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On the Newton Polytope of the Resultant*

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Introduction

The study of Newton polytopes of resultants and discriminants has its orgin in the work of Gelfand, Kapranov, and Zelevinsky on generalized hypergeometric functions (see e.g., [8]). Central to this theory is the notion of the *$\mathcal{A}$-discriminant* $\Delta_{\mathcal{A}}$, which is the discriminant of a Laurent polynomial with specified support set $\mathcal{A}$ (see [6, 7]). Two main results of Gelfand, Kapranov, and Zelevinsky are concerned with their *secondary polytope* $\Sigma(\mathcal{A})$. First, the vertices of this polytope are in bijection with the coherent triangulations of $\mathcal{A}$, and, secondly, the secondary polytope $\Sigma(\mathcal{A})$ approximates the Newton polytope of the $\mathcal{A}$-discriminant $\Delta_{\mathcal{A}}$. It was observed in [6, Proposition 1.3.1] that resultants are special instances of $\mathcal{A}$-discriminants, and this observation was used in [9] to give an explicit combinatorial description of the Newton polytope of the classical Sylvester resultant.

Subsequent papers extended the theory of Gelfand, Kapranov and Zelevinsky into several different directions. In [11] the $\mathcal{A}$-resultant was introduced, and its interpretation as the Chow form of an associated toric variety leads to a refined geometric understanding of the relationship between triangulations of $\mathcal{A}$ and monomials in $\Delta_{\mathcal{A}}$. In [3] the concept of secondary polytopes was extended to the more geometric construction of *fiber polytopes*. Product formulas of Poisson type, first given for the $\mathcal{A}$-discriminant in [6, §2F], were proved in [14] for general Chow forms, for the $\mathcal{A}$-resultant, and for the *sparse mixed resultant*.

The present paper continues this line of research, but it is self-contained. Our main result is a combinatorial construction of the Newton polytope $\mathcal{N}(\mathcal{R})$ of the *sparse mixed resultant* $\mathcal{R}$. To define these terms, we let $\mathcal{A}_0, \mathcal{A}_1, \dots \mathcal{A}_n \subset \mathbf{Z}^n$ be subsets which jointly span the affine lattice $\mathbf{Z}^n$, and $card(\mathcal{A}_i) =: m_i$. Then $\mathcal{R}$ is the unique (up to scaling) irreducible polynomial in $m := m_0 + m_1 + \cdots + m_n$ variables $c_{i,\mathbf{a}}$, which vanishes whenever the Laurent polynomials

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$$f_i(x_1,\ldots,x_n) = \sum_{\mathbf{a}\in\mathcal{A}_i} c_{i,\mathbf{a}}\mathbf{x}^{\mathbf{a}} \qquad (i = 0, 1, \ldots, n) \tag{1}$$

have a common zero in $(\mathbf{C}^*)^n$. The *Newton polytope* $\mathcal{N}(\mathcal{R})$ is the convex hull in $\mathbf{R}^m$ of the exponent vectors of all monomials appearing with nonzero coefficient in $\mathcal{R}$.

This paper is organized as follows. In Section 1 we collect some basics, including the precise definition of the sparse mixed resultant, and a dimension formula for the variety of solvable systems (Theorem 1.1). Section 2 deals with the monomials corresponding to vertices of $\mathcal{N}(\mathcal{R})$, which are called the *extreme monomials*. Let $Q_i := conv(\mathcal{A}_i)$ denote the Newton polytopes of the Laurent polynomials in (1), and let $Q := Q_0 + Q_1 + \cdots + Q_n$ be their Minkowski sum. We present a combinatorial construction for the extreme monomials of $\mathcal{R}$ using mixed polyhedral decompositions of Q (Theorem 2.1).

Canny and Emiris [5] recently gave an efficient algorithm, based on a determinantal formula, for computing the sparse mixed resultant. In Section 3 we generalize the Canny-Emiris formula by showing that for each extreme monomial m of $\mathcal{R}$ there exists a determinant as in [5], for which m appears as a factor of the main diagonal product.

We say that a polytope P is a *resultant polytope* if $P = \mathcal{N}(\mathcal{R})$ for some $\mathcal{A}_0,\ldots,\mathcal{A}_n$. In Section 4 we prove that all faces of resultant polytopes are Minkowski sums of resultant polytopes. We express each initial form $init_\omega(\mathcal{R})$ of the sparse mixed resultant as a product of resultants corresponding to subsets of the $\mathcal{A}_i$ (Theorem 4.1). For each extreme monomial of $\mathcal{R}$ we determine the exact coefficient, which is either -1 or $+1$ (Proposition 4.2).

In Section 5 we examine the relationship between the sparse mixed resultants and the $\mathcal{A}$-discriminant. We give a bijection between the coherent triangulations of the auxiliary set $\mathcal{A} = \cup_{i=0}^n \mathcal{A}_i \times \{\mathbf{e}_i\}$ and the tight coherent mixed decompositions of $\mathcal{A}_0, \mathcal{A}_1,\ldots,\mathcal{A}_n$. Theorem 5.4 states that the secondary polytope $\Sigma(\mathcal{A})$ is strongly isomorphic to a certain fiber polytope, which, in the notation of [3], can be expressed as

$$\Sigma(\Delta_{m_0-1} \times \Delta_{m_1-1} \times \cdots \times \Delta_{m_n-1}, \mathcal{A}_0 + \mathcal{A}_1 + \cdots + \mathcal{A}_n) \tag{2}$$

We show that the resultant polytope $\mathcal{N}(\mathcal{R})$ is a Minkowski summand of (2).

In Section 6 we explore combinatorial properties of resultant polytopes. We characterize the edges of $\mathcal{N}(\mathcal{R})$ in terms of *mixed circuits*, and we use this to show that the resultant polytope has the same dimension as the fiber polytope (2), namely $dim(\mathcal{N}(\mathcal{R})) = m - 2n - 1$. We characterize all resultant polytopes of dimensions 2 and 3 (Corollary 6.3).

For readers familiar with the theory of $\mathcal{A}$-discriminants [6], we summarize our progress:

(a) Our theorems do not require the smoothness hypothesis on the toric variety $X_{\mathcal{A}}$. This restrictive hypothesis makes it impossible to derive our results directly from [6].

(b) The new polyhedral interpretation of the extreme terms of the sparse mixed resultant is in dimension n, while the polyhedral interpretation derived from [6] is in dimension $2n$. In the interesting cases $n = 2, 3$ this increases the practical applicability a lot.
(c) Our proofs are elementary and constructive. Techniques such as intersection cohomology sheaves, determinants of Cayley-Koszul complexes, etc. are not needed.
(d) We give a combinatorial rule for the *Δ-equivalence* of coherent triangulation [6, Remark 3D.21] in the special case of supports arising from the Cayley trick (see (5.41)).

1. Preliminaries on the sparse mixed resultant

Let $\mathcal{A}_0, \ldots, \mathcal{A}_n$ be subsets of $\mathbf{Z}^n$ and $Q_i = conv(\mathcal{A}_i)$ their convex hulls in $\mathbf{R}^n$. For any subset $J \subset \{0, \ldots, n\}$, we consider the affine lattice generated by $\sum_{j\in J} \mathcal{A}_j$, that is,

$$\mathcal{L}_J := \left\{ \sum_{j\in J} \lambda_j \mathbf{a}^{(j)} \mid \mathbf{a}^{(j)} \in \mathcal{A}_j,\ \lambda_j \in \mathbf{Z} \text{ for all } j \in J \text{ and } \sum_{j\in J} \lambda_j = 1 \right\}.$$

Let *rank(J)* denote the rank of $\mathcal{L}_J$. A subcollection of supports $\{\mathcal{A}_i\}_{j\in I}$ is said to be *essential* if

$$rank(I) = card(I) - 1 \text{ and } rank(J) \geq card(J)$$
$$\text{for each proper subset } J \text{ of } I.$$

The vector of coefficients $c_{i,\mathbf{a}}$ of a system (1) defines a point in the product of complex projective spaces $P^{m_0-1} \times \cdots \times P^{m_n-1}$. Let Z denote the subset of those system (1) which have a solution $\mathbf{x}$ in $(\mathbf{C}^*)^n$, and let $\overline{Z}$ be its closure in $P^{m_0-1} \times \cdots \times P^{m_n-1}$.

LEMMA 1.1 [14]. *The projective variety $\overline{Z}$ is irreducible and defined over $\mathbf{Q}$.*

Proof. Let W denote the incidence correspondence in $(\mathbf{C}^*)^n \times (P^{m_0-1} \times \cdots \times P^{m_n-1})$ defined by the equations (1). It is defined over $\mathbf{Q}$ and has codimension $n+1$. Also, since W is a vector bundle over the irreducible variety $(\mathbf{C}^*)^n$, it is irreducible. Let π denote the projection onto the second factor. Then $\pi(W) = \overline{Z}$ is irreducible and defined over $\mathbf{Q}$. □

We now define the *sparse mixed resultant.* If $codim(\overline{Z}) = 1$ then $\mathcal{R}$ is the unique (up to sign) irreducible polynomial in $\mathbf{Z}[\ldots, c_{i,\mathbf{a}}, \ldots]$ which vanishes on the hypersurface $\overline{Z}$. If $codim(\overline{Z}) \geq 2$ then $\mathcal{R}$ is defined to be the constant 1. Using Bernstein's Theorem [1], the following result was derived in [14].

LEMMA 1.2. *Suppose that* $\{\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n\}$ *is essential. For all* $i \in \{0, 1, \ldots, n\}$ *the degree of* $\mathcal{R}$ *in the ith group of variables* $\{c_{i,\mathbf{a}}, \mathbf{a} \in \mathcal{A}_i\}$ *is a positive integer, equal to the mixed volume*

$$\mathcal{M}(Q_0, \ldots, Q_{i-1}, Q_{i+1} \ldots, Q_n) = \sum_{J \subset \{0,\ldots,i-1,i+1\ldots,n\}} (-1)^{card(J)} vol\left(\sum_{j \in J} Q_j\right). \tag{3}$$

We next determine the codimension of the variety $\overline{Z}$ of solvable systems (1).

THEOREM 1.1. *The codimension of* $\overline{Z}$ *in* $P^{m_0-1} \times \cdots \times P^{m_n-1}$ *equals the maximum of the numbers card(I)–rank(I), where I runs over all subsets of* $\{0, 1, \ldots, n\}$.

