every core that is not convectively unstable. Yet no such proof seems to have reached the literature.

Not even understood properly, to my knowledge, is the idealized example where the rotating core consists of just two incompressible gravitating fluids of different uniform densities, with the denser liquid of course residing deeper inside. One would think that such a situation, or the analogous one with a solid spheroidal inner core, should give rise to *two* slowly regressing modes—and that a continuously (and stably) stratified fluid might even exhibit a dense spectrum. In fact, already Jeffreys & Vicente (1957b) obtained two slow eigenmodes from their variational treatment of one so-called Roche model of the core. But on gyroscopic grounds, assuming all torques to be of the restoring sense, it is worrisome that one of their approximate modes travels forward in space. Moreover, Molodenskii (1961), who included an inner core, isolated (perhaps by choice) only a single mode from his own extensive calculations. Hence the cliché that further analysis is needed seems to apply also here.

Still, these reservations are probably more academic than real, involving more the detailed nature and spectrum of possible core nutations—'principal' and otherwise—than concern about the existence of even one such mode. Any dramatic failure of Poincaré's 'gyrostatic rigidity' in our complicated core seems already excluded by the observed small *reduction* (paradoxical though that sounds!) from solid-body estimates of the amplitude of our 18·6-year nutation: this reduction seems explicable only through gyroscopic coupling with the core fluid (Jeffreys 1948; see also Section 6).

5. Excitation and damping

Such worries aside, the likelihood of actually observing the principal core nutation as a free mode obviously hinges on estimates of excitation and damping—now that the claimed detection by Popov and others simply has to be dismissed.

At least on first thought, any excitation from within seems apt to be minute. After all, no process confined *exclusively* to either the core or the mantle can in principle bring about the misalignment of the angular momenta $\mathbf{L}_f$ and $\mathbf{L}_m$ that we have seen to be essential to this particular mode. This excludes, for instance, any sudden change in the axes of inertia of the mantle as the result of an earthquake, or any wind torques or snow loads on its surface. Needed instead are random torques between the two major parts of the Earth.

Of course, that some such interactive torques indeed exist is suggested by the well-known irregular changes in the length of the day (lod), particularly those that have amounted to roughly one part in 10^8 over certain intervals of the order of a decade. It is not yet certain that those lod changes must be attributed to magnetic or other interactions with the core (Rochester 1973). But even assuming that they must be, it is unfortunate that the effects of such implied axial torques seem much more obviously cumulative than those of the torques at right angles needed to misalign the angular momenta: One must remember that any equatorial torque —say, around the Singapore–Quito axis—that appears *steady* to an earthbound observer over some years, months, or even weeks would in our sense accomplish very little. Hence the existence of random internal torques capable of misaligning $\mathbf{L}_f$ and $\mathbf{L}_m$ by even as little as, say, one part in 10^{10} over the time-span of a decade seems by no means assured at present.

Even this pessimism about the likely magnitudes of random torques would of course not exclude a sizeable amplitude of nutation if the damping time were immense. Yet that hope too seems not about to be realized. Three sources of dissipation come to mind at once. One, the anelasticity of especially the lower mantle, seems too complex and enigmatic to discuss here. The other two, involving either viscous or magnetic stresses at the core–mantle interface, are much easier to estimate—and they themselves seem quite rapid.

The viscous time of e-fold decay of the tilt angle β of a fluid of uniform properties within an almost spherical container has already been given by Stewartson & Roberts (1963) as

$$T_{\text{visc}} = 0{\cdot}382a/(\nu\omega)^{1/2}\ , \qquad (9)$$

where ν is the kinematic viscosity. The important point to note is that this 'spin-over' time scale, also familiar from many experiments of Malkus (1968), is far shorter than the time of viscous diffusion across the full radius a. It is so because already the gentle 'Ekman suction' in and out of the boundary layer where the dissipation actually takes place is very efficient in communicating with the bulk of the fluid, essentially through diurnal pressure forces. (Similar remarks no doubt apply to magnetic boundary layers. Thus even there, contrary to first intuition, one need not rely solely on Alfvén waves and the like to carry the 'message' about the proper spin axis into the deep interior).

