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On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane

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Abstract We investigate the growth of the Nevanlinna characteristic of $f(z+\eta)$ for a fixed $\eta \in \mathbf{C}$ in this paper. In particular, we obtain a precise asymptotic relation between $T(r, f(z+\eta))$ and $T(r, f)$, which is only true for finite order meromorphic functions. We have also obtained the proximity function and pointwise estimates of $f(z+\eta)/f(z)$ which is a discrete version of the classical logarithmic derivative estimates of $f(z)$. We apply these results to give new growth estimates of meromorphic solutions to higher order linear difference equations. This also allows us to solve an old problem of Whittaker (Interpolatory Function Theory, Cambridge University Press, Cambridge, 1935) concerning a first order difference equation. We show by giving a number of examples that all of our results are best possible in certain senses. Finally, we give a direct proof of a result in Ablowitz, Halburd and Herbst (Nonlinearity 13:889–905, 2000) concerning integrable difference equations.

Keywords Poisson–Jensen formula · Meromorphic functions · Order of growth · Difference equations

Mathematics Subject Classification (2000) Primary 30D30 · 30D35 · 39A05

Dedicated to the eightieth birthday of Walter K. Hayman.

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1 Introduction

A function $f(z)$ is called meromorphic if it is analytic in the complex plane $\mathbf{C}$ except at isolated poles. In what follows, we assume the reader is familiar with the basic notion of Nevanlinna's value distribution theory (see e.g. [18, 32]).

Recently, there has been renewed interests in difference (discrete) equations in the complex plane $\mathbf{C}$ [1, 6, 8, 20, 22, 26, 36]. In particular, and most noticeably, is the proposal by Ablowitz, Halburd and Herbst [1] to use the notion of order of growth of meromorphic functions in the sense of classical Nevanlinna theory [18] as a detector of *integrability* (i.e., solvability) of second order non-linear difference equations in $\mathbf{C}$. In particular, they showed in [1] that if the difference equation

$$f(z+1)+f(z-1)=R(z,f(z))=\frac{a_0(z)+a_1(z)f(z)+\cdots+a_p(z)f(z)^p}{b_0(z)+b_1(z)f(z)+\cdots+b_q(z)f(z)^q} \quad (1.1)$$

admits a finite order meromorphic solution, then $\max(p,q)\le 2$. It is proposed in [1] that a difference equation admits a *finite order* meromorphic solution is a strong indication of *integrability* of the equation. It is known that when $\max(p,q)\le 2$, (1.1) includes the well-known discrete Painlevé equations which are prime examples of integrable second order difference equations [1, 12]. More discussion about the integrability of difference equations will be relegated to Sect. 10.

In contrast to differential equations, non-linear difference equations often admit global meromorphic solutions [37, 40] and hence Nevanlinna's value distribution theory is applicable. The classical growth comparisons between $f(z)$ and $f'(z)$ have important applications to differential equations (see e.g. [14, 19]). In the case of applying Nevanlinna theory to difference equations, one of the most basic questions is the growth comparison between $T(r,f(z+1))$ and $T(r,f(z))$. It is shown in [10, p. 66], that for an arbitrary $b\neq 0$, the following inequalities[1]

$$(1+o(1))T(r-|b|,f(z))\le T(r,f(z+b))\le (1+o(1))T(r+|b|,f(z)) \quad (1.2)$$

hold as $r\to\infty$ for a general meromorphic function. Let η be a non-zero complex number, we shall prove the following precise asymptotic relation

$$T(r,f(z))\sim T(r,f(z+\eta)), \quad (1.3)$$

holds for finite order meromorphic functions.

In this paper, instead of considering non-linear difference equations, we shall, however, concentrate ourselves on the value distribution properties of $f(z+\eta)$ and related expressions and their applications to *linear difference equations*. It turns out that the results we obtained not only allow us to give new results on linear difference equations, but they also allow us to give a direct proof of the Nevanlinna-type theorems in [1], including that for (1.1).

[1]The inequalities (1.2) can be easily derived from corresponding inequalities for the Ahlfors–Shimizu characteristic function [10].

Although there has been considerable progress of the knowledge on the growth of meromorphic solutions to q-difference equations (e.g. [3–5, 23, 35]), relatively little is known for the growth of meromorphic solutions to even the first order difference equation

$$F(z+1) = \Psi(z)F(z), \tag{1.4}$$

where $\Psi(z)$ is a meromorphic coefficient. If $\Psi(z)$ is a rational function, then a solution is given by

$$F(z) = e^{az}\frac{\prod_{j=1}^{m}\Gamma(z-b_j)}{\prod_{k=1}^{n}\Gamma(z-c_k)} \tag{1.5}$$

for some suitable choices of constants $a, b_j, c_k \in \mathbf{C}$. Thus the solution has order 1.

Given a finite order meromorphic coefficient $\Psi(z)$, Whittaker [39, Sect. 6] explicitly constructed a meromorphic solution $F(z)$ of order $\le\sigma(\Psi)+1$ to (1.4). On the other hand, let $\Pi(z)$ be a periodic meromorphic function of period 1, then the product $\Pi(z)F(z)$ again satisfies (1.4). Thus, we can get at most a lower bound order estimate for a general meromorphic solution to (1.4). In this paper, we shall settle the lower bound order estimate for the Whittaker problem and, moreover, for the higher order linear difference equations

$$A_n(z)f(z+n) + \cdots + A_1(z)f(z+1) + A_0 f(z) = 0 \tag{1.6}$$

with certain entire coefficients $A_j(z)$. It will be shown that the order of growth of a meromorphic solution is one larger than the order of the *dominant* coefficient amongst the $A_j(z)$. Examples are given to demonstrate that the lower bound we obtain is the best possible.

It turns out that the fundamental estimate (1.3) needs both the

$$N(r, f(z+\eta)) \sim N(r, f) \tag{1.7}$$

for finite order meromorphic functions, as well as a version of *discrete* analogue of the classical logarithmic derivative to be discussed below.

It is well-known that the following logarithmic derivative estimate

$$m\left(r, \frac{f'(z)}{f(z)}\right) = O(\log T(r, f)) = S(r, f), \tag{1.8}$$

holds outside a possible set of finite linear measure, where the notation $S(r, f)$ means that the expression is of $o(T(r, f))$. It shows that the proximity function of the logarithmic derivative of $f(z)$ grows much slower than the Nevanlinna characteristic function of $f(z)$. The *logarithm derivative lemma*, as it is often called, has numerous applications in complex differential equations [28] and it also plays a crucial role in proving the celebrated Nevanlinna Second Fundamental theorem [18, 28]. It is generally recognized that the estimate (1.8) is amongst the deepest results in the value distribution theory. One can also find other applications of it in [25, 29].

Let η be a fixed complex number and $f(z)$ a meromorphic function, we ask under what assumption on $f(z)$ do we have the following difference analogue of (1.8)

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = S^*(r, f)? \tag{1.9}$$

Here the notation $S^*(r, f)$ means that the left hand side of (1.9) is of slower growth than $T(r, f)$ in some sense.

We shall give an answer to the question (1.9). More specifically, we show that if $f(z)$ is a meromorphic function of finite order σ, then we have

$$m\left(r, \frac{f(z)}{f(z+\eta)}\right) + m\left(r, \frac{f(z+\eta)}{f(z)}\right) = O\left(r^{\sigma-1+\varepsilon}\right) \tag{1.10}$$

for an arbitrary $\varepsilon > 0$. This estimate holds without any exceptional set. Hence we obtain (1.9) when we choose ε in (1.10) to be sufficiently small. It is not difficult to see that it is impossible for (1.9) to hold for an arbitrary meromorphic function. In fact, (1.9) fails to hold even for the simple entire function $f(z) = e^{e^z}$ and $\eta \neq 2\pi ik$ $(k = 1, 2, 3, \ldots)$. After this paper is completed we learnt[2] that R.G. Halburd and R.J. Korhonen [15] have also obtained an essentially same estimate (1.10) (however, their estimate is valid outside an exceptional set of finite logarithmic measure), and its interesting applications in [16] and [17].