Proof. We first show that $codim(\overline{Z})$ is bounded below by $card(I)-rank(I)$, for each I. Let W and π be as in the proof of Lemma 1.1, and let η be a generic point of $\overline{Z}$. Then

$$codim(\overline{Z}) + codim(\pi^{-1}(\eta)) = codim(W) = n + 1,$$

and hence $codim(\overline{Z}) = dim(\pi^{-1}(\eta)) + 1$. Therefore we need to show that

$$dim(\pi^{-1}(\eta)) \geq card(I) - rank(I) - 1.$$

After relabeling we may assume $I = \{0, 1, \ldots, c-1\}$ and $rank(I) = r$. By a multiplicative change of coordinates $x_i \mapsto \prod_{j=1}^{n} z_j^{\nu_{ij}}$ on $(\mathbf{C}^*)^n$, our system (1) transforms into

$$f_0(z_1, \ldots, z_r) = \cdots = f_{c-1}(z_1, \ldots, z_r) = f_c(z_1, \ldots, z_r, z_{r+1}, \ldots, z_n) = \cdots = f_n(z_1, \ldots, z_r, z_{r+1}, \ldots, z_n) = 0. \tag{4}$$

Now, fixing η amounts to fixing coefficients $c_{i,\mathbf{a}}$ such that (4) is solvable in $(\mathbf{C}^*)^n$. We need to determine the dimension of $\pi^{-1}(\eta)$, which is the solution variety of (4). For any choice of $(z_1, \ldots, z_r) \in (\mathbf{C}^*)^r$ satisfying the first c equations, we are left with $n + 1 - c$ equations in $n - r$ indeterminates $z_{r+1}, \ldots, z_n$. This defines a subvariety of $\pi^{-1}(\eta)$, having dimension $\geq (n - r) - (n + 1 - c) = c - r - 1$. Therefore $dim(\pi^{-1}(\eta)) \geq c - r + 1$.

To show the reverse inequality, we continue to assume that the maximum of $card(I)-rank(I)$ is attained for $I = \{0, 1, \ldots, c-1\}$ and $r = rank(I)$. After relabeling if necessary, we can assume that $rank(\{c - r, c - r + 1, \ldots, k\}) \geq k - c + r + 1$ for all $k = c - r, \ldots, n$. This implies that $rank(J) \geq card(J)$ for each subset J of $\{c - r, \ldots, n\}$. By Bernstein's theorem [1], the generic system of equations $f_{c-r} = \cdots = f_n = 0$ has a solution $\mathbf{x}$ in $(\mathbf{C}^*)^n$. For each of the remaining $r - c$ equations $f_0 = \cdots = f_{c-r-1} = 0$ we can arbitrarily select all but one of the coefficients, while maintaining $\mathbf{x}$ as a common root of all $n + 1$

equations. This shows that all but $c - r$ of the coefficients $c_{i,a}$ in (1) can be chosen arbitrarily, while maintaining solvability. Hence $codim(\overline{Z}) \leq c - r$. □

Here is a combinatorial criterion for the existence of a nontrivial resultant. Note that if each Q_i is n-dimensional then the criterion in Corollary 1.1 holds for $I = \{0, 1, \ldots, n\}$.

COROLLARY 1.1. *The variety $\overline{Z}$ has codimension 1 if and only if there exists a unique subset $\{\mathcal{A}_i\}_{i \in I}$ which is essential. In this case the sparse mixed resultant $\mathcal{R}$ coincides with the resultant of the equations $\{f_i : i \in I\}$, considered with respect to the lattice $\mathcal{L}_I$.*

Example 1.1. For the linear system

$$c_{00}x + c_{01}y = c_{10}x + c_{11}y = c_{20}x + c_{21}y + c_{22} = 0, \tag{5}$$

the variety $\overline{Z}$ has codimension 1 in $P^1 \times P^1 \times P^2$. The unique essential subset consists of the first two equations. Hence the sparse mixed resultant of (5) is *not* the 3×3-determinant (which would be reducible), but it equals its cofactor

$$\mathcal{R} = c_{00}c_{11} - c_{10}c_{01}. \tag{6}$$

This phenomenon has nothing to do with the equations (5) being linear. For instance, let $f(x, y)$ be any polynomial with at least three terms, and consider the nonlinear system:

$$c_{00}x^4y^7 + c_{01}x^8y^2 = c_{10}x^5y^5 + c_{11}x^9 = f(x, y) = 0. \tag{7}$$

Then the sparse mixed resultant of (7) is also equal to (6).

Proof of Corollary 1.1. If $codim(\overline{Z}) = 1$ then, by Theorem 1.1, there exists an index set I with $card(I) = rank(I) + 1$, for instance $I = \{0, 1, \ldots, n\}$. Choose I to be minimal with respect to inclusion. Then I is essential. To show uniqueness, suppose that I and J are distinct essential index sets. Then $I \cap J$ is a proper subset of I, hence

$$\begin{aligned} rank(I \cup I) &= rank(I) + \mathrm{rank}(J) - rank(I \cap J) \\ &= (card(I) - 1 + (card(J) - 1) - card(I \cap J) = card(I \cup J) - 2, \end{aligned}$$

which means that $codim(\overline{Z}) \geq 2$, by Theorem 1.1

This argument is reversible: if there is a unique minimal essential index set I, then I attains the maximum in Theorem 1.1 and we have $codim(\overline{Z}) = 1$.

Let $\mathcal{R}_I$ denote the resultant of the equations $\{f_i : i \in I\}$, which we consider with respect to the lattice $\mathcal{L}_I$. By Lemma 1.2, $\mathcal{R}_I$ is a nonconstant polynomial, which involves coefficients from each of the $card(I)$ groups of variables. It is irreducible and vanishes on $\overline{Z}$, so it defines the irreducible hypersurface $\overline{Z}$. Hence $\mathcal{R} = \pm\mathcal{R}_I$. □

Many of the statements in this paper require the hypothesis that the family of supports $\{\mathcal{A}_0, \ldots, \mathcal{A}_n\}$ is essential. Corollary 1.1 guarantees that this is no loss in generality. The following simple class of resultants will become important later on.

PROPOSITION 1.1. *Suppose that* $\mathcal{A}_i = \{\mathbf{a}_{i1}, \mathbf{a}_{i2}\} c \mu^n$ *for* $i = 0, \ldots, n$, *and rank* $(\sum_{i=0}^n \mathcal{A}_i) = n$.

(i) *There is unique (up to sign) primitive vector* $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ in $\mathbf{Z}^{n+1}$ *satisfying*

$$\sum_{i=0}^{n} \lambda_i (\mathbf{a}_{i1} - \mathbf{a}_{i2}) = 0.$$

(ii) *The sparse mixed resultant equals*

$$\mathcal{R} = (-1)^{\lambda_0 + \lambda_1 + \cdots + \lambda_n} \cdot \prod_{i:\lambda_i > 0} c_{i1}^{\lambda_i} \cdot \prod_{j:\lambda_j < 0} c_{j2}^{-\lambda_j} - \prod_{i:\lambda_i > 0} c_{i2}^{\lambda_i} \cdot \prod_{j:\lambda_j < 0} c_{j1}^{-\lambda_j}. \tag{8}$$

Proof. The map $\mathbf{Z}^{n+1} \mapsto \mathbf{Z}^n$, $\mu \mapsto \sum_i \mu_i (\mathbf{a}_{i1} - \mathbf{a}_{i2})$ is onto a sublattice of rank n. In part (i) we take λ to be a generator of the kernel, which is a rank 1 lattice. If the system

$$c_{01}\mathbf{x}^{\mathbf{a}_{01}} + c_{02}\mathbf{x}^{\mathbf{a}_{02}} = c_{11}\mathbf{x}^{\mathbf{a}_{11}} + c_{12}\mathbf{x}^{\mathbf{a}_{12}} = \cdots = c_{n1}\mathbf{x}^{\mathbf{a}_{n1}} + c_{n2}\mathbf{x}^{\mathbf{a}_{n2}} = 0$$

has a solution $\mathbf{x} \in (\mathbf{C}^*)^n$, then

$$\prod_{i=0}^{n} (-c_{i2}/c_{i1})^{\lambda_i} = \prod_{i=0}^{n} (\mathbf{x}^{\mathbf{a}_{i1} - \mathbf{a}_{i2}})^{\lambda_i} = 1,$$

which means the polynomial in (8) vanishes. Since λ is primitive, it is irreducible. To show that it coincides with the sparse mixed resultant $\mathcal{R}$, it suffices to show that $codim(\overline{Z}) = 1$. But this follows easily from Theorem 1.1, in view of $rank(\sum_{i=0}^n \mathcal{A}_i) = n$. □

In the situation of Proposition 1.1, the unique essential index set equals $I = \{i : \lambda_i \neq 0\}$, the support of λ.

2. The extreme monomials

Each monomial $\prod c_{i,\mathbf{a}}^{\nu_{i,\mathbf{a}}}$ in the coefficients of our system (1) is identified with a nonnegative integer vector $(\ldots, \nu_{i,\mathbf{a}}, \ldots)$ in $\mathbf{R}^m$. Let ω by any linear functional on $\mathbf{R}^m$. We represent ω by a collection of functions $\omega_i : \mathcal{A}_i \to \mathbf{R}$, $i = 0, 1, \ldots, n$. The value of the linear functional ω at the point $(\ldots, \nu_{i,\mathbf{a}}, \ldots)$ equals $\sum \omega_i(\mathbf{a}) \cdot \nu_{i,\mathbf{a}}$.

This number is the *weight* of the monomial $\prod c_{i,\mathbf{a}}^{\nu_{i,\mathbf{a}}}$ with respect to ω. The *initial form* $init_\omega(\mathcal{R})$ is the sum of all terms of maximum weight in the sparse mixed resultant $\mathcal{R}$.

We consider the lifted polytopes

$$Q_{i,\omega} := conv\{(\mathbf{a}, \omega_i(\mathbf{a})) : \mathbf{a} \in \mathcal{A}_i\} \subset \mathbf{R}^{n+1}. \tag{9}$$

The upper envelope of $Q_{i,\omega}$ defines a *coherent polyhedral subdivision* $\Delta_{i,\omega}$ of Q_i with vertices in $\mathcal{A}_i$, for each $i = 0, 1, \ldots, n$. The cells of $\Delta_{i,\omega}$ are the projections of precisely those faces of $Q_{i,\omega}$ on which a linear functional with negative last coordinate is minimized (see [3, 6, 12] for details).