What is more, even if the core fluid is strongly *stratified* (cf. Clark 1968; Higgins & Kennedy 1971; but see also Stacey 1972; Frazer 1973), any tilt-over mode(s) capable of exerting pressure torques on the mantle must by definition imply diurnally varying pressures, and hence also pressure gradients, fluid motions and finally dissipation near any typical spot of the interface. Thus, unlike with purely axial 'spin-up' where stratification can notoriously lengthen the damping process (cf. Greenspan 1968), the time scale of viscous decay of any core nutation of interest to us outsiders can probably never much exceed T_{visc} from equation (9), provided ν refers to the top of the core. Any boundary-layer instabilities (absent for sufficiently small nutation amplitudes) and other eddy viscosity would only shorten that decay time.

The magnetic time of decay of the same 'diurnal wobble' was estimated as 65 years by Rochester (1968), who used 'a simplified model which neglects the time required for magnetic diffusion through the lower mantle'. Unfortunately such a neglect seems self-contradictory, since blatantly the magnetic *fluctuations* relevant to the present damping *are* nearly diurnal. Their daily upward diffusion into the mantle is apt to be limited, for 'skin depth' reasons, to distances of the order of only 10 km rather than 1000. Hence a safer estimate is needed—and follows.

Treating the core tilt-over by the small angle β as if it had occurred within an exactly spherical container, and also neglecting here the sympathetic nutation of the mantle, the energy to be dissipated is roughly $\frac{1}{2}I_{\mathrm{core}}\omega^2\beta^2$, where I_{core} denotes the moment of inertia. The velocities of slip at the interface, in the north–south and east–west directions respectively, are then $V\cos\omega t$ and $V\cos\theta\sin\omega t$, where θ is the co-latitude and $V = a\omega\beta$. The analysis of MacDonald & Ness (1961; slightly corrected by Toomre 1966) shows that this north–south motion, for instance, induces at the interface a diurnally varying horizontal component of magnetic field

$$H' = -(2H_{\mathrm{r}}V/\delta_{\mathrm{m}}\omega)\ \mathrm{Re}\ \{e^{i\omega t}\ /\ [1 + i + (2\sigma_{\mathrm{m}}/\sigma_{\mathrm{c}})^{1/2}\ (C_\alpha + i)^{1/2}]\}, \qquad (10)$$

upon the further neglect both of Coriolis forces and of the fluid viscosity. Here H_{r} refers to the prescribed *vertical* component of the magnetic field, σ_{m} and σ_{c} are the electrical conductivities of the mantle and core near the interface, δ_{m} is the mantle 'skin depth'

$$\delta_{\mathrm{m}} = (2\pi\sigma_{\mathrm{m}}\omega)^{-1/2} \qquad (11)$$

already implied to be of 0(10 km), and C_α = here $[H_{\mathrm{r}}/15\ \mathrm{Gauss}]^2$ is a dimensionless ratio making allowance for the magnetohydrodynamics of the uppermost core fluid.

On the usual assumptions that $\sigma_{\mathrm{m}} \ll \sigma_{\mathrm{c}}$, and that H_{r} = 0(5 Gauss), the square root term in equation (10) is negligible. (Incidentally, this only corroborates that the principal movement or 'slippage' of the magnetic field lines in the present problem occurs not in the fluid but in the lower mantle. Thus even the inclusion of Coriolis forces in the fluid boundary layer would here have made little difference). Hence the Maxwell shear stress in the north–south sense is approximately

$$H_{\mathrm{r}}H'/4\pi \cong -(H_{\mathrm{r}}^2V/4\pi\delta_{\mathrm{m}}\omega)\ (\cos\omega t + \sin\omega t), \qquad (12)$$

and the mean rate of dissipation per unit area, attributable to just that component of velocity, is $(H_{\mathrm{r}}^2V^2/8\pi\delta_{\mathrm{m}}\omega)$.

After repeating the same in the east–west sense and upon some obvious integration, one thus finds in the case of an imagined pure dipole field with

$$H_{\mathrm{r}}(\theta) = H_{\mathrm{p}}\cos\theta \qquad (13)$$

that magnetic stresses alone would cause the tilt β to decrease e-fold in about the time

$$T_{\mathrm{magn}} = (15\omega I_{\mathrm{core}}/4a^4)\ (\delta_{\mathrm{m}}/H_{\mathrm{p}}^2)\ ; \qquad (14)$$

alternatively, for a random field of rms radial strength H_{rms}, replace H_p^2 above by $2{\cdot}5\ H_{rms}^2$. The same magnetic estimates could also have been obtained by integrating torques rather than dissipation; owing to the 45° phase shift inherent in the magnetic diffusion upwards into the mantle, one would then also have discovered that the out-of-phase (or precessional) component of the magnetic torque exactly equals the part responsible for the damping.