Although our problem regarding (1.9) is somewhat weaker than (1.8), we show that it is already sufficient for our applications.

The idea of the proof of (1.10) relies on an application of the Poisson–Jensen formula [18]. The formula also allows us to obtain pointwise estimates for $|f(z+\eta)/f(z)|$.

We recall that Gundersen [13, Corollary 1] has given a precise pointwise estimate for the logarithmic derivative for a meromorphic function $f(z)$ of order σ to be

$$\left|\frac{f'(z)}{f(z)}\right| \leq |z|^{\sigma-1+\varepsilon} \tag{1.11}$$

for all $|z|$ sufficiently large and outside some *small exceptional sets*. Our estimates allow us to show, amongst others, the new upper bound

$$\left|\frac{f(z+\eta)}{f(z)}\right| \leq \exp\{r^{\sigma-1+\varepsilon}\}, \tag{1.12}$$

where $|z| = r$ is sufficiently large and $|z|$ is also outside some *small exceptional sets*. We shall apply these estimates to obtain new growth estimates of entire solutions to (1.6) with *polynomial* coefficients.

This paper is organized as follows. The main results concerning the growth of the Nevanlinna characteristic of $f(z+\eta)$ will be stated in Sect. 2; some preliminary lemmas are stated and proved in Sect. 3. The proof of the main theorems are given

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in Sects. 4 to 7. We will consider pointwise estimates such as (1.12) in Sect. 8. The applications of the main results to difference equations are given in Sect. 9, followed by a discussion of the relation of our results to integrable difference equations in Sect. 10.

2 Main results on Nevanlinna characteristics

When $f(z)$ has a finite order of growth, we shall improve the inequalities (1.2) to the following theorem.

Theorem 2.1 *Let $f(z)$ be a meromorphic function with order $\sigma = \sigma(f)$, $\sigma < +\infty$, and let η be a fixed non zero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r). \tag{2.1}$$

It is interesting to compare the inequalities (1.2) and (2.1) with the following estimate given by Ablowitz, Halburd and Herbst [1, Lemma 1] that given any $\varepsilon > 0$, then for all $r \geq 1/\varepsilon$, we have

$$T(r, f(z \pm 1)) \leq (1+\varepsilon) T(r+1, f(z)) + \kappa, \tag{2.2}$$

where κ is a constant. Thus our (2.1) shows that we have "equality" in (2.2) and that we can choose $\varepsilon = 0$ there, although we have a larger remainder term in (2.1). Although the technique used in [1] to obtain (2.2) is different from that of (1.2) in [10], the latter has already contained (2.2).

The above main theorem on Nevanlinna characteristic depends on the following results.

Theorem 2.2 *Let f be a meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f}) = \lambda < +\infty$, $\eta \neq 0$ be fixed, then for each $\varepsilon > 0$,*

$$N(r, f(z+\eta)) = N(r, f) + O(r^{\lambda-1+\varepsilon}) + O(\log r). \tag{2.3}$$

The following example shows that the above theorem is sharp in the sense that (2.3) no longer hold for infinite order meromorphic functions.

Theorem 2.3 *There exists a meromorphic function $f(z)$ of infinite order such that*

$$\frac{N(r, f(z+1)) - N(r, f(z))}{N(r, f(z))} \geq 1 \tag{2.4}$$

as $r \to \infty$.

Theorem 2.4 *Let α, R, R' be real numbers such that $0 < \alpha < 1, 0 < R$, and let η be a non-zero complex number. Then there is a positive constant C_α depending only on α such that for a given meromorphic function $f(z)$ we have, when $|z| = r$, $\max\{1, r + |\eta|\} < R < R'$, the estimate*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right)$$
$$\leq \frac{2|\eta|R}{(R-r-|\eta|)^2}\left(m(R, f) + m\left(R, \frac{1}{f}\right)\right)$$
$$+ \left(\frac{2R'}{R'-R}\right)\left(\frac{|\eta|}{R-r-|\eta|} + \frac{C_\alpha|\eta|^\alpha}{(1-\alpha)r^\alpha}\right)$$
$$\times \left(N(R', f) + N\left(R', \frac{1}{f}\right)\right). \tag{2.5}$$

We immediately deduce from (2.5) the following corollary for finite order meromorphic functions.

Corollary 2.5 *Let $f(z)$ be a meromorphic function of finite order σ and let η be a non-zero complex number. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O(r^{\sigma-1+\varepsilon}). \tag{2.6}$$

Proof Since $f(z)$ has finite order $\sigma(f) = \sigma < +\infty$, so given ε, $0 < \varepsilon < 2$, we have

$$T(r, f) = O(r^{\sigma+\frac{\varepsilon}{2}})$$

for all r. We obtain (2.6) by choosing $\alpha = 1 - \frac{\varepsilon}{2}$, $R = 2r, R' = 3r$ and $r > \max\{|\eta|, 1/2\}$ in Theorem 2.4. This completes the proof. □

We also deduce from (2.5) the following result.

Corollary 2.6 *Let η_1, η_2 be two complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a finite order meromorphic function. Let σ be the order of $f(z)$, then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\sigma-1+\varepsilon}). \tag{2.7}$$

We note that the estimate (2.6) satisfies (1.9) which is a discrete analogue of (1.8). This answers our question raised in (1.9) in the introduction. We note the above estimates do not hold when the order of $f(z)$ is infinite as indicated in the following example. Hence they are the best possible.

Example 2.7 Let $g(z) = e^{e^z}, z = re^{i\theta}$. We choose η to be real. It follows from [18, p. 7] that

$$m\left(r, \frac{g(z+\eta)}{g(z)}\right) = (e^\eta - 1)m(r, g) = (e^\eta - 1)T(r, g) \sim (e^\eta - 1)\frac{e^r}{\sqrt{2\pi^3 r}}$$

as $r \to +\infty$.

The Example 2.7 demonstrates that $m(r, f(z+\eta)/f(z))$ can grow as fast as the $m(r, f(z))$ itself for an infinite order function, thus showing that the finite order restriction in Corollaries 2.5 and 2.6 cannot be removed. The following example shows that the exponent "$\sigma - 1 + \varepsilon$" that appears in (2.6) cannot be replaced by "$\sigma - 1$".

Example 2.8 Since the order of $\Gamma(z)$ is 1, and that

$$m\left(r, \frac{\Gamma(z+1)}{\Gamma(z)}\right) = \log r,$$

we thus see immediately that we cannot drop the $\varepsilon > 0$ from (2.6). More generally, let $\sigma > 0$, then according to [39, Sect. 6, Theorem 5] for any given meromorphic function $\Psi(z)$ of order σ there is a meromorphic solution $F(z)$ to (1.4) with $\sigma(F) \le \sigma + 1$. In particular, we may choose $\Psi(z)$ so that

$$m(r, \Psi) = T(r, \Psi) \sim r^{\sigma} \log r. \tag{2.8}$$

Thus we see that

$$m\left(r, \frac{F(z+1)}{F(z)}\right) = m(r, \Psi) \sim r^{\sigma} \log r > C r^{\sigma(F)-1},$$

for each positive constant C when we choose r to be sufficiently large, since $\sigma(F) \le \sigma + 1$ holds. We conclude that we cannot drop the $\varepsilon > 0$ from (2.6), and hence (2.6) and (2.7) are the best possible in this sense.

Remark 2.9 Let $f(z)$ be meromorphic of finite order σ. Let $\varepsilon > 0$ be given. Then $T(r, f) < O(r^{\sigma+\varepsilon})$. If we choose $R = 3r$, $R' = 4r$ in (2.5), then we obtain

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O(r^{\sigma+\varepsilon}), \tag{2.9}$$

holds *uniformly* for $|\eta| < r$.

Remark 2.10 Let $f(z)$ be a meromorphic function. We choose $R = 3r$, $R' = 4r$ in (2.5). Then, we have

$$m(r, f(z+\eta)) \le m(r, f) + m\left(r, \frac{f(z+\eta)}{f(z)}\right) = O(T(4r, f)) \tag{2.10}$$

for $|\eta| < r$ to hold *uniformly*.