Similarly, we get a coherent polyhedral subdivision Δ_ω of the Minkowski sum $Q = Q_0 + Q_1 + \cdots + Q_n$ by taking the upper envelope of $Q_{0,\omega} + Q_{1,\omega} + \cdots + Q_{n,\omega}$. Any such subdivision of Q is called a *coherent mixed decomposition*, or *CMD*, for short. Each facet (= cell of codimension 1) in Δ_ω is of the form

$$F = F_0 + F_1 + \cdots + F_n, \tag{10}$$

where F_i is a cell in $\Delta_{i,\omega}$. We have

$$n = dim(Q) = dim(F) \le dim(F_0) + dim(F_1) + \cdots + dim(F_n). \tag{11}$$

If the linear functional ω is sufficiently generic, then equality holds in (11) for all facets F of Δ_ω (cf. [2]). In this case the CMD Δ_ω is called a *tight mixed coherent decomposition*, or *TCMD*, for short. A facet F of a TCMD Δ_ω is said to be *mixed of type* i if $dim(F_i) = 0$, and $dim(F_j) = 1$ for all $j \neq i$. In this case the face F_i is just a point in $\mathcal{A}_i$, say $F_i = \{\mathbf{a}\}$, and we write $c_{iF_i} := c_{i,\mathbf{a}}$ for the corresponding coefficient.

Our first main theorem describes a natural surjection from the set of TCMDs of Q onto the set of extreme monomials of the sparse mixed resultant $\mathcal{R}$. The exact coefficient of each extreme monomial will be determined later (Corollaries 3.1 and Proposition 4.2)

THEOREM 2.1. *Suppose that* $\{\mathcal{A}_0, \ldots, \mathcal{A}_n\}$ *is essential. Then the initial form of the sparse mixed resultant* $\mathcal{R}$ *with respect to a generic* ω *equals the monomial*

$$init_\omega(\mathcal{R}) = const. \cdot \prod_{i=0}^{n} \prod_{F} c_{i,F_i}^{vol(F)}, \tag{12}$$

where vol $(\cdot)$ *denotes ordinary Euclidean volume, and the second product is over all mixed facets of type* i *of the TCMD* Δ_ω.

Proof. Let t denote a new variable. The resultant of the deformed system

$$f_i^t(x_1, \ldots, x_n; t) = \sum_{\mathbf{a} \in \mathcal{A}_i} c_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}} t^{-\omega_i(\mathbf{a})} \quad (i = 0, 1, \ldots, n) \tag{13}$$

equals $\mathcal{R}' = \mathcal{R}(\ldots, c_{i,\mathbf{a}}t^{-\omega_i(\mathbf{a})}, \ldots)$. If we expand $\mathcal{R}'$ as a Laurent polynomial in t then the coefficient of the lowest term equals $init_\omega(\mathcal{R})$. We will show that this lowest coefficient equals $p \cdot \prod_F c_{0,F_0}^{vol(F_0)}$, where p is a rational function in the $c_{i,\mathbf{a}}$ for $i \geq 1$. By symmetry, this implies that $init_\omega(\mathcal{R})$ equals the right-hand side of (12), as desired.

The product formula for the sparse mixed resultant [14, Theorem 1.1] states that

$$\mathcal{R}' = p \cdot \prod_{\gamma(t)} \left(\sum_{a \in \mathcal{A}_0} c_{0,\mathbf{a}} \gamma(t)^{\mathbf{a}} t^{-\omega_0(\mathbf{a})} \right) \tag{14}$$

where p is a certain rational function in $\{c_{i,\mathbf{a}} : i = 1, \ldots, n\}$, and $\gamma(t)$ runs over all roots of $f_1' = \cdots = f_n' = 0$ in $(\overline{\mathbf{C}(t)}^*)^n$. By Bernstein's Theorem [1], the number of roots $\gamma(t)$ equals the mixed volume $\mathcal{M}(Q_1, \ldots, Q_n)$.

We view each root $\gamma(t)$ of $f_1' = \cdots = f_n' = 0$ as an algebraic function $\mathbf{C}^* \to (\mathbf{C}^*)^n$ in t, and we consider the Puiseux series of this algebraic curve for t close to the origin:

$$\gamma(t) = (\gamma_1 t^{\lambda_1}, \ldots, \gamma_n t^{\lambda_n}) + \text{ componentwise higher terms in } t \tag{15}$$

Here $\lambda = (\lambda_1, \ldots, \lambda_n)$ runs over a finite subset of $\mathbf{Q}^n$ which is to be determined. We substitute (15) in to the equation (13) for $i = 1, \ldots, n$:

$$f_i'(\gamma_1 t^{\lambda_1}, \ldots, \gamma_n t^{\lambda_n}; t) = \sum_{\mathbf{a} \in \mathcal{A}_i} c_{i,\mathbf{a}} \gamma^{\mathbf{a}} t^{\lambda \cdot \mathbf{a} - \omega_i(\mathbf{a})} + \text{ higher terms in } t. \tag{16}$$

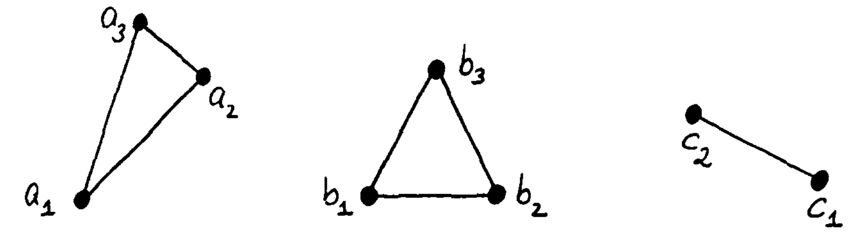
Here $\gamma = (\gamma_1, \ldots, \gamma_n)$. Consider the face of $Q_{i,\omega}$ on which the linear functional $(\lambda, -1) = (\lambda_1, \ldots, \lambda_n, -1)$ attains its minimum, and let F_i denote its projection into $Q_i \subset \mathbf{R}^n$, for $i = 0, 1, \ldots, n$. The Minkowski sum $F = F_0 + F_1 + \cdots + F_n$ is a face (possibly of lower dimension) of the TCMD Δ_ω. Equating the lowest degree coefficient in (16) to zero, we get the identity

$$\sum_{\mathbf{a} \in \mathcal{A}_i \cap F_i} c_{i,\mathbf{a}} \gamma^{\mathbf{a}} = 0 \quad (i = 1, \ldots, n) \tag{17}$$

In order for λ to contribute a branch (15), it is necessary that (17) has a solution γ in $(\mathbf{C}^*)^n$. This implies $dim(F_i) \geq 1$ for $i = 1, \ldots, n$. Since ω is generic, we have equality in (11), and F is a mixed facet of type 0 of Δ_ω. In other words, $dim(F_i) = 1$ for $i = 1, \ldots, n$, and $dim(F_0) = 0$, say $F_0 = \{\bar{\mathbf{a}}\}$.

We now consider the factor of (14) indexed by our specific branch $\gamma(t) = \gamma \cdot t^\lambda + \cdots$. This factor equals (16) for $i = 0$. Its lowest coefficient with respect to t is $c_{o,\bar{\mathbf{a}}}\gamma^{\bar{\mathbf{a}}} = c_{o,F_0}\gamma^{F_0}$. The product of the expressions $c_{o,F_0}\gamma^{F_0}$ over all roots γ of (17) equals $c_{o,F_0}^{vol(F)}$ times a rational function p in $\{c_{i,\mathbf{a}} : i \geq 1\}$. Here we are using the fact that

$$\mathcal{M}(F_1, \ldots, F_n) = |det(F_1, \ldots, F_n)| = vol(F). \tag{18}$$



$Q_0 = conv(\mathcal{A}_0)$ $Q_1 = conv(\mathcal{A}_1)$ $Q_2 = conv(\mathcal{A}_2)$

Figure 1. The Newton polytopes of the system (19).

To get the lowest t-coefficient of $\mathcal{R}'$ in (14) we now take the product over the expressions $c_{o,F_0}^{vol(F)}$, where F runs over all type 0 mixed faces $F = F_0 + F_1 + \cdots + F_n$ of Δ_ω. This completes the proof of Theorem 2.1. □

Remark 2.1. The analysis is steps (15) to (18) of the above proof is used in [10] to give a numerical homotopy algorithm for solving semimixed sparse systems.

We illustrate Theorem 2.1 and our results in the later sections for an easy example of a sparse bivariate system.

Example 2.1. Let $\mathcal{A}_0 = \{(0, 0), (2, 2), (1, 3)\}$, $\mathcal{A}_1 = \{(0, 0), (2, 0), (1, 2)\}$, and $\mathcal{A}_2 = \{(3, 0), (1, 1)\}$, and consider the system (Figure 1)

$$\begin{aligned} f_0 &= a_1 + a_2x^2y^2 + a_3xy^3 \\ f_1 &= b_1 + b_2x^2 + b_3xy^2 \\ f_2 &= c_1x^3 + c_2xy. \end{aligned} \tag{19}$$

Here the sparse mixed resultant equals

$$\begin{aligned} \mathcal{R} = {} & \underline{a_1^5b_3^7c_1^6c_2} + 3a_1^4a_2b_2^2b_3^5c_1^4c_2^3 + 3a_1^3a_2^2b_2^4b_3^3c_1^2c_2^5 - 13a_1^3a_2a_3b_1^2b_2b_3^4c_1^5c_2^2 \\ & - 7a_1^3a_3^2b_1b_2^3b_3^3c_1^4c_2^3 + 6a_1^2a_2^3b_1^3b_2b_3^3c_1^4c_2^3 + \underline{a_1^2a_2^3b_2^6b_3c_2^7} \\ & - a_1^2a_2^2a_3b_1^2b_2^3b_3^2c_1^3c_2^4 + 5a_1^2a_2a_3^2b_1^4b_3^3c_1^6c_2 - a_1^2a_2a_3^2b_1b_2^5b_3c_1^2c_2^5 \\ & + 14a_1^2a_3^3b_1^3b_2^2b_3^2c_1^5c_2^2 + \underline{a_1^2a_3^3b_2^7c_1c_2^6} - 2a_1a_2^4b_1^3b_2^3b_3c_1^2c_2^5 \\ & - 5a_1a_2^3a_3b_1^5b_3^2c_1^5c_2^2 + \underline{a_2^5b_1^6b_3c_1^4c_2^3} + 2a_1a_2^2a_3^2b_1^4b_2^2b_3c_1^4c_2^3 \\ & - 2a_1a_2a_3^3b_1^3b_2^4c_1^3c_2^4 - 7a_1a_3^4b_1^5b_2b_3c_1^6c_2 + \underline{a_2^2a_3^3b_1^6b_2c_1^5c_2^2} + \underline{a_3^5b_1^7c_1^7}. \end{aligned}$$

Note that the degree in each group of variables agrees with the mixed volumes:

$$\begin{aligned} & deg_a(\mathcal{R}) = \mathcal{M}(P_1, P_2) = 5, deg_b(\mathcal{R}) = \mathcal{M}(P_0, P_2) = 7, \\ & deg_c(\mathcal{R}) = \mathcal{M}(P_0, P_1) = 7. \end{aligned}$$

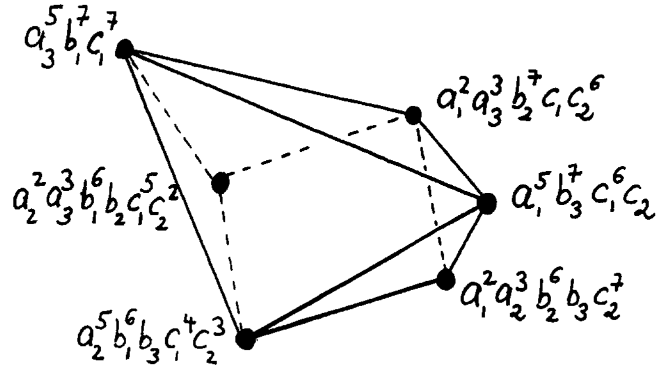


Figure 2. The Newton polytope $\mathcal{N}(\mathcal{R})$.

