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On the New Solutions of the Conformable Time Fractional Generalized Hirota-Satsuma Coupled KdV System

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Abstract. In this paper, generalized Hirota Satsuma coupled KdV system is solved with tanh method and q-Homotopy analysis method. New fractional derivative definition called "conformable fractional derivative" used in the solution procedure. Tanh method with conformable derivative firstly introduced in the literature. By the graphics of analytical and approximate solutions, it is shown that, both methods provide an effective and powerful mathematical tool for solving nonlinear PDEs containing conformable fractional derivative.

AMS Subject Classification (2000). 35R11; 35A20; 35C05.
Keywords. Tanh Method; Hirota-Satsuma Couple KdV System; q-Homotopy Analysis Method; Conformable Fractional Derivative.

1 Introduction

In this paper we apply the tanh method and q-Homotopy Analysis method for solving fractional generalized Hirota-Satsuma couple KdV system

$$\begin{aligned}
\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{1}{4}\frac{\partial^3 u}{\partial x^3} + 3u\frac{\partial u}{\partial x} + 3\frac{\partial(-v^2+w)}{\partial x}\\
\frac{\partial^\alpha v}{\partial t^\alpha} &= -\frac{1}{2}\frac{\partial^3 v}{\partial x^3} - 3u\frac{\partial v}{\partial x}\\
\frac{\partial^\alpha w}{\partial t^\alpha} &= -\frac{1}{2}\frac{\partial^3 w}{\partial x^3} - 3u\frac{\partial w}{\partial x}
\end{aligned} \tag{1.1}$$

where the fractional derivatives are in the conformable sense [1]. The generalized Hirota-Satsuma coupled KdV system is an example of a universal nonlinear model that describes many physical nonlinear systems. So, it is important to find the solutions for this equation. The system corresponds to a special case of the Toda lattice equation, a well-known soliton equation in one space and one time dimension, which can be used for modelling the interaction of neighboring particles of equal mass in a lattice formation with a crystal. The generalized Hirota-Satsuma coupled KdV system has many applications in many branches of nonlinear science. For example this equation can be applied to the field of thermodynamics, where it can be used to exactly calculate partition and correlation functions. Describing generic properties of string dynamics for strings and multi-strings in constant curvature space can be thought as another application of the generalized Hirota-Satsuma coupled KdV system.

The fractional differential equations arises in different branches of applied sciences such as engineering, applied mathematics, biology and physics [2–7]. So scientists are making effort to develop fractional calculus theory. As a result of this effort, a well behaved, applicable, efficient definition for fractional derivative and integral are stated by Khalil *et al.* [1].

Definition 1.1. *Let $f : [0,\infty) \to \mathbb{R}$ be a function. The α^{th} order "conformable fractional derivative" of f is defined by,*

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0, \alpha \in (0,1)$.

Definition 1.2. *If f is α-differentiable in some $(0,a), a > 0$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists then define $f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t)$. The "conformable fractional integral" of a function f starting from $a \geq 0$ is defined as:*

$$I_\alpha^a(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

Here are some properties which satisfied by this new derivative.

Theorem 1.1 ([1]). *Let $\alpha \in (0,1]$ and suppose f, g are α-differentiable at point $t > 0$. Then*

1. *$T_\alpha(cf + dg) = cT_\alpha(f) + cT_\alpha(g)$ for all $a, b \in \mathbb{R}$.*
2. *$T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.*
3. *$T_\alpha(\lambda) = 0$ for all constant functions $f(t) = \lambda$.*
4. *$T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.*
5. *$T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.*
6. *In addition, if f differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$.*

With the help of these definitions scientists can overcome some deficiencies of other definitions. Hence it is worthwhile to work on this subject. Up to now, many scientists have paid great attention to this new definition [8–12]. Practicability and easy applicability of this definition permits us many different applications on this new area. For example; T. Abdeljawad [13] has deduced fractional versions of the chain rule, exponential functions, Gronwalls inequality, integration by parts, Taylor power series expansions and Laplace transform. Conformable time-scale fractional calculus has been pointed out by N. Benkhettoua *et al.* [14]. Moreover , M.A. Hammad and R. Khalil [15] gave the solution for the conformable fractional heat equation and W.S. Chung [16] employed the conformable fractional derivative and integral while discussing on fractional Newtonian mechanics. Also, Atangana *et al.* [17] have investigated in more detail some new properties of conformable derivative and have proved some useful related theorems.