It is instructive to compare (2.10) and the stronger estimate

$$m(r, f(z+\eta)) \le 10T(4r, f)$$

holds *uniformly* for $|\eta| < r$ for all r sufficiently large. It is obtained by one of the authors and Ruijsenaars in [8, Lemma 3.2], by computing directly on the Poisson–Jensen formula [18].

3 Some preliminary results

Lemma 3.1 *Let α be a given constant with $0 < \alpha \leq 1$. Then there exists a constant $C_\alpha > 0$ depending only on α such that*

$$\log(1+x) \leq C_\alpha x^\alpha, \tag{3.1}$$

holds for $x \geq 0$. In particular, $C_1 = 1$.

Proof The case when $\alpha = 1$ is well-known. For α with $0 < \alpha < 1$, we define the function

$$g_\alpha(x) = \frac{\log(1+x)}{x^\alpha}.$$

It is clear that $g_\alpha(x)$ is continuous on $(0, +\infty)$. Since

$$\lim_{x\to 0} g_\alpha(x) = 0, \qquad \lim_{x\to+\infty} g_\alpha(x) = 0$$

hold. We deduce that g_α is bounded on $(0, +\infty)$. So there exists a constant C_α,

$$C_\alpha = \max_{0<x<+\infty} g_\alpha(x) \tag{3.2}$$

depending only on α such that (3.1) holds. □

Lemma 3.2 *Let $\alpha, 0 < \alpha \leq 1$ be given and C_α as given in (3.2). Then for any two complex numbers z_1 and z_2, we have the inequality*

$$\left|\log\left|\frac{z_1}{z_2}\right|\right| \leq C_\alpha\left(\left|\frac{z_1-z_2}{z_2}\right|^\alpha + \left|\frac{z_2-z_1}{z_1}\right|^\alpha\right). \tag{3.3}$$

Proof We deduce from (3.1) that

$$\log\left|\frac{z_1}{z_2}\right| = \log\left|1+\frac{z_1-z_2}{z_2}\right| \leq \log\left(1+\left|\frac{z_1-z_2}{z_2}\right|\right) \leq C_\alpha\left|\frac{z_1-z_2}{z_2}\right|^\alpha, \tag{3.4}$$

and similarly

$$\begin{aligned} -\log\left|\frac{z_1}{z_2}\right| &= \log\left|\frac{z_2}{z_1}\right| = \log\left|1+\frac{z_2-z_1}{z_1}\right| \leq \log\left(1+\left|\frac{z_2-z_1}{z_1}\right|\right) \\ &\leq C_\alpha\left|\frac{z_2-z_1}{z_1}\right|^\alpha. \end{aligned} \tag{3.5}$$

Combining the above two inequalities, we deduce

$$\left|\log\left|\frac{z_1}{z_2}\right|\right| = \max\left\{\log\left|\frac{z_1}{z_2}\right|, -\log\left|\frac{z_1}{z_2}\right|\right\} \leq C_\alpha\left(\left|\frac{z_1-z_2}{z_2}\right|^\alpha + \left|\frac{z_2-z_1}{z_1}\right|^\alpha\right)$$

as required. □

We need the following result which can be found in [21, p. 62] and [27, p. 66].

Lemma 3.3 *Let $\alpha, 0<\alpha<1$ be given, then for every given complex number w, we have*

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|re^{i\theta}-w|^{\alpha}}d\theta \le \frac{1}{(1-\alpha)r^{\alpha}}. \tag{3.6}$$

Lemma 3.4 ([7]; see also [30]) *Let $z_1, z_2, \dots, z_p$ be any finite collection of complex numbers, and let $B>0$ be any given positive number. Then there exists a finite collection of closed disks $D_1, D_2, \dots, D_q$ with corresponding radii $r_1, r_2, \dots, r_q$ that satisfy*

$$r_1+r_2+\cdots+r_q=2B,$$

such that if $z\notin D_j$ for $j=1,2,\dots,q$, then there is a permutation of the points $z_1, z_2, \dots, z_p$, say, $\hat{z}_1, \hat{z}_2, \dots, \hat{z}_p$, that satisfies

$$|z-\hat{z}_l|>B\frac{l}{p}, \quad l=1,2,\dots,p,$$

where the permutation may depend on z.

Lemma 3.5 (A.Z. Mohon'ko [31]; see also Laine [28]) *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z,f)=\frac{P(z,f)}{Q(z,f)}=\frac{\sum_{i=0}^{p}a_i(z)f^i}{\sum_{j=0}^{q}b_j(z)f^j}, \tag{3.7}$$

such that the meromorphic coefficients $a_i(z), b_j(z)$ satisfy

$$\begin{cases} T(r,a_i)=S(r,f), & i=0,1,\dots,p,\\ T(r,b_j)=S(r,f), & j=0,1,\dots,q,\end{cases} \tag{3.8}$$

then we have

$$T(r,R(z,f))=\max\{p,q\}\cdot T(r,f)+S(r,f). \tag{3.9}$$

4 Proof of Theorem 2.1

Proof Since $f(z)$ has finite order σ so that $\lambda(1/f)\le\sigma<+\infty$. We deduce from Theorem 2.2 that

$$N(r,f(z+\eta))=N(r,f)+O(r^{\lambda-1+\varepsilon})+O(\log r)$$

holds for the function $f(z)$. This relation and (2.6) together yield

$$\begin{aligned} T(r,f(z+\eta)) &= m(r,f(z+\eta))+N(r,f(z+\eta))\\ &\le m(r,f)+m\left(r,\frac{f(z+\eta)}{f(z)}\right)+N(r,f)+O(r^{\sigma-1+\varepsilon})+O(\log r)\\ &= T(r,f)+O(r^{\sigma-1+\varepsilon})+O(\log r). \end{aligned}$$

Similarly, we deduce

$$\begin{aligned} T(r,f) &= m(r,f) + N(r,f) \\ &\le m(r, f(z+\eta)) + N(r,f) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) + O(\log r) \\ &= T(r, f(z+\eta)) + O(r^{\sigma-1+\varepsilon}) + O(\log r). \end{aligned}$$

This completes the proof. □

5 Proof of Theorem 2.2

Proof Let $(b_\mu)_{\mu\in N}$ be the sequence of poles of f, with due count of multiplicity. Then $(b_\mu - \eta)_{\mu\in N}$ is the sequence of poles of $f(z+\eta)$. Thus by appealing to the definition of $N(r,f)$, we deduce

$$\begin{aligned} &|N(r, f(z+\eta)) - N(r,f)| \\ &\quad = \Bigg| \sum_{0<|b_\mu-\eta|<r} \log \frac{r}{|b_\mu - \eta|} + n(0, f(z+\eta)) \log r \\ &\qquad - \sum_{0<|b_\mu|<r} \log \frac{r}{|b_\mu|} - n(0,f)\log r \Bigg| \\ &\quad \le \Bigg| \sum_{\substack{0<|b_\mu-\eta|<r,\\ 0<|b_\mu|<r}} \left(\log \frac{r}{|b_\mu-\eta|} - \log \frac{r}{|b_\mu|} \right) \Bigg| + \Bigg(\sum_{\substack{0<|b_\mu-\eta|<r,\\ |b_\mu|\ge r \text{ or } b_\mu=0}} \log \frac{r}{|b_\mu-\eta|} \Bigg) \\ &\qquad + \Bigg(\sum_{\substack{0<|b_\mu|<r,\\ |b_\mu-\eta|\ge r \text{ or } b_\mu-\eta=0}} \log \frac{r}{|b_\mu|} \Bigg) + O(\log r) \\ &\quad \le \Bigg(\sum_{\substack{0<|b_\mu-\eta|<r,\\ 0<|b_\mu|<r}} \left| \log \left| \frac{b_\mu}{b_\mu-\eta} \right| \right| \Bigg) + \Bigg(\sum_{\substack{0<|b_\mu-\eta|<r,\\ |b_\mu|\ge r}} \log \frac{r}{|b_\mu-\eta|} \Bigg) \\ &\qquad + \Bigg(\sum_{\substack{0<|b_\mu|<r,\\ |b_\mu-\eta|\ge r}} \log \frac{r}{|b_\mu|} \Bigg) + O(\log r). \end{aligned} \tag{5.1}$$