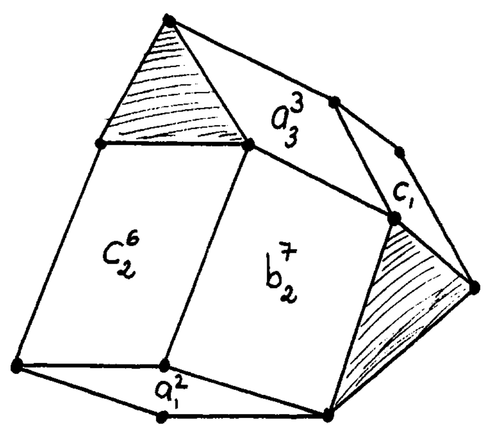


Figure 3. A tight coherent mixed decomposition (TCMD).

The extreme terms of the resultant are precisely the six underlined monomials. The Newton polytope of $\mathcal{R}$ is a 3-dimensional polytope, which looks like Figure 2 (cf. [9, Figure 1]).

The vertices of $\mathcal{N}(\mathcal{R})$ are in one-to-many correspondence with the TCMD's Δ_ω of the octagon $Q = Q_0 + Q_1 + Q_2$. For instance, for $\omega = (1, 0, 0; 7, 13, 0; 0, 0)$ we get the initial monomial $init_\omega(\mathcal{R}) = a_1^2 a_3^3 b_2^7 c_1 c_2^6$ and the TCMD shown in Figure 3.

The initial systems (17) corresponding to the five mixed facets in Figure 3 are

type 0: $\{b_1 + b_2x^2, c_1x^3 + c_2xy\}$,

facet = [(3, 0), (5, 0), (1, 1), (3, 1)], vol = 2, coeff = a_1.

type 0: $\{b_2x^2 + b_3xy^2, c_1x^3 + c_2xy\}$,

facet = [(6, 3), (4, 4), (5, 5), (3, 6)], vol = 3, coeff = a_3.

type 1: $\{a_1 + a_3xy^3, c_1x^3 + c_2xy\}$,

facet = [(6, 3), (5, 0), (3, 1), (4, 4)], vol = 7, coeff = b_2.

type 2: $\{a_2x^2y^2 + a_3xy^3, b_2x^2 + b_3xy^2\}$,

facet = [(6, 3), (7, 2), (6, 4), (5, 5)], vol = 1, coeff = c_1.

type 2: $\{a_1 + a_3xy^3, b_1 + b_2x^2\}$,

facet = [(1, 1), (3, 1), (2, 4), (4, 4)], vol = 6, coeff = c_2.

3. Determinantal formulas of Canny-Emiris type

In [10] a restricted class of coarse mixed decompositions of the Minkowski sum $Q = Q_0 + Q_1 + \cdots + Q_n$ was introduced, and it was applied to give a numerical algorithm for finding all roots of a system of polynomial equations. We say that a CMD Δ_ω is *coarse* if its defining linear functional ω satisfies following system of linear constraints: $(*)$ *For each* i, $\omega_i : \mathcal{A}_i \to \mathbf{R}$ *is the restriction of an affine-linear function on* $\mathbf{R}^n$. If ω is generic relative to these constraints, then we call Δ_ω a *coarse TCMD*.

Canny and Emiris [5] applied these coarse decompositions to give an efficient algorithm for computing the sparse mixed resultant. More precisely, for each coarse TCMD Δ_ω they constructed a square matrix $\mathcal{M}_\omega$ of size roughly $card(Q \cap \mathbf{Z}^n)$ and having entries $c_{i,\mathbf{a}}$ and 0, whose determinant is a nonzero multiple of $\mathcal{R}$. A key point of their construction is that the extreme term $init_\omega(\mathcal{R})$ appears on the main diagonal of the matrix $\mathcal{M}_\omega$.

In what follows we generalize this construction by removing the hypothesis $(*)$. In the light of Theorem 2.1, our new result can be stated as follows: *for every extreme term of the sparse mixed resultant there exists a determinantal formula of Canny-Emiris type.*

Let ω be any linear functional on $\mathbf{R}^m$ such that Δ_ω is a TCMD of Q. Proceeding as in [5, §2], we fix a generic vector $\delta \in \mathbf{Q}^n$ and we set $\mathcal{E} := \mathbf{Z}^n \cap (\delta + Q)$. The *row content* of an element $\mathbf{p} \in \mathcal{E}$ is a pair $[i, \mathbf{a}]$, which is defined as follows: Let $F = F_0 + F_1 + \cdots + F_n$ be the unique facet of Δ_ω which contains $\mathbf{p} - \delta$ in its interior, let i be the largest index such that $dim(F_i) = 0$, and let $F_i = \{\mathbf{a}\}$. Note that if F is a mixed face, then i is its type.

We define a square matrix $\mathcal{M}_{\omega,\delta}$ with both rows and columns indexed by $\mathcal{E}$ as follows: The entry indexed $(\mathbf{p}, \mathbf{p}')$ equals the coefficient of $\mathbf{x}^{\mathbf{p}'}$ in the expansion of the polynomial $\mathbf{x}^{\mathbf{p}-\mathbf{a}} \cdot f_i(\mathbf{x})$, where $[i, \mathbf{a}]$ is the row content of $\mathbf{p}$. The following theorem is a direct generalization of the main result in [5, §3].

THEOREM 3.1. *The determinant of $\mathcal{M}_{\omega,\delta}$ equals the sparse mixed resultant $\mathcal{R}$ times a nonzero polynomial $P_{\omega,\delta}$ in the variables $c_{i,\mathbf{a}}$ for $i \geq 1$.*

Proof. If the system (1) has a root $\mathbf{x}$ in $(\mathbf{C}^*)^n$, then the matrix $\mathcal{M}_{\omega,\delta}$ has the nonzero vector $(\mathbf{x}^{\mathbf{p}} : \mathbf{p} \in \mathcal{E})$ in its kernel. Here we are using the fact that each monomial appearing in $\mathbf{x}^{\mathbf{p}-\mathbf{a}} \cdot f_i(\mathbf{x})$ does lie in $\mathcal{E}$. Therefore the zero set of $det(\mathcal{M}_{\omega,\delta})$ in the space of coefficients $c_{i,\mathbf{a}}$ contains the zero set of $\mathcal{R}$. Since the sparse mixed resultant $\mathcal{R}$ is irreducible (Lemma 1.1), we conclude that $\mathcal{R}$ divides $det(\mathcal{M}_{\omega,\delta})$.

We next show that $det(\mathcal{M}_{\omega,\delta}(c_{i,\mathbf{a}}))$ is not the zero polynomial. To this end we replace (1) by the deformed system (13) and consider the deformed matrix $\mathcal{M}_{\omega,\delta}(c_{i,\mathbf{a}}t^{-\omega_i(\mathbf{a})})$. For each $\mathbf{p} \in \mathcal{E}$, we multiply the row indexed $\mathbf{p}$ by $t^{h(\mathbf{p})-\omega_i(\mathbf{a})}$, where $h(\mathbf{p})$ is defined to be the smallest rational number such that $(\mathbf{p}-\delta, h(\mathbf{p})) \in Q_{0,\omega} + Q_{n,\omega} + \cdots + Q_{n,\omega}$. Call the resulting matrix $\mathcal{M}'(t)$. Its entry indexed $(\mathbf{p}, \mathbf{p}') \in \mathcal{E} \times \mathcal{E}$ equals

$$c_{i,\mathbf{a}'} \cdot t^{h(\mathbf{p})-\omega_i(\mathbf{a})+\omega_i(\mathbf{a}')} \quad \text{if } \mathbf{a}' := \mathbf{a} + \mathbf{p}' - \mathbf{p} \text{ lies in } \mathcal{A}_i, \tag{20}$$

and 0 otherwise. Here $[i, \mathbf{a}]$ is the row content of p.

By a convexity argument as in [5, Lemma 3.4] we see that, for $\mathbf{p}' \neq \mathbf{p}$,

$$h(\mathbf{p}') < h(\mathbf{p}) - \omega_i(\mathbf{a}) + \omega_i(\mathbf{a}').$$

Hence among the nonzero entries in each column the unique lowest power in t occurs on the main diagonal. The product over all main diagonal terms is the lowest term of the determinant:

$$det(\mathcal{M}'(t)) = \pm \prod_{\mathbf{p}\in\mathcal{E}} c_{i,\mathbf{a}} t^{h(\mathbf{p})} = \text{ higher terms in } t. \tag{21}$$

This proves that $det(\mathcal{M}'(1)) = det(\mathcal{M}_{\omega,\delta})$ is not the zero polynomial in $c_{i,\mathbf{a}}$.