Let us finally translate both decay times into 'quality factors' Q, defined as usual as the reciprocals of the fractional energy losses per radian period, here taken to equal $(460/2\pi)$ days. So treated, they yield

$$Q_{\mathrm{visc}} = 1200 / \nu^{1/2} \tag{15}$$

and

$$Q_{\mathrm{magn}} = 2 \times 10^5 / H_p^2 S , \tag{16}$$

where ν and H_p are to be expressed in stokes ($=\mathrm{cm}^2\ \mathrm{s}^{-1}$) and Gauss; moreover, the conductivity σ_m of the mantle just above the core has here been written as S^2 times the nominal value of 10^{-9} cgs emu.

Probably the best-existing estimate of the viscosity at the top of the core is the one by Gans (1972) based on the Andrade hypothesis; it suggests that $\nu = 0(10^{-2}$ stokes) or even slightly smaller. However, if Backus (1968), in an earlier extrapolation from other experimental data, had adopted $T = 3400$°K (Stacey 1972) for the uppermost core, he would have obtained $\nu \cong 1$; as it was, with an assumed temperature of 2000°K, Backus reckoned that $\nu \cong 50$ stokes. Thus adopting $0{\cdot}01 < \nu < 1$ for the sake of argument, one might guess that

$$10^3 < Q_{\mathrm{visc}} < 10^4 \ ; \tag{17}$$

yet it should also be borne in mind that, in the continuing absence of measurements at high pressures, even past theoretical estimates as high as 10^6 stokes (here implying $Q_{\mathrm{visc}} \cong 1$, or a damping time of 150 days) cannot strictly be said to have been refuted.

For a lower bound on the magnetic damping, one should probably adopt $S^2 = 2$ (cf. Rochester 1970*b*), $H_p = 3$ Gauss, and also add $H_{rms} = 3$ Gauss (cf. Booker 1969). These values yield

$$Q_{\mathrm{magn}} \cong 5000 \text{ and } T_{\mathrm{magn}} \cong 2000 \text{ years.} \tag{18}$$

In fact, however, any likely increase in the assumed conductivity and/or the high-harmonic magnetic fields at the bottom of the mantle (as may be needed to rationalize certain changes in the length of day—cf. Rochester 1970b again) may shrink these values dramatically.

All told, it seems hard to believe that the overall Q of any core nutation can much exceed a few thousand—and conceivably it may be much less.

6. Periodic forcing

Far more certain, of course, than any random internal torques are various periodic torques exerted by the Sun and the Moon. Roughly three-quarters of any such tidal torque acts *proportionately* on the mantle and on our three-quarters-as-oblate core, causing the angular momenta of both to precess or to 'nutate' at the same rate without any need for coupling. The other quarter or so, however, acts solely on the mantle: It alone manages to 'shake' that container relative to the fluid, and conversely *only for this fraction* of any total torque can the astronomical nutation of the mantle be altered from fully-solid expectations by the fluidity of the core.

With one exception, the present writer has nothing to add to such frictionless estimates of this core-modified response as have already been given by Lamb, Jeffreys, Vicente, Molodenskii, and most recently by Melchior (1971). However, the relatively low values of Q just obtained—or at least not excluded—for the free core nutation force us to reconsider a question first asked and partly answered by Jeffreys (1948, 1950): Does the *lack* of any perceptible phase lag and other frictional effects in our observed 18·6-year and other forced nutations place any useful upper limits on the emergent magnetic fields and/or the poorly-known viscosity of the core?

To answer that, it probably suffices to consider the pair of model equations

$$\frac{d\alpha}{dt} + \mu(i+F)(\alpha-\beta) = 4i\sigma\, e^{i\sigma t} / (4-\mu)$$

$$\frac{d\beta}{dt} + (1-\mu)(i+F)(\beta-\alpha) = 3i\sigma\, e^{i\sigma t} / (4-\mu). \tag{18}$$

These simple equations are offered in the same spirit as the ones used recently by Stacey (1973) to discuss aspects of the steady precession: Suppressing even the possibility of any Chandler wobble, they again mimic only the *slow*, coupled gyroscopic motions in space of what may loosely be termed the mantle and core spin axes, subject to torques of constant amplitude (though of different effectiveness, in the ratio 4 : 3) that simply revolve in space with a period $2\pi/\sigma$ much longer than one day.