The rest of this article is organized as follows; in section (2), the analytical solution of conformable time fractional generalized Hirota-Satsuma coupled KdV system using tanh method is presented, in section (3), q-HAM is applied to obtain series solution of the equation involved. In section (4) the graphics for analytical and approximate solutions are given and in section (5), we give some conclusion.

2 Analytical Solution of Time Fractional Generalized Hirota-Satsuma coupled KdV system by Tanh Method

In this section, firstly we give brief summary of tanh method first and then employ it to solve the time fractional generalized Hirota-Satsuma coupled KdV system.

2.1 The Tanh Method

Now lets summarize the method [18] step by step.
Step1. Consider the nonlinear conformable fractional partial differential equation in the form

$$P\left(\frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial u}{\partial x}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \frac{\partial^2 u}{\partial x^2}, ...\right) = 0 \tag{2.1}$$

where P is a polynomial and in its arguments and subscripts denote partial derivatives.
Step2. Regarding the transformation

$$u(x,t) = u(\xi), \xi = mx + n\frac{t^\alpha}{\alpha} \tag{2.2}$$

in which m is wave number and n is the travelling wave velocity. Depending on this:

$$\frac{\partial^\alpha(.)}{\partial t^\alpha} = n\frac{d(.)}{d\xi}, \frac{\partial(.)}{\partial x} = m\frac{d(.)}{d\xi}, \ldots . \tag{2.3}$$

Using Eq.(2.2), Eq. (2.1) becomes a nonlinear ordinary differential equation

$$G(U, U', U'', U''', ...) = 0 \tag{2.4}$$

where the derivatives is with respect to ξ.

Step3. Now, introduce a new independent variable

$$Y = \tanh(\xi) \tag{2.5}$$

which results in the following equalities.

$$\begin{aligned} \frac{\partial}{\partial \xi} &= (1-Y^2)\frac{\partial}{\partial Y}, \\ \frac{\partial^2}{\partial \xi^2} &= -2Y(1-Y^2)\frac{\partial}{\partial Y} + (1-Y^2)^2\frac{\partial^2}{\partial Y^2}, \\ \frac{\partial^3}{\partial \xi^3} &= 2(1-Y)^2(3Y^2-1)\frac{\partial}{\partial Y} - 6Y(1-Y^2)^2\frac{\partial^2}{\partial Y^2} + ((1-Y^2)^3)\frac{\partial^3}{\partial Y^3}. \end{aligned} \tag{2.6}$$

Step4. Introduce the ansatz

$$U(\xi) = S(Y) = \sum_{i=0}^{r} a_i Y^i \tag{2.7}$$

were r is a positive integer, in most cases, that will be determined. Substituting (2.7) into the ODE (2.4) yields an equation in powers of Y.

Step5. To determine the parameter r, we balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms. After determining r, equate the coefficients of Y to zero in the eventuating equation. This yields a system of algebraic equations including m, n and $a_i, (i = 0, 1, \ldots, r)$. Having determined these parameters, considering that r is a positive integer, and using (2.7) we have an analytic solution in a closed form.
To understand the details of the tanh method the analysis are presented in useful discussions in [19, 20].