It remains to be shown that the polynomial $P_{\omega,\delta} = det(\mathcal{M}_{\omega,\delta}(c_{i,a}))/\mathcal{R}$ contains none of the variables $c_{0,\mathbf{a}}$. Both the denominator and the numerator are homogeneous with respect to each group of variables $\{c_{i,\mathbf{a}} : \mathbf{a} \in \mathcal{A}_i\}$, and hence so is their quotient. It therefore suffices to consider the initial monomial. By Theorem 2.1 and (21), we have

$$init_\omega(P_{\omega,\delta}) = init_\omega(det(\mathcal{M}_{\omega,\delta}(c_{i,\mathbf{a}})))/init_\omega(\mathcal{R}) = \pm \prod_{\mathbf{p}} c_{i,\mathbf{a}},$$

where the product is over those $\mathbf{p} \in \mathcal{E}$ such that the facet $F = F_0 + F_1 + \cdots + F_n$ of Δ_ω which contains $\mathbf{p} - \delta$ is not mixed. In each such case there are at least two indices $i' < i$ satisfying $dim(F_{i'}) = dim(F_i) = 0$, and therefore the row content $[i, \mathbf{a}]$ of $\mathbf{p}$ satisfies $i \geq 1$. This completes the proof. □

Since $det(\mathcal{M}_{\omega,\delta})$ has integer coefficients and its factor $\mathcal{R}$ is irreducible over $\mathbf{Z}$, we can apply Gauss' lemma to conclude that their quotient $P_{\omega,\delta}$ has integer coefficients. The formula (21) implies the following result of Gelfand, Kapranov, and Zelevinsky.

COROLLARY 3.1. (cf. [6, Theorem 3A.2.b]) *All extreme monomials of the sparse mixed resultant have coefficient* -1 *or* $+1$.

A classical formula for the resultant of $n+1$ forms in $n+1$ variables is due to Macaulay [13]. It can be shown that Macaulay's matrix is a special case of the above matrix $\mathcal{M}_{\omega,\delta}$, for suitable choice of δ and ω. What is remarkable about Macaulay's paper is that he succeeds in giving an explicit irreducible factorization of the extraneous factor $P_{\omega,\delta}$ in terms of smaller determinants of the same type as $\mathcal{M}_{\omega,\delta}$.

It is an important open problem to find a more explicit formula for $P_{\omega,\delta}$ in the general sparse case. Does there exist such a formula in terms of some smaller resultants?

This problem is closely related to the following empirical observation. For suitable choice of δ and ϵ, the matrix $\mathcal{M}_{\delta,\epsilon}$ seems to have a block structure which allows to extract the resultant from a proper submatrix. This leads to faster algorithms for computing the sparse mixed resultant. J. Canny (personal communication) has reported some progress in this direction. We illustrate this phenomenon for our bivariate example.

Example 2.1 (continued). Let ω as before and $\delta = (0, 1/3)$. Then the set $\mathcal{E}$ contains 23 elements. Nineteen of these lie in the mixed cells of $\Delta_\omega + \delta$. The four remaining points are (2, 5), (2, 6), (1, 3), (1, 4). If we order the set $\mathcal{E}$ such that these four extraneous points come first, then our matrix has the structure

$$\mathcal{M}_{\omega,\delta} = \begin{pmatrix} c_2 \cdot \mathbf{I}_4 & * \\ \mathbf{0} & \mathcal{N} \end{pmatrix}, \tag{22}$$

where $\mathbf{I}_4$, denotes the 4×4-unit matrix and $\mathbf{0}$ denotes the 19×4-zero matrix. Here the extraneous factor equals simply $P_{\omega,\delta} = c_2^4$, and the resultant $\mathcal{R}$ can be computed exactly as the determinant of the 19×19-matrix $\mathcal{N}$. □

4. The initial forms

In this section we describe all initial forms of the sparse mixed resultant, that is, we consider the more general case when ω need not be generic. Our main result, Theorem 4.1, is a direct generalization of Theorem 2.1. We first need to recall some fine print of polytope theory. A polyhedral subdivision (such as a mixed decomposition, or a triangulation) is always a collection of labeled subsets of the given labeled multiset of points [3, 12]. Thus each facet $F = F_0 + F_1 + \cdots + F_n$ of a CMD Δ_ω is equipped with additional combinatorial data, consisting in a sequence of subsets $(\mathcal{A}'_0, \mathcal{A}'_1, \ldots, \mathcal{A}'_n)$, where $\mathcal{A}'_i \subset \mathcal{A}_i$ and $F_i = conv(\mathcal{A}'_i)$. Obviously, different subsets $\mathcal{A}'_i$ might have the same convex hull, so one has to be cautious.

Keeping this in mind, we now return to the usual (more sloppy) notation. For any facet $F = F_0 + \cdots + F_n$ of a CMD Δ_ω, where $F_i = conv(\mathcal{A}'_i)$, we define the restrictions:

$$f_i|_{F_i} := \sum_{\mathbf{a} \in \mathcal{A}'_i} c_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}. \tag{23}$$

With F we associate an integer d_F as follows. If $\{\mathcal{A}'_i : i = 0, 1, \ldots, n\}$ has the unique essential subset $\{A'_i\}_{i \in I}$, then

$$d_F := \frac{\mathcal{M}(F_0, \ldots, F_{i-1}, F_{i+1}, \ldots, F_n)}{\mathcal{M}_{\mathcal{L}_I}(\{F_j : j \in I \backslash i\})}, \quad \text{for any } i \in I.$$

Otherwise $d_F := 0$. In this formula $\mathcal{M}_{\mathcal{L}_I}$ is defined as follows. Let $\mathcal{L}_I$ denote the affine lattice spanned by $\sum_{i \in I} F_i \cap \mathcal{A}_i$, and consider the induced volume form on its real span, that is, an elementary simplex with vertices in $\mathcal{L}_I$ has volume 1. Then $\mathcal{M}_{\mathcal{L}_I}$ denotes the mixed volume associated with the normalized volume on $\mathcal{L}_I$.

By Corollary 1.1, d_F equals the unique integer such that $\mathcal{R}(f_0|_{F_0}, f_1|_{F_1}, \ldots, f_n|_{F_n})^{d_F}$ has total degree $\sum_{t=0}^{n} \mathcal{M}(F_0, \ldots, F_{l-1}, F_{l+1}, \ldots, F_n)$.

THEOREM 4.1. *Let $\{\mathcal{A}_0, \ldots, \mathcal{A}_n\}$ be essential, and let ω be any linear functional on $\mathbf{R}^m$. The initial form of the sparse mixed resultant equals*

$$init_\omega(\mathcal{R}) = \pm \prod_F \mathcal{R}(f_0|_{F_0}, f_1|_{F_1}, \ldots, f_n|_{F_n})^{d_F}, \tag{24}$$

where F runs over all facets of Δ_ω.

Each factor on the right-hand side of (24) is a sparse mixed resultant with respect to a different choice of supports, which are proper subsets of $\mathcal{A}_0, \ldots, \mathcal{A}_n$ respectively. At this point we recall that the sparse mixed resultant equals the constant 1 if the corresponding variety $\overline{Z}$ of solvable systems has codimension $\geq$ 2 in the coefficient space.

Let us illustrate the formula (24) in the case when ω is generic:

Alternative proof of Theorem 2.1, using Theorem 4.1. Since ω is generic, we have equality in (11) for each facet F of Δ_ω. Let F be any facet which is not mixed. There are at least two indices $i' < i'$ such that $dim(F_{i'}) = dim(F_i) = 0$. The

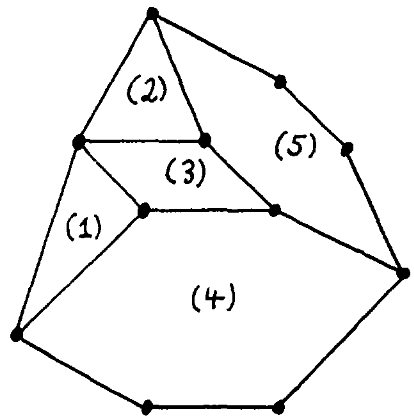


Figure 4. CMD corresponding to a facet of the Newton polytope $\mathcal{N}(\mathcal{R})$.

corresponding equations (23) are simply monomials $c_{F_i}\mathbf{x}^{F_i}$ and $c_{F_{i'}}\mathbf{x}^{F_{i'}}$, which have no zeros in the torus $(\mathbf{C}^*)^n$, unless $c_{F_i} = c_{F_{i'}} = 0$. This amounts to a condition of codimension ≥ 2. The corresponding factor on the right-hand side of (24) is simply 1. Hence the nonmixed facets do not contribute anything to the product in (24).

Now, let F be a mixed facet, say of type 0. Then (24) equals the system (17) augmented by the monomial equation $c_{F_0}\mathbf{x}^{c_{F_0}} = 0$. The resultant of that system equals the irreducible polynomial c_{F_0}, and the index d_F coincides with the determinant in (18). Thus each mixed facet of type 0 contributes $c_{F_0}^{vol(F)}$ to the right-hand product in (23). □

Before proving Theorem 4.1 let us first return to our example.

Example 2.1 (continued). To illustrate our formula (24), we consider the specific vector $\omega = (69, 0, 0; 11, -12, 0; 0, 0)$. This vector supports a facet of the 3-dimensional polytope $\mathcal{N}(\mathcal{R})$. The corresponding CMD Δ_ω looks like Figure 4.

Each of the five facets supports an initial system (24):

(1) $a_1 + a_2x^2y^2 + a_3xy^3 = b_1 = c_2xy = 0$;
(2) $a_3xy^3 = b_1 + b_2x^2 + b_3xy^2 = c_2xy = 0$;
(3) $a_2x^2y^2 + a_3xy^3 = b_1 + b_2x^2 = c_2xy = 0$;
(4) $a_1 + a_2x^2y^2 = b_1 + b_2x^2 = c_1x^3 + c_2xy = 0$;
(5) $a_2x^2y^2 + a_3xy^3 = b_2x^2 + b_3xy^2 = c_1x^3 + c_2xy = 0$.

The systems (1) and (2) each have the resultant 1, while the systems (3), (4), and (5) each contribute a nontrivial factor to the product

$$init_\omega(\mathcal{R}) = c_2^2 \cdot (a_1 b_2^3 c_2^2 - a_2 b_1^3 c_1^2)^2 \cdot (a_3^3 b_2 c_1 + a_2^3 b_3 c_2). \tag{25}$$

The multiplicities 2, 2, and 1 can be read off as lattice indices from Figure 4. Note that the monomial $a_1^2 a_3^3 b_2^7 c_1 c_2^6$ in Figure 3 appears as an extreme monomial in (25). □

Proof of Theorem 4.1. We first consider the following special case:

$$\begin{aligned} \mathcal{A} &:= \mathcal{A}_0 = \mathcal{A}_1 = \cdots = \mathcal{A}_n, \\ \varpi &:= \omega_0 = \omega_1 = \cdots = \omega_n. \end{aligned} \tag{26}$$

In other words, we assume that the system (1) is *unmixed* and all lifting functions are equal. In this case the sparse mixed resultant is called the *$\mathcal{A}$-resultant* and denoted $\mathcal{R}_\mathcal{A}$.