Applying Lemma 3.2 with $\alpha = 1$ to an individual term in the first summand of the last inequality (5.1), we deduce

$$\begin{aligned} \left| \log \left| \frac{b_\mu}{b_\mu - \eta} \right| \right| &\le \left| \frac{b_\mu - (b_\mu - \eta)}{b_\mu - \eta} \right| + \left| \frac{(b_\mu - \eta) - b_\mu}{b_\mu} \right| \\ &= \left| \frac{\eta}{b_\mu - \eta} \right| + \left| \frac{\eta}{b_\mu} \right|. \end{aligned} \tag{5.2}$$

We now consider the second summand in the last line of (5.1). More specifically, we apply the Lemma 3.1 and inequalities $0<|b_\mu-\eta|<r$, $|b_\mu|\ge r$ that restrict the summation to obtain the inequalities

$$\begin{aligned}\log\frac{r}{|b_\mu-\eta|}&=\log\left(\frac{r-|b_\mu-\eta|}{|b_\mu-\eta|}+1\right)\le\frac{r-|b_\mu-\eta|}{|b_\mu-\eta|}\\&\le\frac{|\eta|+r-|b_\mu|}{|b_\mu-\eta|}\le\frac{|\eta|}{|b_\mu-\eta|}.\end{aligned}\tag{5.3}$$

Let us consider the third summand in the last line of (5.1). We similarly consider the inequalities $|b_\mu-\eta|\ge r$, $|b_\mu|<r$ that restrict the summation. This gives $|\eta|\ge r-|b_\mu|>0$. We conclude from Lemma 3.1 the inequalities

$$\log\frac{r}{|b_\mu|}=\log\left(\frac{r-|b_\mu|}{|b_\mu|}+1\right)\le\frac{r-|b_\mu|}{|b_\mu|}\le\left|\frac{\eta}{b_\mu}\right|.\tag{5.4}$$

Combining the above inequalities (5.1)–(5.3) and (5.4), we deduce

$$\begin{aligned}&|N(r,f(z+\eta))-N(r,f)|\\&\quad\le|\eta|\left\{\sum_{\substack{0<|b_\mu-\eta|<r,\\0<|b_\mu|<r}}\left(\frac{1}{|b_\mu|}+\frac{1}{|b_\mu-\eta|}\right)+\sum_{\substack{0<|b_\mu-\eta|<r,\\|b_\mu|\ge r}}\frac{1}{|b_\mu-\eta|}+\sum_{\substack{0<|b_\mu|<r,\\|b_\mu-\eta|\ge r}}\frac{1}{|b_\mu|}\right\}\\&\qquad+O(\log r)\\&\quad=|\eta|\left\{\sum_{0<|b_\mu-\eta|<r}\frac{1}{|b_\mu-\eta|}+\sum_{0<|b_\mu|<r}\frac{1}{|b_\mu|}\right\}+O(\log r).\end{aligned}\tag{5.5}$$

We turn to estimate the first summand in the last line of (5.5). In particular, we divide the summation range $0<|b_\mu-\eta|<r$ into two ranges, namely the $0<|b_\mu-\eta|\le|\eta|$ and $|\eta|<|b_\mu-\eta|<r$. We notice that when $|b_\mu-\eta|>|\eta|$, then

$$\frac{1}{|b_\mu-\eta|}=\frac{1}{|b_\mu|}\cdot\left|1+\frac{\eta}{b_\mu-\eta}\right|\le\frac{1}{|b_\mu|}\cdot\left(1+\left|\frac{\eta}{b_\mu-\eta}\right|\right)<\frac{2}{|b_\mu|}.\tag{5.6}$$

Thus when $r>|\eta|$ the first summand on the last line of (5.5) becomes

$$\begin{aligned}\sum_{0<|b_\mu-\eta|<r}\frac{1}{|b_\mu-\eta|}&=\sum_{0<|b_\mu-\eta|\le|\eta|}\frac{1}{|b_\mu-\eta|}+\sum_{|\eta|<|b_\mu-\eta|<r}\frac{1}{|b_\mu-\eta|}\\&\le2\cdot\left(\sum_{|\eta|<|b_\mu-\eta|<r}\frac{1}{|b_\mu|}\right)+O(1)\\&\le2\cdot\left(\sum_{0<|b_\mu|<r+|\eta|}\frac{1}{|b_\mu|}\right)+O(1).\end{aligned}\tag{5.7}$$

Combining (5.5) and (5.7), we get

$$|N(r, f(z+\eta)) - N(r, f)| \le 3|\eta| \Bigg(\sum_{0<|b_\mu|<r+|\eta|} \frac{1}{|b_\mu|} \Bigg) + O(\log r). \tag{5.8}$$

We distinguish two cases:

(1) Case 1: $\lambda \ge 1$. By the Hölder inequality, we have for any $\varepsilon > 0$,

$$\begin{aligned} \sum_{0<|b_\mu|<r+|\eta|} \frac{1}{|b_\mu|} &\le \Bigg(\sum_{0<|b_\mu|<r+|\eta|} \frac{1}{|b_\mu|^{\lambda+\varepsilon}} \Bigg)^{\frac{1}{\lambda+\varepsilon}} \cdot \Bigg(\sum_{0<|b_\mu|<r+|\eta|} 1^{\frac{\lambda+\varepsilon}{\lambda+\varepsilon-1}} \Bigg)^{\frac{\lambda+\varepsilon-1}{\lambda+\varepsilon}} \\ &\le O(1) \cdot n(r+|\eta|, f)^{\frac{\lambda+\varepsilon-1}{\lambda+\varepsilon}}. \end{aligned} \tag{5.9}$$

But

$$n(r+|\eta|, f) = O((r+|\eta|)^{\lambda+\varepsilon}) = O(r^{\lambda+\varepsilon}). \tag{5.10}$$

Therefore, inequalities (5.9) and (5.10) give

$$\sum_{0<|b_\mu|<r+|\eta|} \frac{1}{|b_\mu|} = O(r^{\lambda-1+\varepsilon}). \tag{5.11}$$

(2) Case 2: $\lambda < 1$. We have, by the definition of exponent of convergence,

$$\sum_{0<|b_\mu|<r+|\eta|} \frac{1}{|b_\mu|} = O(1). \tag{5.12}$$

We finally obtain from (5.8), (5.11) and (5.12) the desired result

$$|N(r, f(z+\eta)) - N(r, f)| = O(r^{\lambda-1+\varepsilon}) + O(\log r). \qquad \square$$

6 Proof of Theorem 2.3

Proof Let $\alpha, 0 < \alpha \le 1$, and let a sequence of numbers located at positive integers k, $k = 2, 3, 4, \dots$, each with multiplicity γ_k. Then according to Weierstrass' theorem [38, Sect. 8.1], there is an entire function $g(z)$ that has zeros precisely at the sequence defined above. We now take $f(z) = 1/g(z)$ to be the meromorphic function that we consider below. We then write

$$N(r, f) = \sum_{2\le k<r} \gamma_k \log \frac{r}{k}. \tag{6.1}$$

Since the poles of $f(z+1)$ are those of $f(z)$ but shifted to the left by one unit, so let us write

$$N(r, f(z+1)) = \sum_{1\le k<r} \beta_k \log \frac{r}{k},$$

where $\beta_k = \gamma_{k+1}$ for $k = 1, 2, 3, 4, \ldots$. We deduce

$$\begin{aligned} N(r, f(z+1)) - N(r, f) &= \sum_{1 \le k < r} \beta_k \log \frac{r}{k} - \sum_{2 \le k < r} \gamma_k \log \frac{r}{k} \\ &= \sum_{1 \le k < r} \gamma_{k+1} \log \frac{r}{k} - \sum_{2 \le k < r} \gamma_k \log \frac{r}{k} \\ &= \gamma_2 \log r + \sum_{2 \le k < r} (\gamma_{k+1} - \gamma_k) \log \frac{r}{k}. \end{aligned} \tag{6.2}$$