2.2 Application of the Tanh Method on Time Fractional Generalized Hirota-Satsuma coupled KdV system

Considering the time fractional generalized Hirota-Satsuma coupled KdV system (1.1) where $\alpha \in (0, 1)$ and the fractional derivatives of $u(x, t)$ are in conformable sense. Using (2.2) and (2.3), the system (1.1) becomes

$$\begin{aligned} nu' &= \frac{1}{4}m^3u''' + 3muu' + 3m(-v^2 + w)', \\ nv' &= -\frac{1}{2}m^3v''' - 3muv', \\ nw' &= -\frac{1}{2}m^3w''' - 3muw' \end{aligned} \tag{2.8}$$

where the prime denotes the derivation with respect to ξ. Then denoting

$$Y = \tanh(\xi) \tag{2.9}$$

and

$$\begin{aligned} u = S(Y) &= \sum_{i=0}^{p} a_i Y^i, \\ v = R(Y) &= \sum_{i=0}^{q} b_i Y^i, \qquad (2.10) \\ w = T(Y) &= \sum_{i=0}^{j} c_i Y^i \end{aligned}$$

and using (2.10), (2.9) and (2.6) in resulting system (2.8), we get

$$\begin{aligned} n(1-Y^2)\frac{dS}{dY} &= \frac{1}{4}m^3\left(2(1-Y)^2(3Y^2-1)\frac{dS}{dY} - 6Y(1-Y^2)^2\frac{d^2S}{dY^2} + ((1-Y^2)^3)\frac{d^3S}{dY^3}\right) \\ &+ 3mS(1-Y^2)\frac{dS}{dY} - 6mR(1-Y^2)\frac{dR}{dY} + 3m(1-Y^2)\frac{dT}{dY}, \\ n(1-Y^2)\frac{dR}{dY} &= -\frac{1}{2}m^3\left(2(1-Y)^2(3Y^2-1)\frac{dR}{dY} - 6Y(1-Y^2)^2\frac{d^2R}{dY^2} + ((1-Y^2)^3)\frac{d^3R}{dY^3}\right) \\ &- 3mS(1-Y^2)\frac{dR}{dY}, \qquad (2.11) \\ n(1-Y^2)\frac{dT}{dY} &= -\frac{1}{2}m^3\left(2(1-Y)^2(3Y^2-1)\frac{dT}{dY} - 6Y(1-Y^2)^2\frac{d^2T}{dY^2} + ((1-Y^2)^3)\frac{d^3T}{dY^3}\right) \\ &- 3mS(1-Y^2)\frac{dR}{dY}, \end{aligned}$$

Now balancing the highest order linear terms in the resulting equation with the highest order nonlinear terms in system (2.11), we obtain $p = 2, q = 2, j = 2$. Using obtained values in (2.10), substituting into (2.11), equating all coefficients yields an algebraic equation system for $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2, m$ and n. Solving this system with aid of Mathematica we obtain two cases;

Case A:

$$b_0 = \frac{\sqrt{3}c_1\sqrt{m\left(m^3+2n\right)}}{2\left(-m^4-2mn\right)}, \qquad (2.12)$$

$$\begin{aligned} c_2 &= 0, \\ a_0 &= \frac{m^3-n}{3m}, \\ b_1 &= -\frac{\sqrt{m^4+2mn}}{\sqrt{3}}, \\ b_2 &= 0, \qquad (2.13) \\ a_2 &= -m^2, \\ a_1 &= 0. \end{aligned}$$

So using (2.2), (2.9), (2.10) and (2.13) the analytical solutions can be found as

$$u(x,t) = \frac{m^3-n}{3m} - m^2\left(\tanh\left(mx+\frac{nt^\alpha}{\alpha}\right)\right)^2,$$

$$v(x,t) = \frac{\sqrt{3}c_1\sqrt{m\left(m^3+2n\right)}}{2\left(-m^4-2mn\right)} - \frac{\sqrt{m^4+2mn}\tanh\left(mx+\frac{nt^\alpha}{\alpha}\right)}{\sqrt{3}},$$

$$w(x,t) = c_0 + c_1\tanh\left(mx+\frac{nt^\alpha}{\alpha}\right).$$

Case B:

$$\begin{aligned} b_0 &= \frac{-3c2+4m^4-4mn}{6m^2}, && (2.14)\\ a_0 &= \frac{4m^3-n}{3m},\\ c_1 &= 0,\\ b_1 &= 0,\\ b_2 &= -m^2, && (2.15)\\ a_2 &= -2m^2,\\ a_1 &= 0. \end{aligned}$$