More exactly, $Re(\alpha)$ and $Im(\alpha)$ in equations (18) should be regarded as the small direction cosines of that mantle 'spin axis', with respect to either of two space-fixed orthogonal directions; the complex $\beta(t)$ refers to some typical core vorticity and decomposes similarly. All frictional effects are embodied in the complex constant

$$F = \frac{1}{2Q}(1 + i\tan\epsilon), \tag{19}$$

in which the phase angle ϵ ($= +45°$ for magnetic torques from above; $= +5{\cdot}7°$ for laminar viscous torque, from Stewartson & Roberts 1963) recognizes explicitly that even such frictional coupling may not be entirely 'passive'. The unit of time t above has been selected to make exactly -1 the angular speed of that homogeneous, $F = 0$ solution which models the free and undamped core nutation. Moreover, the applied torques in equations (18) have already been so normalized as to let $\alpha(t) \to \beta(t) \to$ just $e^{i\sigma t}$ in the completely-rigid $Q \to 0$ limit. Finally, the ratio of the core angular momentum to that of the entire system has here been abbreviated as μ.

The 'visible' half of the steady-state solution of equations (18) is

$$\alpha(t) = \left[1 + \frac{\mu\sigma}{4-\mu}(1+\sigma-iF)^{-1}\right] e^{i\sigma t} \equiv A e^{i\sigma t}, \tag{20}$$

whereas the relative displacement of the 'core axis' from that of the mantle is

$$\beta(t) - \alpha(t) = -\frac{\sigma}{4-\mu}(1+\sigma-iF)^{-1} e^{i\sigma t} \equiv B\, e^{i\sigma t}. \tag{21}$$

As intended, let us apply equation (20) mostly to our principal lunar nutation. This familiar retrograde forced motion of 18·6-year or 6798-day period has an observed semi-amplitude in obliquity of 9″.198 ± 0·002 (Fedorov 1959), as opposed to 9″.225 ± perhaps also 0·002 (Jeffreys 1970, p. 294; Melchior 1971) expected from rigid-body celestial mechanics. In space this nutation actually traces out an ellipse, with an observed axis of 6″.85, for reasons stemming from the finite 23·5° obliquity of the Earth. Upon decomposition into purely circular nutations, that ellipse implies a forward-travelling (or 'direct') component of amplitude $(9{\cdot}198 - 6{\cdot}85)/2 \cong 1''.17$, besides the main retrograde component with radius 8″.03. In our terms, these two components have dimensionless frequencies $\sigma = 460/6798$ and $-460/6798$, respectively.

Using a realistic $\mu = 0{\cdot}11$, we observe that equation (20) claims the $F = 0$ amplitude of this retrograde component to be reduced by about 2·1 parts in 1000 from the fully-solid nominal value. At first sight, such an estimate clashes badly with Jeffreys' (1948) finding that the reduction due to a plausible volume of fluid within an otherwise *rigid* Earth should be more like six parts per thousand. Now the observed difference between 9·198 and 9·225, divided by 6·85 rather than 9·20 for simple reasons related to the presence of the direct component, is about 3·9 parts per thousand; moreover, that difference agrees quite well with the later and more realistic theoretical models of Jeffreys and Vicente (1957*a* only) and especially of Molodenskii (1961), which indicated about 3·7 parts. Hence one can appreciate why Jeffreys's 1948 conclusion, based directly on formulas taken from Lamb (1932, p. 726), gave rise to a lasting impression that the elasticity of the mantle was here needed to diminish what would otherwise have been an overcorrection. Yet the truth is just opposite: Lamb was downright misleading when he chose 'as a typical representation of astronomical disturbing forces' certain tidal torques that he applied *only* to the shell—and Jeffreys was duly misled! That 1948 estimate by Jeffreys must be multiplied by approximately the factor one-quarter of which we spoke earlier, and the resulting 1·5 parts in 1000 will in turn rise to about 2·0 when the elastically lengthened free period of 460 days is used instead of the rigid 340.