We now choose

$$\gamma_{k+1} = 2\gamma_k, \quad k = 2, 3, 4, \ldots \tag{6.3}$$

then

$$\begin{aligned} \frac{N(r, f(z+\eta)) - N(r, f(z))}{N(r, f(z))} &\ge \frac{\gamma_2 \log r + \sum_{2 \le k < r} (\gamma_{k+1} - \gamma_k) \log \frac{r}{k}}{\sum_{2 \le k < r} \gamma_k \log \frac{r}{k}} \\ &= 1 + \frac{\gamma_2 \log r}{\sum_{2 \le k < r} \gamma_k \log \frac{r}{k}} \ge 1 \end{aligned} \tag{6.4}$$

for all $r \ge 3$. On the other hand, it is easy to see from the meromorphic function $g(z)$ constructed above that it has an infinite order of growth. □

7 Proof of Theorem 2.4

Proof Let $z = re^{i\theta}$ such that $|z| < R - |\eta|$. The Poisson–Jensen formula yields

$$\begin{aligned} \log|f(z)| = \frac{1}{2\pi} \int_0^{2\pi} \log|f(\mathrm{R}e^{i\phi})| \Re\left(\frac{\mathrm{R}e^{i\phi} + z}{\mathrm{R}e^{i\phi} - z}\right) d\phi \\ - \sum_{|a_\nu| < R} \log\left|\frac{R^2 - \bar{a}_\nu z}{R(z - a_\nu)}\right| + \sum_{|b_\mu| < R} \log\left|\frac{R^2 - \bar{b}_\mu z}{R(z - b_\mu)}\right|, \end{aligned} \tag{7.1}$$

where $(a_\nu)_{\nu \in N}$ and $(b_\mu)_{\mu \in N}$, denote respectively, and with due count of multiplicity, the zeros and poles of f in $\{|z| < R\}$. Since $|z + \eta| < R$, so (7.1) also yields

$$\begin{aligned} \log|f(z+\eta)| = \frac{1}{2\pi} \int_0^{2\pi} \log|f(\mathrm{R}e^{i\phi})| \Re\left(\frac{\mathrm{R}e^{i\phi} + z + \eta}{\mathrm{R}e^{i\phi} - z - \eta}\right) d\phi \\ - \sum_{|a_\nu| < R} \log\left|\frac{R^2 - \bar{a}_\nu (z+\eta)}{R(z + \eta - a_\nu)}\right| \\ + \sum_{|b_\mu| < R} \log\left|\frac{R^2 - \bar{b}_\mu (z+\eta)}{R(z + \eta - b_\mu)}\right|. \end{aligned} \tag{7.2}$$

Subtracting (7.1) from (7.2) yields

$$\log\left|\frac{f(z+\eta)}{f(z)}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(\mathrm{Re}^{i\phi})|\Re\left(\frac{2\eta \mathrm{Re}^{i\phi}}{(\mathrm{Re}^{i\phi}-z-\eta)(\mathrm{Re}^{i\phi}-z)}\right)d\phi$$
$$-\sum_{|a_\nu|<R}\log\left|\frac{R^2-\bar{a}_\nu(z+\eta)}{R^2-\bar{a}_\nu z}\right| + \sum_{|b_\mu|<R}\log\left|\frac{R^2-\bar{b}_\mu(z+\eta)}{R^2-\bar{b}_\mu z}\right|$$
$$+\sum_{|a_\nu|<R}\log\left|\frac{R(z+\eta-a_\nu)}{R(z-a_\nu)}\right|$$
$$-\sum_{|b_\mu|<R}\log\left|\frac{R(z+\eta-b_\mu)}{R(z-b_\mu)}\right|. \tag{7.3}$$

We deduce from (7.3) that

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \le \left(\frac{2|\eta|R}{(R-|z|-|\eta|)(R-|z|)}\right)\cdot\frac{1}{2\pi}\int_0^{2\pi}|\log|f(\mathrm{Re}^{i\phi})||\,d\phi$$
$$+\sum_{|a_\nu|<R}\left|\log\left|\frac{R^2-\bar{a}_\nu(z+\eta)}{R^2-\bar{a}_\nu z}\right|\right| + \sum_{|b_\mu|<R}\left|\log\left|\frac{R^2-\bar{b}_\mu(z+\eta)}{R^2-\bar{b}_\mu z}\right|\right|$$
$$+\sum_{|a_\nu|<R}\left|\log\left|\frac{z+\eta-a_\nu}{z-a_\nu}\right|\right| + \sum_{|b_\mu|<R}\left|\log\left|\frac{z+\eta-b_\mu}{z-b_\mu}\right|\right|$$
$$\le \frac{2|\eta|R}{(R-|z|-|\eta|)^2}(m(R,f)+m(R,1/f))$$
$$+\sum_{|a_\nu|<R}\left|\log\left|\frac{R^2-\bar{a}_\nu(z+\eta)}{R^2-\bar{a}_\nu z}\right|\right| + \sum_{|b_\mu|<R}\left|\log\left|\frac{R^2-\bar{b}_\mu(z+\eta)}{R^2-\bar{b}_\mu z}\right|\right|$$
$$+\sum_{|a_\nu|<R}\left|\log\left|\frac{z+\eta-a_\nu}{z-a_\nu}\right|\right| + \sum_{|b_\mu|<R}\left|\log\left|\frac{z+\eta-b_\mu}{z-b_\mu}\right|\right|. \tag{7.4}$$

We apply (3.3) with $\alpha = 1$ in Lemma 3.2 to the second and third summands in (7.4). This yields, for $|a_\nu| < R$,

$$\left|\log\left|\frac{R^2-\bar{a}_\nu(z+\eta)}{R^2-\bar{a}_\nu z}\right|\right| \le \left|\frac{\bar{a}_\nu\eta}{R^2-\bar{a}_\nu z}\right| + \left|\frac{\bar{a}_\nu\eta}{R^2-\bar{a}_\nu(z+\eta)}\right|$$
$$\le \frac{|\eta|}{R-|z|} + \frac{|\eta|}{R-|z|-|\eta|} \le \frac{2|\eta|}{R-|z|-|\eta|}. \tag{7.5}$$

Similarly, we have for $|b_\mu| < R$,

$$\left|\log\left|\frac{R^2-\bar{b}_\mu(z+\eta)}{R^2-\bar{b}_\mu z}\right|\right| \le \frac{2|\eta|}{R-|z|-|\eta|}. \tag{7.6}$$

We then choose $0<\alpha<1$ in (3.3) and this yields

$$\left|\log\left|\frac{z+\eta-a_\nu}{z-a_\nu}\right|\right| \le C_\alpha|\eta|^\alpha\left(\frac{1}{|z-a_\nu|^\alpha}+\frac{1}{|z+\eta-a_\nu|^\alpha}\right), \tag{7.7}$$

and

$$\left|\log\left|\frac{z+\eta-b_\mu}{z-b_\mu}\right|\right| \le C_\alpha|\eta|^\alpha\left(\frac{1}{|z-b_\mu|^\alpha}+\frac{1}{|z+\eta-b_\mu|^\alpha}\right). \tag{7.8}$$

Combining the inequalities (7.4)–(7.8), we get

$$\begin{aligned}\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \le{}& \frac{2|\eta|R}{(R-|z|-|\eta|)^2}\left(m(R,f)+m\left(R,\frac{1}{f}\right)\right)\\ &+\frac{2|\eta|}{R-|z|-|\eta|}\left(n(R,f)+n\left(R,\frac{1}{f}\right)\right)\\ &+C_\alpha|\eta|^\alpha\sum_{|a_\nu|<R}\left(\frac{1}{|z-a_\nu|^\alpha}+\frac{1}{|z+\eta-a_\nu|^\alpha}\right)\\ &+C_\alpha|\eta|^\alpha\sum_{|b_\mu|<R}\left(\frac{1}{|z-b_\mu|^\alpha}+\frac{1}{|z+\eta-b_\mu|^\alpha}\right).\end{aligned} \tag{7.9}$$