Hence again using (2.2), (2.9), (2.10) and (2.15) the analytical solutions can be found as

$$u(x,t) = \frac{4m^3-n}{3m} - 2m^2\left(\tanh\left(mx+\frac{nt^\alpha}{\alpha}\right)\right)^2,$$

$$v(x,t) = \frac{-3c_2+4m^4-4mn}{6m^2} - m^2\left(\tanh\left(mx+\frac{nt^\alpha}{\alpha}\right)\right)^2, \qquad (2.16)$$

$$w(x,t) = c_0 + c_2\left(\tanh\left(mx+\frac{nt^\alpha}{\alpha}\right)\right)^2.$$

3 Approximate Analytical Solution of Time Fractional Generalized Hirota-Satsuma coupled KdV system

In this section, we apply the q-homotopy analysis method to time fractional generalized Hirota-Satsuma coupled KdV system. We make this particular choice of parameters for comparison purposes.

Consider the time fractional Generalized Hirota-Satsuma coupled KdV system (1.1) with initial conditions

$$u(x,0) = 1 - 2(\tanh x)^2$$

$$v(x,0) = -\frac{1}{2} - (\tanh x)^2 \qquad (3.1)$$

$$w(x,0) = (\tanh x)^2$$

where $\alpha \in (0,1)$ and the derivative by means of conformable fractional derivative. For convenience and to shorten the article, the exact solutions are taken as (2.16) and $m = 1$, $n = 1$, $c_2 = 1$, $c_0 = 0$ are used for all calculations. One can easily make the calculations for all of the other solutions and other values of m, n, c_2, c_0.
To obtain the series solution of system (1.1) with initial conditions (3.1),we choose the linear operators as

$$\mathcal{L}\left[\varphi(x,t;q)\right] = D_t^{\alpha}\varphi(x,t;q)$$
$$\mathcal{L}\left[\psi(x,t;q)\right] = D_t^{\alpha}\psi(x,t;q)$$
$$\mathcal{L}\left[\omega(x,t;q)\right] = D_t^{\alpha}\omega(x,t;q)$$

with the property

$$\mathcal{L}\left[s\right] = 0$$

where s is constant. Lets define the nonlinear operators as follows,

$$\mathcal{N}\left[\varphi(x,t;q),\psi(x,t;q),\omega(x,t;q)\right] =$$
$$\frac{\partial^{\alpha}\varphi(x,t;q)}{\partial t^{\alpha}} - \frac{1}{4}\frac{\partial^3\varphi(x,t;q)}{\partial x^3} - 3\varphi(x,t;q)\frac{\partial\varphi(x,t;q)}{\partial x} - 3\frac{\partial(-\psi(x,t;q)^2+\omega(x,t;q))}{\partial x}.$$

$$\mathcal{N}\left[\varphi(x,t;q),\psi(x,t;q),\omega(x,t;q)\right] =$$
$$\frac{\partial^{\alpha}\psi(x,t;q)}{\partial t^{\alpha}} + \frac{1}{2}\frac{\partial^3\psi(x,t;q)}{\partial x^3} + 3\varphi(x,t;q)\frac{\partial\psi(x,t;q)}{\partial x}. \tag{3.2}$$
$$\mathcal{N}\left[\varphi(x,t;q),\psi(x,t;q),\omega(x,t;q)\right] = \frac{\partial^{\alpha}\omega(x,t;q)}{\partial t^{\alpha}} + \frac{1}{2}\frac{\partial^3\omega(x,t;q)}{\partial x^3} + 3\varphi(x,t;q)\frac{\partial\omega(x,t;q)}{\partial x}.$$