By the results of [11] (see also [15]), the $\mathcal{A}$-resultant coincides with the Chow form of the projective toric variety $X_\mathcal{A} \subset P^n$, and the initial form $init_\varpi(\mathcal{A})$ coincides with the Chow form of the algebraic cycle $init_\varpi(X_\mathcal{A})$. This cycle has the irreducible decomposition

$$init_\varpi(X_\mathcal{A}) = \sum_{\bar{F}} [\mathbf{Z}^n : \mathcal{A} \cap \bar{F}] \cdot X_{\mathcal{A}\cap\bar{F}}, \tag{27}$$

where the sum is over all facets $\bar{F}$ of the coherent polyhedral subdivision Δ_ϖ of $\mathcal{A}$. By the multiplicativity of Chow forms, the initial term of the $\mathcal{A}$-resultant factors as

$$init_\omega(\mathcal{R}_\mathcal{A}) = \pm \prod_{\bar{F}} \mathcal{R}_{\mathcal{A}\cap\bar{F}}^{[\mathbf{Z}^n:\mathcal{A}\cap\bar{F}]}, \tag{28}$$

where $\bar{F}$ runs over all facets of Δ_ϖ. A proof of (28) via the Cayley-Koszul complex will appear in [7].

We observe that the CMD Δ_ω of Q is simply $n+1$ times the subdivision Δ_ϖ of $\frac{1}{n+1}Q = conv(\mathcal{A})$. Each facet of the former equals $F = (n+1) \cdot \bar{F}$ for some facet $\bar{F}$ of the latter. It is easy to check that $d_F = [Z^n : \mathcal{A} \cap \bar{F}]$, the index of the affine lattice generated by $\mathcal{A} \cap \bar{F}$ in $\mathbf{Z}^n$. This proves the formula (24) under the assumption (26).

In the second part of our proof of Theorem 4.1 we reduce the general case to (26), using the factorization technique in [14, §7]. We form $n+1$ duplicates of each given form using new indeterminate coefficients, and we multiply these together as follows:

$$\begin{aligned} u_0 &= f_{00} f_{01} f_{02} \cdots f_{0n} \\ u_1 &= f_{10} f_{11} f_{12} \cdots f_{1n} \\ &\cdots \quad \cdots \quad \cdots \quad \cdots \\ u_n &= f_{n0} f_{n1} f_{n2} \cdots f_{nn} \end{aligned} \tag{29}$$

Each polynomial f_{ij}, $0 \le i \le n$, appearing in the jth column has the support $\mathcal{A}_j$. Therefore each row product u_i has the same support

$$\mathcal{A} := \mathcal{A}_0 + \mathcal{A}_1 + \cdots + \mathcal{A}_n. \tag{30}$$

Here we consider $\mathcal{A}$ as a multiset, having cardinality $m_0 m_1 \cdots m_n$.

According to [14, Proposition 7.1], the sparse mixed resultant of (29) factors into expressions $\mathcal{R}(f_{0,\sigma(0)}, \ldots, f_{n,\sigma(n)})^{D_\sigma}$, where σ runs over all functions σ : $\{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$. Here the diagonal term appears with exponent $D_{id} = 1$:

$$\begin{aligned}&\mathcal{R}(u_0, u_1, \ldots, u_n)\\ &\quad = \mathcal{R}(f_{00}, f_{11}, f_{22}, \ldots, f_{nn}) \cdot (\text{other non-diagonal resultants}).\end{aligned} \tag{31}$$

We now consider the given linear functional $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$, and we replace each polynomial $f_{ij}(x)$ by the corresponding deformation $f'_{ij}(\mathbf{x}, t)$ defined by ω_j as in (13). Then the deformation of the product $u'_i = f'_{i0} f'_{i1} \cdots f'_{in}$ is given similarly by the lifting

$$\tilde{\omega} : \mathcal{A} \to \mathbf{Z},\ \mathbf{a}^{(0)} + \cdots + \mathbf{a}^{(n)} \mapsto \omega_0(\mathbf{a}^{(0)}) + \cdots + \omega_n(\mathbf{a}^{(n)}). \tag{32}$$

Note that (32) is well defined because $\mathcal{A}$ is considered as a multiset.

The coherent polyhedral subdivision of $(Q, \mathcal{A})$ defined by $\tilde{\omega}$ equals the CMD defined by $\omega = (\omega_0, \ldots, \omega_n)$. As before, each facet F of $\Delta_\omega = \Delta_{\tilde{\omega}}$ has the form $F = F_0 + F_1 + \cdots + F_n$, where F_i is a subpolytope of Q_i. From our above special case (28) we derive

$$\begin{aligned} init_\omega(\mathcal{R}) &= \text{lowest } t\text{-coefficient of } \mathcal{R}(u'_0, \ldots, u'_n)\\ &= \prod_{F \in \Delta_\omega} \mathcal{R}(u_0|_F, \ldots, u_n|_F)^{[\mathbf{Z}^n : F \cap \mathcal{A}]}\end{aligned} \tag{33}$$

where

$$u_i|_F = (f_{i0})|_{F_0} \cdot (f_{i1})|_{F_1} \cdot (f_{i2})|_{F_2} \cdots \cdots (f_{in})|_{F_n}. \tag{34}$$

Applying the product formula (31) to the factors on the right-hand side of (33) we get

$$\begin{aligned}&\mathcal{R}(u_0|_F, \ldots, u_n|_F)\\ &\quad = \mathcal{R}(f_{00}|_{F_0}, f_{11}|_{F_1}, \ldots, f_{nn}|_{F_n}) \cdot (\text{nondiagonal factors}).\end{aligned} \tag{35}$$

We now pass to ω-initial terms in (31), and we collect all diagonal factors, using (33) and (35). The result is the desired formula

$$init_\omega(\mathcal{R}(f_{00}, f_{11}, \ldots, f_{nn})) = \prod_{F \in \Delta_\omega} \mathcal{R}(f_{00}|_{F_0}, f_{11}|_{F_1}, \ldots, f_{nn}|_{F_n})^{d_F}. \tag{36}$$

To see that the identification of the diagonal factors is unique, we use a degree count and induction on the cardinality of the occurring multisubsets $\{\mathcal{A}_{\sigma(0)}, \ldots, \mathcal{A}_{\sigma(n)}\}$. This completes the proof of Theorem 4.1. □

COROLLARY 4.1. *Each face of a resultant polytope is a Minkowski sum of resultant polytopes.*

Proof. This follows immediately from (24) since the Newton polytope of $init_\omega(\mathcal{R})$ equals the face of $\mathcal{N}(\mathcal{R})$ supported by ω. □

Each resultant obtained by restriction of supports appears on a suitable face of $\mathcal{N}(\mathcal{R})$.

COROLLARY 4.2. *Let $\mathcal{A}'_0 \subset \mathcal{A}_0, \ldots, \mathcal{A}'_n \subset \mathcal{A}_n$, having resultants $\mathcal{R}'$ and $\mathcal{R}$, and let $\omega : \cup_i \mathcal{A}_i \to \{0, 1\}$ be the indicator function of $\cup_i \mathcal{A}'_i$. Then $\mathcal{R}'$ is a factor of $init_\omega(\mathcal{R})$.*

Proof. Consider the CMD Δ_ω defined by the 0-1-vector ω. Let $F_i := conv(\mathcal{A}'_i)$ for $i = 0, 1, \ldots, n$. It is easy to see that $F = F_0 + F_1 + \cdots + F_n$ appears as a cell in Δ_ω. If $\mathcal{R}'$ is not a constant, then (by Theorem 1.1) the cell F is a facet of Δ_ω, and d_F is a positive integer. By Theorem 4.1, the restricted resultant $\mathcal{R}' = \mathcal{R}(f_0|_{F_0}, f_1|_{F_1}, \ldots, f_n|_{F_n})$ appears as a factor in the initial form $init_\omega(\mathcal{R})$. □

In the remainder of this section we study the initial forms $init_\omega(\mathcal{R})$, which are supported on the edges of the resultant polytope $\mathcal{N}(\mathcal{R})$. To this end we first characterize one-dimensional resultant polytopes, in analogy to the approach in [11, §2.C].

PROPOSITION 4.1. *Let $\{\mathcal{A}_0, \mathcal{A}_i, \cdots, \mathcal{A}_n\}$ be an essential family of subsets of $\mathbf{Z}^n$. The resultant polytope $\mathcal{N}(\mathcal{R})$ has dimension 1 if and only if $card(\mathcal{A}_0) = \cdots = card(\mathcal{A}_n) = 2$.*

Proof. The if direction was proved in Proposition 1.1. For the only-if-direction, we assume that $\mathcal{N}(\mathcal{R})$ has dimension 1. By Theorem 1.1, each of the sets $\mathcal{A}_i$ has cardinality at least 2. Suppose that one of them, say $\mathcal{A}_0$, as cardinality ≥ 3. Then we can select a proper subset $\mathcal{A}'_0$ of $\mathcal{A}_0$ such that the family $\{\mathcal{A}'_0, \mathcal{A}_1, \ldots, \mathcal{A}_n\}$ is still essential. Let $\mathcal{R}'$ be the corresponding resultant. By Corollary 1.1, $\mathcal{R}'$ is a nonconstant irreducible polynomial in more than one variable, hence $dim(\mathcal{N}(\mathcal{R}')) \geq 1$. Corollary 4.2 implies that the polytope $\mathcal{N}(\mathcal{R}')$ is a Minkowski summand of a proper face of $\mathcal{N}(\mathcal{R})$. Therefore $dim(\mathcal{N}(\mathcal{R})) > dim(\mathcal{N}(\mathcal{R}'))$, which contradicts our hypothesis $dim(\mathcal{N}(\mathcal{R})) = 1$. □

Let E be any edge of the resultant polytope $\mathcal{N}(\mathcal{R})$. Let v_1 and v_2 be the two vertices connected by E, with corresponding extreme monomials $init_{\omega_1}(\mathcal{R})$ and $init_{\omega_2}(\mathcal{R})$. Let $\delta(E)$ denote the ratio of the coefficient of $init_{\omega_1}(\mathcal{R})$ in $\mathcal{R}$ and the coefficient of $init_{\omega_2}(\mathcal{R})$ in $(\mathcal{R})$. By Corollary 3.1 we know that $\delta(E) \in \{-1, +1\}$.

It is our objective to give a combinatorial formula for $\delta(E)$, the *parity* of the edge E. Since the 1-skeleton (edge graph) of $\mathcal{N}(\mathcal{R})$ is connected, this will imply a combinatorial formula for the exact extreme monomials of the sparse mixed resultant.