Integrating (7.9) on $|z|=r$, and applying Lemma 3.3 gives

$$\begin{aligned}&m\left(r,\frac{f(z+\eta)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+\eta)}\right)\\ &\le\frac{2|\eta|R}{(R-r-|\eta|)^2}\left(m(R,f)+m\left(R,\frac{1}{f}\right)\right)+\frac{2|\eta|}{R-r-|\eta|}(n(R,f)+n(R,1/f))\\ &\quad+C_\alpha|\eta|^\alpha\sum_{|a_\nu|<R}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|re^{i\theta}-a_\nu|^\alpha}\,d\theta+\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|re^{i\theta}+\eta-a_\nu|^\alpha}\,d\theta\right)\\ &\quad+C_\alpha|\eta|^\alpha\sum_{|b_\mu|<R}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|re^{i\theta}-b_\mu|^\alpha}\,d\theta+\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|re^{i\theta}+\eta-b_\mu|^\alpha}\,d\theta\right)\\ &\le\frac{2|\eta|R}{(R-r-|\eta|)^2}\left(m(R,f)+m\left(R,\frac{1}{f}\right)\right)\\ &\quad+\left(\frac{2|\eta|}{R-r-|\eta|}+\frac{2C_\alpha|\eta|^\alpha}{(1-\alpha)r^\alpha}\right)\left(n(R,f)+n\left(R,\frac{1}{f}\right)\right).\end{aligned} \tag{7.10}$$

Since $R'>R>1$, we deduce

$$\begin{aligned}N(R',f)&\ge\int_R^{R'}\frac{n(t,f)-n(0,f)}{t}\,dt+n(0,f)\log R'\\ &\ge n(R,f)\int_R^{R'}\frac{dt}{t}-n(0,f)\int_R^{R'}\frac{dt}{t}+n(0,f)\log R'\\ &\ge n(R,f)\frac{R'-R}{R'}.\end{aligned}$$

Hence

$$n(R, f) \leq \frac{R'}{R' - R} N(R', f). \tag{7.11}$$

Similarly, we have

$$n\left(R, \frac{1}{f}\right) \leq \frac{R'}{R' - R} N\left(R', \frac{1}{f}\right). \tag{7.12}$$

We deduce (2.5) after combining (7.10), (7.11) and (7.12). □

8 Pointwise estimates

It is well-known that pointwise logarithmic derivative estimates of finite order meromorphic functions play an important role in complex differential equations (see e.g. [14]). In particular, the following estimate of Gundersen [13, Corollary 2] gives a sharp upper bound of logarithm derivatives.

Theorem 8.1 *Let $f(z)$ be a meromorphic function, and let $k \geq 1$ be an integer, $\alpha > 1$, and $\varepsilon > 0$ be given real constants, then there exists a set $E \subset (1, \infty)$ of finite logarithmic measure,*

(a) *and a constant $A > 0$ depending only on α, such that for all $|z| \notin E \cup [0, 1]$, we have*

$$\left|\frac{f'(z)}{f(z)}\right| \leq A\left(\frac{T(\alpha r, f)}{r} + \frac{n(\alpha r)}{r} \log^{\alpha} r \log^{+} n(\alpha r)\right), \tag{8.1}$$

where $n(t) = n(t, f) + n(t, 1/f)$;

(b) *and if in addition that $f(z)$ has finite order σ, and such that for all $|z| \notin E \cup [0, 1]$, we have*

$$\left|\frac{f'(z)}{f(z)}\right| \leq |z|^{\sigma - 1 + \varepsilon}. \tag{8.2}$$

We first give pointwise estimates for our difference quotient which are counterparts to Gundersen's logarithmic derivative estimates.

Theorem 8.2 *Let $f(z)$ be a meromorphic function, η a non-zero complex number, and let $\gamma > 1$, and $\varepsilon > 0$ be given real constants, then there exists a subset $E \subset (1, \infty)$ of finite logarithmic measure,*

(a) *and a constant A depending only on γ and η, such that for all $|z| \notin E \cup [0, 1]$, we have*

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq A\left(\frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r} \log^{\gamma} r \log^{+} n(\gamma r)\right); \tag{8.3}$$

(b) *and if in addition that* $f(z)$ *has finite order* σ, *and such that for all* $|z| = r \notin E \cup [0, 1]$, *we have*

$$\exp\left(-r^{\sigma-1+\varepsilon}\right) \le \left|\frac{f(z+\eta)}{f(z)}\right| \le \exp\left(r^{\sigma-1+\varepsilon}\right). \tag{8.4}$$

We remark that the example $f(z) = e^{z^n}$ shows that the $\varepsilon > 0$ in (8.4) cannot be dropped, and so (8.4) is the best possible.

The forms of logarithmic derivative estimates almost always depend on how we remove the "exceptional set" consisting of the zeros and poles of the function in the complex plane, such as the proof given by Hille [24, Theorem 4.5.1]. More precise estimates usually depend on application of the Cartan lemma [7] (see also [30]) such as [28, Proposition 5.12] and Theorem 8.1 above. We shall make use of the same lemma to prove our theorem.

Proof Let z be such that $|z| = r < R - |\eta|$. We deduce from (7.3), (7.4) the inequality (7.9).

Let $\beta > 1$ and $R = \beta r + |\eta|$. We choose r_1 so that $2|\eta| < \beta(\beta - 1)r$ for $r > r_1$. We apply Lemma 3.2 with $\alpha = 1$ (note that $C_1 = 1$) to (7.9) and this yields

$$\begin{aligned}\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| &\le \frac{4|\eta|(\beta r+|\eta|)}{(\beta-1)^2r^2}T(\beta r+|\eta|, f)\\ &\quad + |\eta|\cdot\sum_{|c_k|<\beta r+|\eta|}\left(\frac{1}{|z-c_k|}+\frac{1}{|z+\eta-c_k|}\right)\\ &\le \frac{4|\eta|\beta^2}{(\beta-1)^2}\frac{T(\beta^2 r, f)}{r} + |\eta|\cdot\sum_{|c_k|<\beta r+|\eta|}\left(\frac{1}{|z-c_k|}+\frac{1}{|z+\eta-c_k|}\right)\\ &= 4|\eta|\left(\frac{\beta}{\beta-1}\right)^2\frac{T(\beta^2 r, f)}{r} + |\eta|\cdot\sum_{|d_k|<\beta^2 r}\frac{1}{|z-d_k|},\end{aligned} \tag{8.5}$$

where $(c_k)_{k\in N} = (a_\nu)_{\nu\in N} \cup (b_\mu)_{\mu\in N}$ and $(d_k)_{k\in N} = (c_k)_{k\in N} \cup (c_k-\eta)_{k\in N}$, where the sequence d_k is listed according to multiplicity and ordered by increasing modulus.

We now let $\gamma = \beta^2$ and apply Lemma 3.4 to the second summand of (8.5) with $|d_k| < R = \gamma r$, where $r > \max\{r_1, |d_1|\}$. The argument then follows the same argument as [13, (7.6)–(7.9)] (with their α replaced by our γ) so that we deduce for all $|z| \notin E \cup [0, 1]$, where the E has finite logarithmic measure,

$$\sum_{|d_k|<\gamma r}\frac{1}{|z-d_k|} \le \gamma^2\frac{n(\gamma^2 r)}{r}\log^\gamma r\log n(\gamma^2 r). \tag{8.6}$$

Combining (8.5) and (8.6) we obtain

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \le 4|\eta|\left(\frac{\beta}{\beta-1}\right)^2\frac{T(\gamma r, f)}{r} + |\eta|\cdot\left(\gamma^2\frac{n(\gamma^2 r)}{r}\log^\gamma r\log n(\gamma^2 r)\right)$$

$$\leq |\eta| \left[4 \left(\frac{\beta}{\beta - 1} \right)^2 \frac{T(\gamma^2 r, f)}{r} + \gamma^2 \frac{n(\gamma^2 r)}{r} \log^{\gamma^2} r \log n(\gamma^2 r) \right] \tag{8.7}$$

which gives (8.3) with γ^2 replaced by γ.