From Theorem 1.3, the nonlinear operators can be written as follows,

$$\mathcal{N}\left[\varphi(x,t;q),\psi(x,t;q),\omega(x,t;q)\right] =$$
$$t^{1-\alpha}\frac{\partial\varphi(x,t;q)}{\partial t} - \frac{1}{4}\frac{\partial^3\varphi(x,t;q)}{\partial x^3} - 3\varphi(x,t;q)\frac{\partial\varphi(x,t;q)}{\partial x} - 3\frac{\partial(-\psi(x,t;q)(x,t;q)^2+\omega(x,t;q))}{\partial x},$$
$$\mathcal{N}\left[\varphi(x,t;q),\psi(x,t;q),\omega(x,t;q)\right] =$$
$$t^{1-\alpha}\frac{\partial\psi(x,t;q)}{\partial t} + \frac{1}{2}\frac{\partial^3\psi(x,t;q)}{\partial x^3} + 3\varphi(x,t;q)\frac{\partial\psi(x,t;q)}{\partial x}, \tag{3.3}$$
$$\mathcal{N}\left[\varphi(x,t;q),\psi(x,t;q),\omega(x,t;q)\right] =$$
$$t^{1-\alpha}\frac{\partial\omega(x,t;q)}{\partial t} + \frac{1}{2}\frac{\partial^3\omega(x,t;q)}{\partial x^3} + 3\varphi(x,t;q)\frac{\partial\omega(x,t;q)}{\partial x}.$$

Thus the zero-order deformation equations are set up as:

$$(1-nq)\mathcal{L}\left[\varphi(x,t;q) - u_0(x,t)\right] = qh\mathcal{N}\left[\varphi(x,t;q),\psi(x,t;q),\omega(x,t;q)\right].$$
$$(1-nq)\mathcal{L}\left[\psi(x,t;q) - v_0(x,t)\right] = qh\mathcal{N}\left[\varphi(x,t;q),\psi(x,t;q),\omega(x,t;q)\right].$$
$$(1-nq)\mathcal{L}\left[\omega(x,t;q) - w_0(x,t)\right] = qh\mathcal{N}\left[\varphi(x,t;q),\psi(x,t;q),\omega(x,t;q)\right].$$

Choosing $H(x,t) = 1$, the mth-order deformation equations are

$$\mathcal{L}\left[u_m(x,t) - \chi_m^* u_{m-1}(x,t)\right] = hR_{m,1}\left(\mathbf{u}_{m-1},\mathbf{v}_{m-1},\mathbf{w}_{m-1}\right)$$

$$\mathcal{L}\left[v_m(x,t)-\chi_m^* v_{m-1}(x,t)\right]=hR_{m,2}\left(\mathbf{u}_{m-1},\mathbf{v}_{m-1},\mathbf{w}_{m-1}\right) \tag{3.4}$$
$$\mathcal{L}\left[w_m(x,t)-\chi_m^* w_{m-1}(x,t)\right]=hR_{m,3}\left(\mathbf{u}_{m-1},\mathbf{v}_{m-1},\mathbf{w}_{m-1}\right)$$

with initial condition for $m \geq 1$, $u_m(x,0)=0, v_m(x,0), w_m(x,0)$, χ_m^* is defined as

$$\chi_m^* = \begin{cases} 0, & m \leqslant 1 \\ n, & otherwise, \end{cases} \tag{3.5}$$

and

$$\begin{aligned} R_{m,1}\left(\mathbf{u}_{m-1},\mathbf{v}_{m-1},\mathbf{w}_{m-1}\right) &= t^{1-\alpha}\frac{\partial u_{m-1}(x,t)}{\partial t}-\frac{1}{4}\frac{\partial^3 u_{m-1}(x,t)}{\partial x^3} \\ &-3\sum_{n=0}^{m-1} u_n(x,t)\frac{\partial u_{m-1-n}(x,t)}{\partial x} \\ &+6\sum_{n=0}^{m-1} v_n(x,t)\frac{\partial v_{m-1-n}(x,t)}{\partial x}-3\frac{\partial w_{m-1}(x,t)}{\partial x}, \end{aligned}$$