Much less clear to this writer is why in fact the amplitude change due to core fluidity should be consistently larger—by an almost constant factor 1·8 not only for the principal nutation, but also for the annual and semi-annual ones—in the case of the elastic and self-gravitating mantles treated by Jeffreys, Vicente and Molodenskii, as opposed to equation (20). Something vaguely akin to 'virtual mass' from classical hydrodynamics may be involved here: We recall that any *accelerating* rigid sphere that is totally surrounded by ideal fluid (which must itself be sped up nearby) seems more massive by one-half the mass of fluid it displaces. Somewhat conversely, it may be that the diurnally varying elastic motions induced especially in the lower mantle represent a small but significant net *relative* angular momentum which somewhat complements that of the core. Thus there might be no contradiction in suspecting

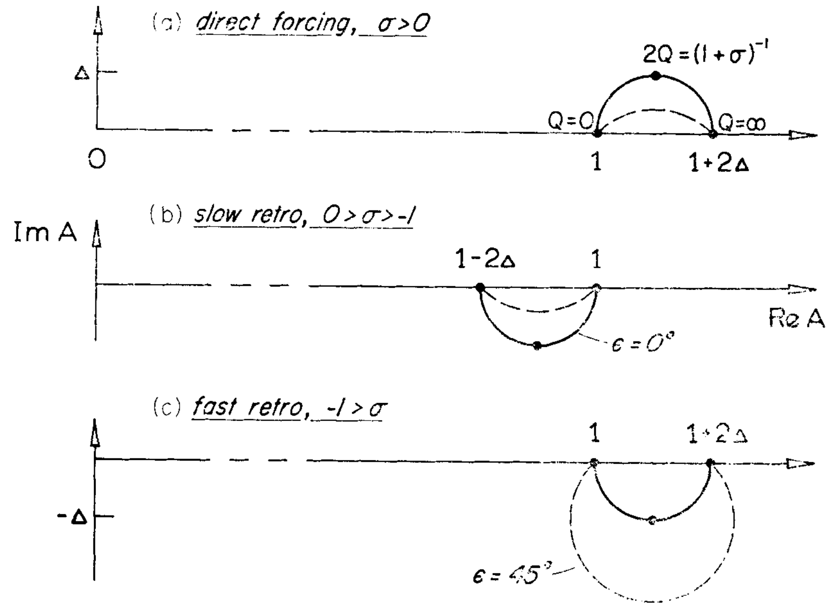

FIG. 2. Dependence of the complex amplitude A on the damping parameters Q and ϵ. The cases (a), (b) and (c) refer to the three qualitatively different choices of forcing frequency σ.

that, whereas the total (i.e. non-rigid) angular momentum of the mantle in any free nutation must indeed swing only about one-tenth as far as that of the core, the outer surface tilts 1·8 times farther. Whatever the exact reason, let us now simply adopt an *effective* core ratio $\mu = 0{\cdot}11 \times 1{\cdot}8 \cong 0{\cdot}2$ for the remainder of this paper.

Viewed solely as a function of Q, for fixed μ, ϵ and σ, the complex amplitude A of the outer solution (20) of course traces out the arc of a circle, much as shown in Fig. 2. If the frictional phase angle ϵ is indeed zero (as we might as well assume, for simplicity, for the viscous case from now on), that arc is a semicircle; its radius then is

$$\Delta = \left|\frac{\sigma/38}{1+\sigma}\right|, \tag{22}$$

provided $\mu = 0{\cdot}2$. However, if $\epsilon = +45°$ (as for magnetic coupling), the arc in question is either a quarter or a three-quarter circle, as denoted by the broken curves. In every instance apart from $Q = 0$ and $Q = \infty$, notice that these theoretical arcs all imply that the fluid-filled container should *lead*—not lag behind—the nutational motion of a solid Earth subjected to the same forcing.

Fig. 2 makes very clear what Jeffreys (1950) had in mind when he wrote that 'the maximum phase shift for the (principal) nutation as a whole would be about 0·2°': No choice of Q_{visc}, when $\sigma = -460/6798$ and $\epsilon = 0°$, makes the temporal phase shift in solution (20) greater than about $1{\cdot}9 \times 10^{-3}$ radian or 0·11°. Similar limits for the direct semi-annual and the retrograde annual nutations are 1·1° and 6·5° (and all three respective limits become 0·05°, 0·5° and 16° when $\epsilon = 45°$). Worse still, we observe from Fig. 2 that even if a usefully low limit on the phase shift could be established from observations, that *by itself* would still admit a serious ambiguity: One could then infer either that the damping is weak, or else that it is so severe as to annihilate much of the relative fluid motion in just a fraction of a single forced nutation period.