If $f(z)$ has finite order σ, then given $\varepsilon > 0$, it is now easy to deduce (8.4) holds from the estimate (8.3). □

We easily obtain the following result.

Corollary 8.3 *Let η_1, η_2 be two arbitrary complex numbers, and let $f(z)$ be a meromorphic function of finite order σ. Let $\varepsilon > 0$ be given, then there exists a subset $E \subset (0, \infty)$ with finite logarithmic measure such that for all $r \notin E \cup [0, 1]$, we have*

$$\exp\big(-r^{\sigma-1+\varepsilon}\big) \leq \left| \frac{f(z+\eta_1)}{f(z+\eta_2)} \right| \leq \exp\big(r^{\sigma-1+\varepsilon}\big). \tag{8.8}$$

We can replace the linear exceptional set by "radial exceptional" set (see also [13, Lemma 2 and Corollary 4] and [13, Theorem 2 and Corollary 1]).

Theorem 8.4 *Let $f(z)$ be a meromorphic function, η a non-zero complex number, and let $\gamma > 1$, and $\varepsilon > 0$ be given real constants, then there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $z = re^{\psi_0}$ satisfying $\psi_0 \notin E$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$,*

(a) *we have*

$$\left| \log \left| \frac{f(z+\eta)}{f(z)} \right| \right| \leq B \left(\frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r} \log^{\gamma} r \log^{+} n(\gamma r) \right),$$

and the constant B depending only on γ and η;

(b) *and if in addition that $f(z)$ has finite order σ, we have*

$$\exp\big(-r^{\sigma-1+\varepsilon}\big) \leq \left| \frac{f(z+\eta)}{f(z)} \right| \leq \exp\big(r^{\sigma-1+\varepsilon}\big).$$

We shall omit the proof. Similarly we have

Corollary 8.5 *Let η_1, η_2 be two arbitrary complex numbers, and let $f(z)$ be a meromorphic function of finite order σ. Let $\varepsilon > 0$ be given, then there exists a subset $E \subset [0, 2\pi)$ of linear measure zero such that if $z = re^{\psi_0}$ satisfying $\psi_0 \notin E$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$, we have*

$$\exp\big(-r^{\sigma-1+\varepsilon}\big) \leq \left| \frac{f(z+\eta_1)}{f(z+\eta_2)} \right| \leq \exp\big(r^{\sigma-1+\varepsilon}\big).$$

9 Applications to difference equations

We first apply the Theorem 2.1 to give a direct proof of the following theorem.

Theorem 9.1 ([1, 22]) *Let* $c_1, \dots, c_n$ *be non-zero complex numbers. If the difference equation*

$$\sum_{i=j}^{n} y(z+c_j) = R(z, y(z)) = \frac{a_0(z) + a_1(z)y(z) + \cdots + a_p(z)y(z)^p}{b_0(z) + b_1(z)y(z) + \cdots + b_q(z)y(z)^q} \tag{9.1}$$

with polynomial coefficients a_i, b_j*, admits a finite order meromorphic solution* $f(z)$*, then we have* $\max\{p, q\} \le n$.

This theorem was first given in [1, Theorem 3] with $n = 2$ and was written in the above generalized form in [22, Proposition 2.1].

Proof Without loss of generality, we assume $f(z)$ to be a finite order transcendental meromorphic solution to (9.1). The estimate on the right side of (9.1) is easily handled by applying Lemma 3.5, as in the proofs in [1] and [22], to give (3.9). Then (2.1) of our Theorem 2.1 and (9.1) yield

$$\begin{aligned} \max\{p,q\}T(r,f) &= T(r, R(z,f)) + S(r,f) \\ &\le T\left(r, \sum_{j=1}^{n} f(z+c_j)\right) + S(r,f) \\ &\le nT(r,f) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r) + S(r,f) \end{aligned} \tag{9.2}$$

since (2.1) is independent of c_j. This yields the asserted result. □

We remark that the above argument also allows us to handle the case when we replace the left side of (9.1) by $\prod_{i=1}^{n} y(z+c_i)$, which gives the same conclusion that $\max\{p,q\} \le n$. This case was also considered in [1] and [22].

We now consider the growth of meromorphic solutions to general linear difference equation (1.6).

Theorem 9.2 *Let* $A_0(z), \dots, A_n(z)$ *be entire functions such that there exists an integer* $\ell, 0 \le \ell \le n$*, such that*

$$\sigma(A_\ell) > \max_{\substack{0\le\ell\le n \\ j\ne\ell}} \{\sigma(A_j)\}. \tag{9.3}$$

If $f(z)$ *is a meromorphic solution to*

$$A_n(z)y(z+n) + \cdots + A_1(z)y(z+1) + A_0 y(z) = 0, \tag{9.4}$$

then we have $\sigma(f) \ge \sigma(A_\ell) + 1$.

Proof Let us choose σ in relation to (9.3) so that

$$\max_{\substack{0\le \ell\le n\\ \ell\ne j}}\{\sigma(A_i)\}<\sigma<\sigma(A_\ell) \tag{9.5}$$

holds. Let us suppose that $f(z)$ is a finite order meromorphic solution to (9.4) such that

$$\sigma(f)<\sigma(A_\ell)+1. \tag{9.6}$$

We divide through the equation (9.4) by $f(z+\ell)$ to get

$$A_n(z)\frac{f(z+n)}{f(z+\ell)}+\cdots+A_\ell(z)+\cdots+A_0(z)\frac{f(z)}{f(z+\ell)}=0. \tag{9.7}$$

Since (9.5) and (9.6) hold, so we may choose $\varepsilon>0$ such that the inequalities

$$\sigma(f)+2\varepsilon<\sigma(A_\ell)+1 \quad \text{and} \quad \sigma+2\varepsilon<\sigma(A_\ell), \tag{9.8}$$

hold simultaneously. With the $\varepsilon>0$ as given in (9.8), then (2.7) gives, when $0\le j<\ell$ or $\ell<j\le n$,

$$m\left(r,\frac{f(z+j)}{f(z+\ell)}\right)\le O\big(r^{\sigma(f)-1+\varepsilon}\big). \tag{9.9}$$

Then we deduce from (2.6), (9.9) and (9.8) that

$$\begin{aligned} m(r,A_\ell)&\le\sum_{\substack{0\le j\le n,\\ j\ne\ell}} m\left(r,\frac{f(z+j)}{f(z+\ell)}\right)+\sum_{j\ne\ell}m(r,A_j)\\ &\le O\big(r^{\sigma(f)-1+\varepsilon}\big)+O\big(r^{\sigma+\varepsilon}\big)\\ &\le o\big(r^{\sigma(A_\ell)-\varepsilon}\big). \end{aligned} \tag{9.10}$$

A contradiction. □

We next show how to use the Theorem 9.2 to settle a problem of Whittaker [39] concerning linear difference equations.

Corollary 9.3 *Let σ be a real number, and let $\Psi(z)$ be a given entire function with order $\sigma(\Psi)=\sigma$. Then the equation*

$$F(z+\eta)=\Psi(z)F(z), \tag{9.11}$$

admits a meromorphic solution of order $\sigma(F)=\sigma+1$.

Proof Whittaker [39, Sect. 6] constructed a meromorphic solution $F(z)$ to (9.11) and the solution has order $\sigma(F)\le\sigma(\Psi)+1$. Since Ψ is entire, and it certainly satisfies the assumption (9.3) and this leads to the conclusion that $\sigma(F)\ge\sigma(\Psi)+1$. This completes the proof. □

Theorem 9.4 *Let $P_0(z), \dots, P_n(z)$ be polynomials such that there exists an integer ℓ, $0 \le \ell \le n$ so that*

$$\deg(P_\ell) > \max_{\substack{0\le \ell\le n\\ j\neq \ell}} \{\deg(P_j)\} \tag{9.12}$$

holds. Suppose $f(z)$ is a meromorphic solution to

$$P_n(z)y(z+n) + \cdots + P_1(z)y(z+1) + P_0 y(z) = 0, \tag{9.13}$$

then we have $\sigma(f) \ge 1$.