$$\begin{aligned} R_{m,2}\left(\mathbf{u}_{m-1},\mathbf{v}_{m-1},\mathbf{w}_{m-1}\right) &= t^{1-\alpha}\frac{\partial v_{m-1}(x,t)}{\partial t}+\frac{1}{2}\frac{\partial^3 v_{m-1}(x,t)}{\partial x^3} \\ &+3\sum_{n=0}^{m-1} u_n(x,t)\frac{\partial v_{m-1-n}(x,t)}{\partial x}, \end{aligned}$$

$$\begin{aligned} R_{m,3}\left(\mathbf{u}_{m-1},\mathbf{v}_{m-1},\mathbf{w}_{m-1}\right) &= t^{1-\alpha}\frac{\partial w_{m-1}(x,t)}{\partial t}+\frac{1}{2}\frac{\partial^3 v_{m-1}(x,t)}{\partial x^3} \\ &+3\sum_{n=0}^{m-1} u_n(x,t)\frac{\partial w_{m-1-n}(x,t)}{\partial x}. \end{aligned}$$

The solutions of the mth-order deformation equations (3.4) for $m \geq 1$ result in

$$\begin{aligned} u_m(x,t) &= \chi_m^* u_{m-1}(x,t)+h\mathcal{L}^{-1}\left[R_{m,1}\left(\mathbf{u}_{m-1},\mathbf{v}_{m-1},\mathbf{w}_{m-1}\right)\right], \\ v_m(x,t) &= \chi_m^* v_{m-1}(x,t)+h\mathcal{L}^{-1}\left[R_{m,2}\left(\mathbf{u}_{m-1},\mathbf{v}_{m-1},\mathbf{w}_{m-1}\right)\right], \\ w_m(x,t) &= \chi_m^* w_{m-1}(x,t)+h\mathcal{L}^{-1}\left[R_{m,3}\left(\mathbf{u}_{m-1},\mathbf{v}_{m-1},\mathbf{w}_{m-1}\right)\right]. \end{aligned} \tag{3.6}$$

By using equations (3.6) with initial conditions given by (3.1) we successively obtain the solutions

$$\begin{aligned} u_0(x,t) &= -\frac{8ht^\alpha(-1+2\cosh(2x))(\mathrm{sech}x)^4\tanh(x)}{\alpha}, \\ u_1(x,t) &= \frac{h^2t^\alpha(\mathrm{sech}x)^8\left(t^\alpha(368-201\cosh(2x)-60\cosh(4x)+5\cosh(6x))\right)}{4\alpha^2} \\ &- \frac{h^2t^\alpha(\mathrm{sech}x)^8\left(32\alpha(\cosh x)^3(-2\sinh(x)+\sinh(3x))\right)}{4\alpha^2} \\ &- \frac{8hnt^\alpha(-1+2\cosh(2x))(\mathrm{sech}x)^4\tanh(x)}{\alpha}, \\ &\vdots \end{aligned}$$

$$
\begin{aligned}
v_0(x,t) &= \frac{2ht^{\alpha}(\mathrm{sech}x)^2\tanh(x)}{\alpha},\\
v_1(x,t) &= \frac{h^2t^{\alpha}(\mathrm{sech}x)^8\left(t^{\alpha}(352-585\cosh(2x)+216\cosh(4x)+\cosh(6x))\right)}{16\alpha^2}\\
&+ \frac{h^2t^{\alpha}(\mathrm{sech}x)^8\left(32\alpha(\cosh x)^5\sinh(x)\right)}{16\alpha^2}\\
&+ \frac{2hnt^{\alpha}(\mathrm{sech}x)^2\tanh(x)}{\alpha},\\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
w_0(x,t) &= -\frac{2ht^{\alpha}(\mathrm{sech}x)^2\tanh(x)}{\alpha},\\
w_1(x,t) &= -\frac{h^2t^{\alpha}(\mathrm{sech}x)^8\left(t^{\alpha}(352-585\cosh(2x)+216\cosh(4x)+\cosh(6x))\right)}{16\alpha^2}\\
&- \frac{h^2t^{\alpha}(\mathrm{sech}x)^8\left(32\alpha(\cosh x)^5\sinh(x)\right)}{16\alpha^2}\\
&- \frac{2hnt^{\alpha}(\mathrm{sech}x)^2\tanh(x)}{\alpha},\\
&\vdots \quad .
\end{aligned}
$$