To our rescue, however, comes the mutual success of Jeffreys and Molodenskii in accounting for the observed deficiency in amplitude. It seems quite ungenerous to believe that more than perhaps 3 or 4 parts in 1000 of that retrograde amplitude of the principal nutation—or more than 10 per cent of its entire theoretical correction —could still be attributed to any neglected dissipation. Hence

$$Q_{\text{visc}} > 1{\cdot}1 \text{ and } Q_{\text{magn}} > 5. \tag{23}$$

from equation (20) with $\mu = 0{\cdot}2$. Armed with those lower bounds, one can more sensibly examine the existing phase data.

To this day, the best and indeed almost the only study of the phase of the 18·6-year nutation seems still to be that of Fedorov (1959, 1963; see also Jeffreys 1950, 1959 for comments). Fedorov determined that this principal nutation showed, in obliquity, a totally insignificant 90°-out-of-phase amplitude of 0″·001 ± 0·002, which translates into a tiny temporal phase *lag* of $(1 \pm 2) \times 10^{-4}$ radian. Yet in the minor axis direction or longitude, Fedorov claimed an out-of-phase amplitude 0″·008 ± 0·002, or actually a phase *lead* of $(11 \pm 3) \times 10^{-4}$ radian! Superficially it may be tempting to average these two estimates into a 'detected' phase lead of perhaps 5×10^{-4} radian. If that exercise were trustworthy, it would now imply that $Q \cong 4$, assuming $\epsilon = 0°$. Unfortunately, there is one serious hitch with Fedorov's disparate findings in those two orthogonal directions: equation (20) demands that the two circular components of that slow elliptical nutation should both lead the motion of the Moon's node by almost identical phase angles—and the same ought to be true of their sums.

That dilemma probably only reaffirms what Jeffreys (1959) already suspected, namely that Fedorov's error estimates were too optimistic, and perhaps especially so in the historically less well-studied longitude. Yet at the same time it seems most unlikely that Fedorov's study of the (more reliable) obliquity changes alone could have obscured any true phase lead in excess of 10^{-3} radian. Even on that presumably conservative hypothesis, our viscous estimate (23) can now be improved slightly, to $Q_{\mathrm{visc}} > 1{\cdot}9$; but since even half as much phase lead would probably not have escaped detection in the obliquity data, a more reasonable lower bound seems

$$Q_{\mathrm{visc}} > 4. \qquad (24)$$

Whatever one may think of their utility, one need not distrust these low bounds (23) and (24) on Q as representing a rate of damping so rapid as to disown the theory: The viscous and magnetic dissipations that we tried to assess earlier really only supposed the damping to be slow compared with one day, not with 460. Translated into upper bounds via equations (15) and (16), our conclusions (23) and (24) thus imply

$$\nu \lesssim 10^5 \text{ stokes} \qquad (25)$$

and

$$H_{\mathrm{rms}} \lesssim 125\, S^{-1/2} \text{ Gauss}. \qquad (26)$$

This upper bound (26) on the random poloidal field strength H_{rms} intruding from the core into the lowest mantle seems relatively *un*-newsworthy. Yet large though it is, the bound (25) on the kinematic viscosity ν of the uppermost core fluid appears lower by several orders of magnitude than any firm upper limit claimed previously, except possibly in the recent seismic analyses by Qamar & Eisenberg (1974).

Looking perhaps far ahead, imagine a time when it will finally be possible to set firm upper limits of, say, 0″.002 on the observed 90°-*out-of-phase* spatial amplitudes of the various forced nutations. It follows from equation (20) and again $\mu = 0{\cdot}2$ and $\epsilon = 0°$ that the retrograde principal nutation, the direct semi-annual nutation, and the retrograde annual nutation (with frequencies $\sigma \cong -0{\cdot}068$, $+2{\cdot}51$ and $-1{\cdot}26$, and rigid-body theoretical amplitudes of 8″.03, 0″.529 and 0″.025; cf. Woolard 1953; Melchior 1971) will then separately raise the lower bound (24) on Q_{visc} to about 9, 1·5 and 5·5. This gives some idea where, if at all, to direct future data gathering and analysis. By contrast, no comparable improvement on Q_{visc} appears possible at all from more accurate *in-phase* observations, given the inevitable uncertainties in the dynamics of even a frictionless Earth.