Fix a support vector ω for the edge E, for instance

$$\omega := \langle \omega_1, v_1 - v_2 \rangle \cdot \omega_2 + \langle \omega_2, v_2 - v_1 \rangle \cdot \omega_1,$$

and consider the CMD Δ_ω is corresponding to the edge E. We say that a facet $F = F_0 + F_1 + \cdots + F_n$ of Δ_ω is *nontrivial* of $dim(F_i) \geq 1$ for $i = 0, 1, \ldots, n$.

PROPOSITION 4.2. *The parity of an edge E of the resultant polytope $\mathcal{N}(\mathcal{R})$ equals*

$$\delta(E) = (-1)^{\sum_F (vol(F) + d_F)}, \tag{37}$$

where the sum is over all nontrivial facets of Δ_ω.

This formula was proved in [6, Theorem 3A.11] for the principal $\mathcal{A}$-determinant. The resultant version, Proposition 4.2, can easily be derived from that theorem of Gelfand, Kapranov, and Zelevinsky. In what follows we give an alternative, self-contained proof.

Proof. The initial form $init_\omega(\mathcal{R})$ has a unique irreducible factor $\mathcal{R}'$ which is not a monomial. This factor is the resultant of an essential family $\{\mathcal{A}_i'\}_{i \in I}$ with $I \subset \{0, 1, \ldots, n\}$ and $\mathcal{A}_i' \subset \mathcal{A}_i$ for all $i \in I$. By Proposition 4.1, each of the sets $\mathcal{A}_i'$ has cardinality 2, and the resultant $\mathcal{R}'$ equals (8) with $I = \{i : \lambda_i \neq 0\}$. To see that $\mathcal{R}'$ must be unique, it suffices to note that two irreducible polynomials of the form (8) cannot have parallel Newton segments unless they are identical.

Each nontrivial facet F of Δ_ω contributes a factor of $(\mathcal{R}')^{d_F}$ to the product (24). All other factors are monomials, hence $init_\omega(\mathcal{R})$ equals $\prod_F (\mathcal{R}')^{d_F}$ times a monomial.

The ratio of the coefficients of the two monomials of $\mathcal{R}'$ equals $(-1)^{1+\sum_i |\lambda_i|}$. The expression $(\mathcal{R}')^{d_F}$ is a polynomial of degree $d_F \cdot \sum_{i=1}^n |\lambda_i| = vol(F)$. Therefore the ratio of coefficients of the two extreme monomials of $\mathcal{R}'^{d_F}$ equals

$$((-1)^{1+\sum_{i=1}^n |\lambda_i|})^{d_F} = (-1)^{d_F + d_F \sum_{i=1}^n |\lambda_i|} = (-1)^{d_F + vol(F)}.$$

We now take the product over all nontrivial facets F to get the ratio of the coefficients of the two extreme monomials in $init_\omega(\mathcal{R}')$. □

5. The Cayley trick, fiber polytopes, and R-equivalence

We recall the definition of the $\mathcal{A}$-discriminant due to Gelfand, Kapranov, and Zelevinsky [6, 7, 9]. Fix a set $\mathcal{A} \subset \mathbf{Z}^N$ of cardinality m. For any choice of complex coefficients $c_\mathbf{a}$, $\mathbf{a} \in \mathcal{A}$, the Laurent polynomial

$$f(z_1, \ldots, z_N) := \sum_{\mathbf{a} \in \mathcal{A}} c_{\mathbf{a}} \cdot \mathbf{z}^{\mathbf{a}} \tag{38}$$

defines a hypersurface $\{f = 0\}$ in the torus $(\mathbf{C}^*)^N$. Consider the set $\mathcal{Z} \subset P^{m-1}$ of all coefficient vectors $(c_{\mathbf{a}})$ for which $\{f = 0\}$ fails to be smooth. In other words, $\mathcal{Z}$ is the set of all $(c_{\mathbf{a}})$ for which the system of equations

$$f(\mathbf{z}) = \frac{\partial f}{\partial z_1}(\mathbf{z}) = \frac{\partial f}{\partial z_2}(\mathbf{z}) = \cdots = \frac{\partial f}{\partial z_N}(\mathbf{z}) = 0 \tag{39}$$

has a solution $\mathbf{z}$ in $(\mathbf{C}^*)^N$. In analogy to Lemma 1.1, the closure $\overline{\mathcal{Z}}$ is an irreducible subvariety of P^{m-1} defined over the rationals. But in contrast to Theorem 1.1, there is no easy combinatorial rule for its codimension. The *$\mathcal{A}$-discriminant* $\Delta_{\mathcal{A}}$ is the unique (up to sign) irreducible polynomial in $\mathbf{Z}[c_{\mathbf{a}}, \mathbf{a} \in \mathcal{A}]$ which vanishes on $\overline{\mathcal{Z}}$, provided $codim(\overline{\mathcal{Z}}) = 1$, and $\Delta_{\mathcal{A}} := 1$ otherwise.

It seems as if the $\mathcal{A}$-discriminant can be computed by means of a sparse mixed resultant. Fix $\mathcal{A}_0 := \mathcal{A}$, and $\mathcal{A}_i := \mathcal{A}\backslash\{0\} - \mathbf{e}_i$ for $i = 1, \ldots, N$, and let $\mathcal{R}$ be the corresponding resultant. Substitute the coefficients of (39) into $\mathcal{R}$. The resulting polynomial in $\mathbf{Z}[c_{\mathbf{a}}, \mathbf{a} \in \mathcal{A}]$ is denoted $\mathcal{E}_{\mathcal{A}}$ and called the *principal $\mathcal{A}$-determinant*. For the experts we note that this definition of $\mathcal{E}_{\mathcal{A}}$ is equivalent to the one given in [6] by [11, Theorem 5.10]. The following observation is an immediate consequence of the definitions.

Observation 5.1. The $\mathcal{A}$-discriminant $\Delta_{\mathcal{A}}$ divides the principal $\mathcal{A}$-determinant $\mathcal{E}_{\mathcal{A}}$.

Unfortunately, the $\mathcal{A}$-discriminant is almost always a proper factor of the principal $\mathcal{A}$-determinant, usually of much smaller degree. Under a certain smoothness hypothesis it is possible to explicitly express the quotient $\mathcal{E}_{\mathcal{A}}/\Delta_{\mathcal{A}}$ as a product of other $\mathcal{A}$-discriminants supported on the faces of $P = conv(\mathcal{A})$. This reduces the computation of $\mathcal{A}$-discriminants to the computation of resultants.

However, one can also express the sparse mixed resultant as a suitable $\mathcal{A}$-discriminant. The following construction is sometimes called the *Cayley trick* of elimination theory.

Let $f_0(\mathbf{x}), \ldots, f_n(\mathbf{x})$ be polynomials in (1), having supports $\mathcal{A}_0, \ldots, \mathcal{A}_n \subset \mathbf{Z}^n$, and let $\mathcal{R}$ denote their resultant. We introduce $n+1$ new variables $\mathbf{y} = (y_0, \ldots, y_n)$ and we form the auxiliary polynomial

$$f(\mathbf{x}, \mathbf{y}) = y_0 f_0(\mathbf{x}) + y_1 f_1(\mathbf{x}) + \cdots + y_n f_n(\mathbf{x}). \tag{40}$$

Its support is the $2n$-dimensional set

$$\mathcal{A} := \bigcup_{i=0}^{n} \{\mathbf{e}_i\} \times \mathcal{A}_i \quad \text{in} \quad \mathbf{Z}^{n+1} + \mathbf{Z}^n = \mathbf{Z}^{2n+1}. \tag{41}$$

We identify $\mathbf{z} = (\mathbf{x}, \mathbf{y}), N = 2n + 1$, $m = m_0 + \cdots + m_n$, and we consider the $\mathcal{A}$-discriminant $\Delta_{\mathcal{A}}$. Both $\Delta_{\mathcal{A}}$ and the principal $\mathcal{A}$-determinant $\mathcal{E}_{\mathcal{A}}$ are polynomials in $\mathbf{Z}[\ldots, c_{i,\mathbf{a}}, \ldots]$.

LEMMA 5.1 [6, Proposition 1.3.1]. *The sparse mixed resultant $\mathcal{R}$ equals the $\mathcal{A}$-discriminant $\Delta_{\mathcal{A}}$.*

Our first theorem in this section is a purely combinatorial result about secondary polytopes and fiber polytopes (cf. [3, 6]). It might be of interest independently from its algebraic motivation. Let Δ_{m-1} denote the regular $(m-1)$-simplex, consisting of all nonnegative vectors $(\lambda_{i,\mathbf{a}})$ with coordinate sum 1. Consider the canonical projection

$$\pi : \Delta_{m-1} \to P = conv(\mathcal{A}), \quad (\lambda_{i,\mathbf{a}}) \mapsto \sum_{i=0}^{n} \lambda_{i,\mathbf{a}} \cdot (\mathbf{e}_i, \mathbf{a}). \tag{42}$$

The secondary polytope of $\mathcal{A}$ equals the fiber polytope

$$\Sigma(\mathcal{A}) := \Sigma(\Delta_m, P) = \int_{p\in P} \pi^{-1}(p)dp.$$

Gelfand, Kapranov, and Zelevinsky have shown that the polytope $\Sigma(\mathcal{A})$ coincides with the Newton polytope of the principal $\mathcal{A}$-determinant $\mathcal{E}_{\mathcal{A}}$ [6, §3A; 7; 11, Theorem 5.1]. The faces of $\Sigma(\mathcal{A})$ are in natural bijection with the coherent polyhedral subdivisions of $(P, \mathcal{A})$. The vertices of $\Sigma(\mathcal{A})$ correspond to the coherent triangulations of $\mathcal{A}$.

Maintaining the notation from the previous sections, we set $Q_i := conv(\mathcal{A}_i)$ and $Q = Q_0 + \cdots + Q_n$. Let Δ denote the product of simplices $\Delta_{m_0-1} \times \cdots \times \Delta_{m_n-1}$. Its points are the nonnegative vectors $(\lambda_{i,\mathbf{a}})$ satisfying $\sum_{\mathbf{a}\in\mathcal{A}_i} \lambda_{i,\mathbf{a}} = 1$, for each $i = 0, 1, \ldots, n$ (separately). Consider the canonical projections of polytopes

$$\sigma_1 : Q_0 \times Q_1 \times \cdots \times Q_n \mapsto Q, (q_0, q_1, \ldots, q_n) \mapsto q_0 + q_1 + \cdots + q_n$$

and

$$\sigma_2 : \Delta \to Q_0 \times Q_1 \times \cdots \times Q_n, (\lambda_{i,\mathbf{a}}) \to \left(\sum_{\mathbf{a}\in\mathcal{A}_0} \lambda_{0,\mathbf{a}}\mathbf{a}, \ldots, \sum_{\mathbf{a}\in\mathcal{A}_n} \lambda_{n,\mathbf{a}}\mathbf{a}\right).$$

The composition $\sigma := \sigma_1 \circ \sigma_2$ maps Δ onto Q in a canonical fashion. Each of the three maps $\sigma, \sigma_1, \sigma_2$ defines a class of coherent polyhedral subdivision. The following lemma relates these to the polyhedral subdivisions introduced earlier. Part (b) concerns coarse decompositions as defined in (*) at the beginning of Section 3. The proof of Lemma 5.2 is straightforward using the methods in [3].