We can also obtain $u_m(x,t)$, $v_m(x,t)$, $w_m(x,t)$ for $m = 2,3,4,\cdots$, following the same approximation, using Mathematica, Maple or MATLAB.

Finally the series solution expression by q-HAM can be written in the form

$$u(x,t,n,h) = 1 - 2(\tanh x)^2 + \sum_{n=1}^{\infty} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i, \tag{3.7}$$

$$v(x,t,n,h) = -\frac{1}{2} - (\tanh x)^2 + \sum_{n=1}^{\infty} v_i(x,t;n;h)\left(\frac{1}{n}\right)^i, \tag{3.8}$$

$$w(x,t,n,h) = (\tanh x)^2 + \sum_{n=1}^{\infty} w_i(x,t;n;h)\left(\frac{1}{n}\right)^i. \tag{3.9}$$

Equations (3.8) is an appropriate solution to the problem (2.10) in terms of convergence parameter h and n.

4 Graphics of the Solutions

In this section, we give the graphics for both solutions. We explain the way how the parameter h can be to get to a good approximation.

4.1 The $\hbar$-curve

The auxiliary parameter h and n, which are contained in our q-HAM solution series, provide us with a simple way to arrange and check the convergence of the solution series. To obtain a proper range for $\hbar$, we consider the so-called $\hbar$-curves which are shown in Figures (1)-(3). We choose an appropriate value of $\hbar$ which guarantee that the series solution is convergent, as pointed by Liao [21], by finding the valid region of $\hbar$ which corresponds to the line segments nearly parallel to the horizontal axis.

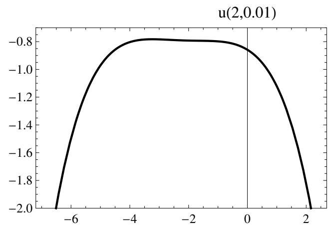

Figure 1: The $\hbar$-curve of $u(x, t)$ for 4th-order approximate solutions obtained by the q-HAM for $\alpha = 0.8$ and $n = 2$.

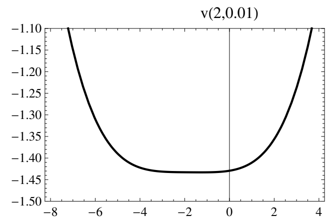

Figure 2: The $\hbar$-curve of $v(x, t)$ for 4th-order approximate solutions obtained by the q-HAM for $\alpha = 0.8$ and $n = 2$.

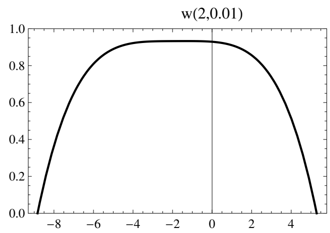

Figure 3: The $\hbar$-curve of $w(x, t)$ for 4th-order approximate solutions obtained by the q-HAM for $\alpha = 0.8$ and $n = 2$.

Now lets give the graphics of all the solutions.

5 Conclusion

In this paper, we present new exact and approximate solutions of time conformable fractional generalized Hirota-Satsuma coupled KdV system, which are found by using tanh method and q-HAM respectively. Authors show that the q-HAM solution, converges very rapidly to the exact one which is obtained using tanh method by choosing a

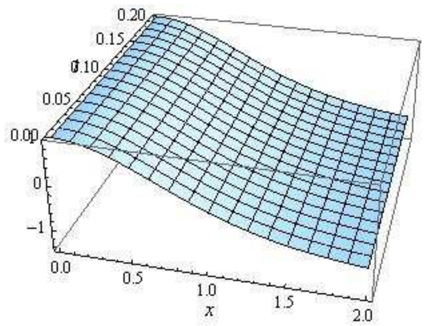

Figure 4: Graphic of $u(x,t)$ for the 4th-order approximate solutions obtained by the q-HAM for $\alpha = 0.8$, $\hbar = -0.0001$ and $n = 2$.