7. An exact resonance in the past

Finally looking much farther back—some 2×10^8 years, perhaps —there must once have been a time when the principal core mode resonated precisely with the *retrograde* annual nutation caused by torques from the Sun. Although difficult to sleuth out from Woolard's (1953) treatise alone, this particular forced notion stems from the finite eccentricity e ($=0{\cdot}0167$ at present; in the past perhaps as large as 0·07 but rarely zero; cf. Cohen, Hubbard & Oesterwinter 1973) of the Earth's orbit. Its expected rigid-body amplitude is very nearly $3e$ times that of the more familiar direct semi-annual nutation, or currently about 1″.5e; moreover that amplitude is almost independent of the longitude of perihelion.

Evidently our year has hardly changed over the aeons. What has changed is the length $2\pi/\omega$ of our day, and with it also the period $2\pi/\Omega$ of the free nutation of our core. It is easy to see that $\Omega \propto \omega^3$: One factor ω is plain in equation (6), and the equilibrium ellipticity $(a-c)/a \propto \omega^2$ contributes the rest. Thus the tidally decelerating Earth must have passed this annual resonance when our day was about $(365/460)^{1/3} \cong 0{\cdot}93$ times as long as it is now.

What was the nutation amplitude $|\alpha|$ of the mantle in that auspicious era? Alas, even with $e = 0{\cdot}05$ and $Q = 10^4$, equation (20) with $\sigma = 0{\cdot}2$ and $\epsilon = 0$ tells that it could at most have amounted to $(2Q/19) \times 1''.6e \cong 75$ seconds of arc relative to the stars. More likely—with turbulent friction, etc.—that maximum would perhaps have been one-tenth as great; ironically, the accompanying body-fixed wobble might *then* have had an amplitude of 0″.01 or 0″.02. On that note let us end this quest for the 'nearly diurnal wobble'.

Acknowledgments

For thoughtful advice here I am especially indebted to Drs H. Jeffreys, W. v. R. Malkus, W. H. Munk and M. G. Rochester, and to an anonymous referee. Financially, this work was supported in part by the National Science Foundation.

Department of Mathematics,
Massachusetts Institute of Technology

References

Aldridge, K. D. & Toomre, A., 1969. Axisymmetric inertial oscillations of a fluid in a rotating spherical container, *J. fluid Mech.*, **37,** 307.

Backus, G. E., 1968. Kinematics of geomagnetic secular variation in a perfectly conducting core, *Phil. Trans. R. Soc. London*, **263,** 239.

Booker, J. R., 1969. Geomagnetic data and core motions, *Proc. R. Soc. London*, **309,** 27.

Clark, S. P., 1968. Temperature gradients in the Earth's core (abstract), *Trans. A.G.U.*, **49,** 284.

Cohen, C. J., Hubbard, E. C. & Oesterwinter, C., 1973. Planetary elements for 10 million years, *Cel. Mech.*, **7,** 438.

Fedorov, E. P., 1959. Nutation as derived from latitude observations, *Astr. J.*, **64,** 81.

Fedorov, E. P., 1963. *Nutation and forced motion of the Earth's pole*, Macmillan, New York.

Frazer, M. C., 1973. Temperature gradients and the convective velocity in the Earth's core, *Geophys. J. R. astr. Soc.*, **34,** 193.

Gans, R. F., 1972. Viscosity of the Earth's core, *J. geophys. Res.*, **77,** 360.

Greenhill, A. G., 1880. On the general motion of a liquid ellipsoid under the gravitation of its own parts, *Proc. Camb. phil. Soc.*, **4,** 4.

Greenspan, H. P., 1968. *The theory of rotating fluids*, Cambridge University Press, Cambridge.

Higgins, G. & Kennedy, G. C., 1971. The adiabatic gradient and the melting point gradient in the core of the Earth, *J. geophys. Res.*, **76,** 1870.

Hopkins, W., 1839. Researches in physical geology, *Phil. Trans. R. Soc. London*, **129,** 381.

Hough, S. S., 1895. The oscillations of a rotating ellipsoidal shell containing fluid, *Phil. Trans. R. Soc. London*, **186,** 469.

Jeffreys, H., 1948. The Earth's core and the lunar nutation, *Mon. Not. R. astr. Soc.*, **108,** 206.