Proof We assume that (9.13) admits a meromorphic solution $f(z)$ with $\sigma(f) < 1$. We now divide through the difference equation (9.13) by $f(z+\ell)$ to obtain

$$P_n(z)\frac{f(z+n)}{f(z+\ell)} + \cdots + P_\ell(z) + \cdots + P_0(z)\frac{f(z)}{f(z+\ell)} = 0. \tag{9.14}$$

We note that since $\sigma(f) < 1$, so let us choose an $\varepsilon > 0$ so that $\varepsilon < 1 - \sigma(f)$, and Corollary 8.3 implies that both when $0 \le j < \ell$ or $\ell < j \le n$ hold, then

$$\left|\frac{f(z+j)}{f(z+\ell)}\right| \le \exp\left(r^{\sigma-1+\varepsilon}\right) = \exp(o(1)) \tag{9.15}$$

also holds outside a possible set of r of finite logarithmic measure. We deduce that (9.15) is bounded outside a possible set of r of finite logarithmic measure.

We now apply (9.15) to (9.14) and this gives

$$|P_\ell(z)| \le \sum_{\substack{0\le j\le n,\\ j\neq \ell}} |P_j(z)|\left|\frac{f(z+j)}{f(z+\ell)}\right| \le O(1)\sum_{\substack{0\le j\le n,\\ j\neq \ell}} |P_j(z)|, \tag{9.16}$$

as $|z| \to \infty$, outside a possible set of r of finite logarithmic measure. A contradiction to the assumption (9.12). □

We consider the following examples showing the sharpness of the above theorems.

Example 9.5 Ruijsenaars [36] considers the equation

$$F(z + ia/2) = \Phi(z)F(z - ia/2), \tag{9.17}$$

where $a > 0$, $\Phi(z) = 2\cosh \pi z/b$ and $b > 0$. The solution

$$G_{\text{hyp}}(a, b; z) = \exp\left(i\left(\int_0^\infty \frac{\sin(2yz)}{2\sinh(ay)\sinh(by)} - \frac{a}{aby}\right)\frac{dy}{y}\right),$$
$$|\Im z| < (a+b)/2, \tag{9.18}$$

which has no zeros and poles in $|\Im z| < (a+b)/2$, can be continued meromorphically to the whole complex plane via (9.17). The poles and zeros of (9.18) are given, respectively, by

$$\begin{aligned} z &= -i(k+1/2)a - i(\ell+1/2)b, \\ z &= -i(k+1/2)a + i(\ell+1/2)b, \quad k,\ell \in \mathbf{N}. \end{aligned} \tag{9.19}$$

The function (9.18) is called the *hyperbolic gamma function*. It follows from (9.19) that the order of $G_{\mathrm{hyp}}(r,a;z)$ is 2. Thus we have $\sigma(G_{\mathrm{hyp}}) = \sigma(\Phi)+1$. We would like to mention that Ruijsenaars [36] also considers (9.17) where Φ is the *trigonometric gamma function* and the *elliptic gamma function* respectively. We again have $\sigma(G_{\mathrm{ell}}) = \sigma(\Phi)+1$, and $\sigma(G_{\mathrm{trig}}) = \sigma(\Phi)+1$ to hold. We also remark that all the three types of generalized gamma functions mentioned above converge to the Euler Gamma function while taking suitable limits of the parameters.

Example 9.6 The following equation was considered in Hayman and Thatcher [20], which has a different form from (9.17). Let $H>0$, then the equation

$$F(z) = (1+H^z)F(z+1) \tag{9.20}$$

admits a meromorphic solution of the form

$$F_1(z) = \prod_{n=1}^{\infty} \left(1+H^{z-n}\right)^{-1} \tag{9.21}$$

with simple poles at

$$z_{k,n} = n + \frac{(2k+1)\pi i}{\log H}, \tag{9.22}$$

where $n = 1,2,3,\ldots$ and k is an integer [20, Theorem 1]. It follows from (9.22) that F_1 has order 2, giving $\sigma(F_1) = \sigma(1+H^z)+1$ so that the "equality" holds in Theorem 9.2 again.

The next example shows that the assumption (9.12) where only one coefficient is allowed to have the highest degree is the best possible.

Example 9.7 ([26]) Let

$$\Delta^n f(z) = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} f(z+j) \tag{9.23}$$

and hence

$$f(z+n) = \sum_{j=0}^{n} \binom{n}{j} \Delta^j f(z). \tag{9.24}$$

Then the equation

$$z(z-1)(z-2)\Delta^3 f(z-3)+z(z-1)\Delta^2 f(z-2)+z\Delta f(z-1)+(z+1)f(z)=0 \tag{9.25}$$

admits an entire solution of order 1/3. In fact, it is shown in [26] that

$$\log M(r,f)=Lr^{1/3}((1+o(1)).$$

By making use the relation (9.24), we can rewrite (9.23) to an equation of the form (9.13) with

$$\deg P_3=\deg P_2=\deg P_1=\deg P_0=3.$$

Thus, there are more than one polynomial coefficients having the same degree (>0) and the equation admits an entire solution of order <1. The above example shows that we cannot drop the assumption (9.12) in Theorem 9.4.

We finally remark that (1.4) and its solution (1.5) show that the lower order one estimate in the Theorem 9.4 is again the best possible.

10 Discussion

In this paper, we have discussed in detail some basic properties of $T(r,f(z+\eta))$, for a fixed η. In particular, we have shown in Theorem 2.1 that the relation (1.3) holds for finite order meromorphic functions and that the Theorem 2.3 shows that no such relation (1.3) can hold for infinite order meromorphic functions. The proof of (1.3) depends on Theorem 2.4 which can be viewed as a discrete analogue of the classical logarithmic derivative estimate given by Nevanlinna [18, 32] and the relation (2.3) in Theorem 2.2 on the counting function. These special properties of finite order meromorphic functions distinguish themselves from general meromorphic functions. They are in strong agreement with the integrability detector of difference equations proposed in [1].

It is worthwhile to note that the integrability test by Nevanlinna theory proposed in [1] is being complex analytic in nature, which is in stark contrast when compared to several other major integrability tests for difference equations [9, 11, 33, 34] proposed in the last decade. In fact, the *Nevanlinna test* seems more natural when compared to the well-known complex analytic *Painlevé test* as an integrability test for second order ordinary differential equations; see [2, p. 362]. We mention that the prime integrable difference equations are the discrete Painlevé equations which can be obtained from the classical Painlevé differential equations [12] via suitable discretizations.

Although the investigation in [1] is for non-linear second order difference equations, we have found that it natural to consider *linear difference equations*. This is based on the following facts. First our investigation leads us to give an answer of a Whittaker's problem (Corollary 9.3), which is amongst the most basic results of first order difference equations from the viewpoint of Nevanlinna theory. Second, we use our main result (Theorem 2.1) to give a simple proof of the main result (Theorem 9.1) in [1].

Linear difference equations are generally accepted as integrable. From the viewpoint in [1], it is therefore natural to demand that meromorphic solutions to linear equations should also be of finite order of growth. However, the discussion in Sect. 1 and Theorem 9.2 indicate that meromorphic solutions to (1.4) could have an arbitrarily fast growth. We give a lower bound order estimate of a finite order meromorphic solution, if any, of a linear equation (Theorem 9.2; see also Theorem 9.4). Thus one must impose certain *minimal growth* condition to single out the minimal solution (and finite order, if any). The question here is that what determines a *minimal solution*. The Whittaker theorem (Corollary 9.3) shows that minimal solution always exist for first order equation with an arbitrary entire coefficient in terms of order of growth. The problem of *minimal solution* is investigated in [8] for certain first order difference equations where the meromorphic solutions has prior growth restriction in an infinite strip. The distinction of different *minimal solutions* is also discussed in [20].

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