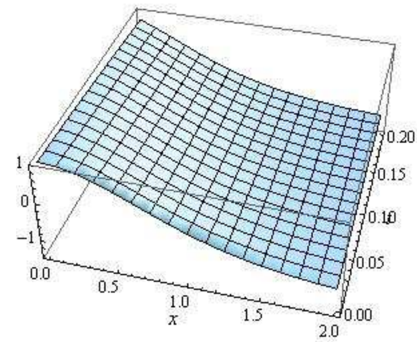

Figure 5: Graphic of $u(x,t)$ for the analytical solution for $\alpha = 0.8$.

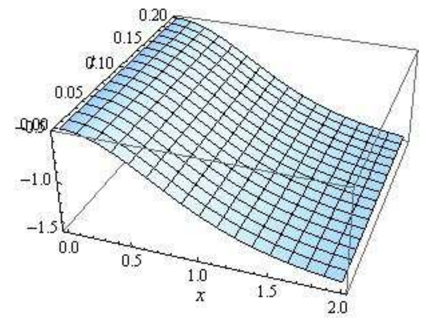

Figure 6: Graphic of $v(x,t)$ for the 4th-order approximate solutions obtained by the q-HAM for $\alpha = 0.8$, $\hbar = -0.0001$ and $n = 2$.

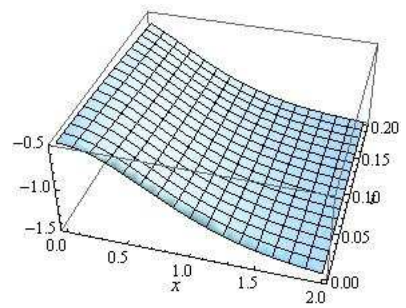

Figure 7: Graphic of $v(x,t)$ for the analytical solution for $\alpha = 0.8$.

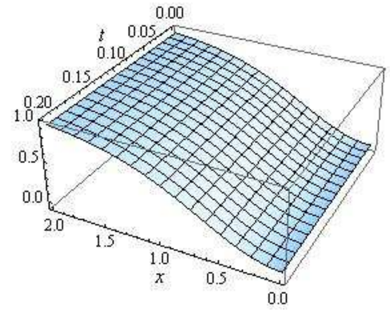

Figure 8: Graphic of $w(x,t)$ for the 4th-order approximate solutions obtained by the q-HAM for $\alpha = 0.8$, $\hbar = -0.0001$ and $n = 2$.

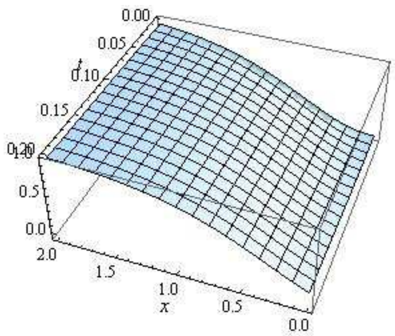

Figure 9: Graphic of $w(x,t)$ for the analytical solution for $\alpha = 0.8$.

proper auxiliary parameter from given figures. Thus, it is deducted that both methods give reliable and effective results for solving conformable fractional nonlinear equations. From the Figures 4-9, it is seen that the solutions are compatible with each other. In this way, we conclude that the applied methods can be used to solve many nonlinear time-fractional partial differential equation systems. So called conformable fractional derivative definition is a convenient definition in the exact solution procedure of fractional differential equations. Conformable fractional derivative provides convenience both in applicability of methods and solution procedures as its derivative definition does not include any integral term.

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