Jeffreys, H., 1950. Dynamic effects of a liquid core (II), *Mon. Not. R. astr. Soc.*, **110,** 460.

Jeffreys, H., 1959. Nutation: comparison of theory and observation, *Mon. Not. R. astr. Soc.*, **119,** 75.

Jeffreys, H., 1970. *The Earth*, 5th edition, Cambridge University Press, Cambridge.

Jeffreys, H. & Vicente, R. O., 1957*a*. The theory of nutation and the variation of latitude, *Mon. Not. R. astr. Soc.*, **117,** 142.

Jeffreys, H. & Vicente, R. O., 1957*b*. The theory of nutation and the variation of latitude: the Roche model core, *Mon. Not. R. astr. Soc.*, **117,** 162.

Kelvin, Lord (W. Thomson), 1876. *Mathematical Papers*, **4,** 193.

Lamb, H., 1932. *Hydrodynamics*, 6th edition, Cambridge University Press, Cambridge.

MacDonald, G. J. F. & Ness, N. F., 1961. A study of the free oscillations of the Earth, *J. geophys. Res.*, **66,** 1865.

Malkus, W. v. R., 1968. Precession of the Earth as the cause of geomagnetism, *Science*, **160,** 259.

Melchior, P., 1971. Precession-nutations and tidal potential, *Cel. Mech.*, **4,** 190.

Molodenskii, M. S., 1961. The theory of nutation and diurnal Earth tides, *Comm. Obs. R. Belgique*, **188,** 25.

Poincaré, H., 1910. Sur la précession des corps déformables, *Bull. Astr.*, **27,** 321.

Popov, N. A., 1963a. Nutational motion of the Earth's axis, *Nature*, **198,** 1153.

Popov, N. A., 1963b. Determination of diurnal nutation from observation of bright zenith stars at Poltava, *Sov. Astr.*, **7,** 422 (=*Astr. Zh.*, **40,** 553).

Qamar, A. & Eisenberg, A., 1974. The damping of core waves, *J. geophys. Res.*, **79,** 758.

Rochester, M. G., 1968. Perturbations in the Earth's rotation and geomagnetic core-mantle coupling, *J. Geomagn. Geoelectr.*, **20,** 387.

Rochester, M. G., 1970a. Polar wobble and drift: a brief history, *Earthquake displacement fields and the rotation of the Earth*, eds L. Mansinha, D. E. Smylie and A. E. Beck, D. Reidel Publishing Company, Dordrecht, Holland.

Rochester, M. G., 1970b. Core-mantle interactions: geophysical and astronomical consequences, *ibid.*

Rochester, M. G., 1973. The Earth's rotation, *EOS: Trans. A.G.U.*, **54,** 769.

Rochester, M. G., Jenson, O. G. & Smylie, D. E., 1974. A search for the Earth's 'nearly diurnal free wobble', *Geophys. J. R. astr. Soc.*, **38,** 349–363.

Sludskii, F., 1896. De la rotation de la Terre supposeé fluide à son intérieur, *Bull. Soc. Natur. Moscou*, **9,** 285.

Stacey, F. D., 1972. Physical properties of the Earth's core, *Geophys. Surveys*, **1,** 99.

Stacey, F. D., 1973. The coupling of the core to the precession of the Earth, *Geophys. J. R. astr. Soc.*, **33,** 47.

Stewartson, K. & Roberts, P. H., 1963. On the motion of a liquid in a spheroidal cavity of a precessing rigid body, *J. fluid Mech.*, **17,** 1.

Takagi, S. & Murakami, G., 1968. Numerical analysis of the Oppolzer terms, *Publ. Int. Lat. Obs. Mizusawa*, **6,** 225.

Toomre, A., 1966. On the coupling of the Earth's core and mantle during the 26,000 year precession, *The Earth–Moon system*, eds B. G. Marsden and A. G. W. Cameron, Plenum Press, New York.

Vicente, R. O. & Jeffreys, H., 1964. Nearly diurnal nutation of the Earth, *Nature*, **204,** 120.

Woolard, E. W., 1953. Theory of the rotation of the Earth around its center of mass, *Astr. Papers Am. Ephem. Naut. Almanac*, **15,** 3.

Yatskiv, Ya. S., 1972. On the comparison of diurnal nutation derived from separate series of latitude and time observations, *Rotation of the Earth*, eds P. Melchior and S. Yumi, D. Reidel Publishing Company, Dordrecht, Holland.