LEMMA 5.2
(a) The σ-coherent subdivisions of Q are the coherent mixed decompositions (CMDs).
(b) The σ_1-coherent subdivisions of Q are the coarse CMDs.

Part (a) shows that the face lattice of the fiber polytope $\Sigma(\Delta, Q)$ is isomorphic to the poset of all CMDs, ordered by refinement. Under this isomorphism the vertices of $\Sigma(\Delta, Q)$ correspond to the TCMDs of Q. The subposet of coarse CMDs is isomorphic to the poset of the face lattice of $\Sigma(Q_0 \times \cdots \times Q_n, Q)$. The inclusion of posets is realized geometrically by the fact that $\Sigma(Q_0 \times \cdots \times Q_n, Q)$ equals the projection of $\Sigma(\Delta, Q)$ under σ_2 (cf. [3, Lemma 2.3]).

We now come to the first main theorem in this section. Two polytopes are called *strongly isomorphic* if they lie in the same affine space and they have the same normal fan. (In [3] we used the term *normally equivalent*). This implies that they have the same face lattice, but it is stronger.

THEOREM 5.1. *The fiber polytope $\Sigma(\Delta, Q)$ is strongly isomorphic to the secondary polytope $\Sigma(\mathcal{A})$.*

Theorem 5.1 implies that the poset of CMDs is isomorphic to the poset of coherent subdivisions $(\mathcal{A}, P)$. In particular, the TCMDs are in natural bijection with the coherent triangulations of $(\mathcal{A}, P)$.

First note that the strong isomorphism in Theorem 5.1 has the potential to make sense because both polytopes lie in the same ambient affine space: $\Sigma(\Delta, Q) \subset \Delta \subset \mathbf{R}^m$ and $\Sigma(\mathcal{A}) \subset \Delta_{m-1} \subset \mathbf{R}^m$. Note also the both polytopes have the same dimension:

$$dim(\Sigma(\Delta, Q)) = dim(\Delta) - dim(Q) = m - 2n - 1 = dim(\Sigma(\mathcal{A})). \tag{43}$$

Proof of Theorem 5.1. Fix an arbitrary vector $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ in the interior of the unit simplex Δ_n. Let $\Delta_{m_i-1}^{\lambda_i}$ denote the regular $(m_i - 1)$-simplex consisting of nonnegative vectors with coordinate sum λ_i. We get the canonical projection

$$\sigma_\lambda : \Delta^\lambda := \Delta_{m_0-1}^{\lambda_0} \times \cdots \times \Delta_{m_n-1}^{\lambda n} \to Q^\lambda := \lambda_0 Q_0 + \cdots + \lambda_n Q_n, \tag{44}$$

which is isomorphic to the projection σ defined above. In particular, the fiber polytope of (44) is strongly isomorphic of $\Sigma(\Delta, Q)$.

Consider the following commutative diagram of polytopes:

$$\begin{array}{ccccccccc} 0 & \to & \Delta^\lambda & \to & \Delta_{m-1} & \to & \Delta_n & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & Q^\lambda & \to & P & \to & \Delta_n & \to & 0. \end{array} \tag{45}$$

The horizontal maps are the obvious inclusions and surjections, while the vertical maps are σ^λ, π and the identity. The secondary polytope $\Delta(\mathcal{A})$ is computed by integrating the fibers of the middle vertical map π. This integral can be decomposed into integrating the fibers of the left and the right vertical map in (45):

$$\begin{aligned} \Sigma(\mathcal{A}) = \int_{p \in P} \pi^{-1}(p) dp &= \int_{\lambda \in \Delta_n} \int_{q \in Q_\lambda} \sigma_\lambda^{-1}(q) dq\, d\lambda \\ &= \int_{\lambda \in \Delta_n} \Sigma(\Delta, Q) d\lambda = \Sigma(\Delta, Q). \end{aligned} \tag{46}$$

Each of the equations in (46) is a strong isomorphism. □

The isomorphism in Theorem 5.1 defines a bijection between the coherent triangulations of $\mathcal{A}$ and the TCMDs of Q. The combinatorial rule for this bijection is as follows. A triangulation T of $(\mathcal{A}, P)$ is a collection of $(2n+1)$-element subsets σ of $\mathcal{A}$. Under the natural identification of $\mathcal{A}$ with the disjoint union of $\mathcal{A}_0, \ldots, \mathcal{A}_n$, each cell σ of T is given as the disjoint union of its subsets $\sigma_i := \sigma \cap \mathcal{A}_i$ for $i = 0, 1, \ldots, n$. Here each σ_i is nonempty, because otherwise σ would not span a $2n$-dimensional affine space. The corresponding TCMD T' of Q has the maximal cells $F = F_0 + F_1 + \cdots + F_n$, where $F_i = conv(\sigma_i)$.

Theorem 2.1 gave a one-to-many correspondence between the extreme monomials of $\mathcal{R}$ and the TCMD's of Q. This correspondence has the following geometric refinement.

COROLLARY 5.1. *The resultant polytope $\mathcal{N}(\mathcal{R})$ is a summand of the fiber polytope $\Sigma(\Delta, Q)$.*

Two proofs. Corollary 5.1 is a direct consequence of Observation 5.1, Lemma 5.1, Theorem 5.1 and [6, Theorem 2E.1].

An alternative, self-contained proof goes as follows. We need to show that the normal fan of $\Sigma(\Delta, Q)$ refines the normal fan of $\mathcal{N}(\mathcal{R})$. Let ω and ω' be linear functional on $\mathbf{R}^m$ which define the same vertex of $\Sigma(\Delta, Q)$, i.e., they lie in the same open cone of the normal fan of $\Sigma(\Delta, Q)$. By Lemma 5.2 (a), they define the same TCMD $\Delta_\omega = \Delta_{\omega'}$. By Theorem 2.1, they define the same initial monomial of the sparse mixed resultant: $init_\omega(\mathcal{R}) = init_{\omega'}(\mathcal{R})$. Hence ω and ω' lie in the same open cone of the normal fan of $\mathcal{N}(\mathcal{R})$. □

Two vertices of the fiber polytope $\Sigma(\Delta, Q)$, or two TCMDs of Q, are said to be *R-equivalent* if they correspond to the same extreme monomial of the sparse mixed resultant $\mathcal{R}$. For sets $\mathcal{A}$ arising from the Cayley trick, this notion of R-equivalence is exactly the notion of Δ-equivalence introduced by Gelfand, Kapranov, and Zelevinsky. In what follows we give a combinatorial characterization of R-equivalence, thus providing a partial answer to a question raised in [6, Remark 3D.21].

Corollary 5.1 implies that any two R-equivalent vertices of $\Sigma(\Delta, Q)$ are connected by a sequence of edges. Therefore we need to identify those edges of $\Sigma(\Delta, Q)$ whose endpoints are R-equivalent. For this task we utilize the known general construction of the edges of any fiber polytope. A face F of Δ is called *critical if* $dim(\pi(F)) = dim(F) - 1$ and $dim(\pi(G)) = dim(G)$ for each proper face $G \subset F$.

LEMMA 5.3

(a) *Each edge of the fiber polytope $\Sigma(\Delta, Q)$ is parallel to a fiber segment $\Sigma(F, \pi(F))$, for some critical face F of Δ.*

(b) *For each critical face F of Δ, there exists an edge of $\Sigma(\Delta, Q)$ which is parallel to $\Sigma(F, \pi(F))$.*

For fiber polytopes in general, the critical face F in part (a) of Lemma 5.3 need not be unique. However, we claim that in our situation it must be unique. First note that each face of the product of simplices $\Delta = \Delta_{m_0} \times \cdots \times \Delta_{m_n}$ is itself a product of simplices $F = \Delta_{l_0} \times \Delta_{l_1} \times \cdots \times \Delta_{l_n}$. We call $(l_0, l_1, \ldots, l_n)$ the *type* of F. Hence each face F of Δ is gotten as an intersection $F = \Delta \cap L$, where L is a coordinate subspace L in $\mathbf{R}^m$. Given the fiber polytope $\Sigma(F, \pi(F))$, we can recover L (and hence F): it is the smallest coordinate subspace containing $\Sigma(F, \pi(F))$. Now, if Δ had two critical faces F_1 and F_2 for which the fiber segments $\Sigma(F_1, \pi(F_1))$ and $\Sigma(F_2, \pi(F_2))$ were parallel, then the corresponding subspaces L_1 and L_2 would coincide, and hence $F_1 = F_2$. We conclude that each edge of $\Sigma(\Delta, Q)$ equals a translate of $\Sigma(F, \pi(F))$ for a unique critical face F of Δ.

Note that the type of a critical face F satisfies $l_0 + l_1 + \cdots + l_n \leq n + 1$. We say that F is an *affine cube* if $0 \leq l_0, l_1, \ldots, l_n \leq 1$.

THEOREM 5.2. *The two endpoints of an edge of $\Sigma(\Delta, Q)$ are R-equivalent if and only if the corresponding critical face F is not an affine cube.*

Proof. Let ω be a linear functional on $\mathbf{R}^m$ which supports the given edge of $\Sigma(\Delta, Q)$. We need to show that $init_\omega(\mathcal{R})$ is a not a monomial if and only if the critical face F is an affine cube. The image of F under π appears as one of the faces in the CMD Δ_ω. We will identify the critical face F and its image in Δ_ω, say $F = F_0 + F_1 + \cdots + F_n$. All other factors in (24) are supported on faces with equality in (11), so they must be monomials. It suffices to consider the specific factor $\mathcal{R}(f_0|_{F_0}, f_1|_{F_1}, \ldots, f_n|_{F_n})$ which is supported on the critical face F. By Proposition 4.1, its Newton polytope has dimension ≥ 1 if and only if $dim(F_0), dim(F_1), \ldots, dim(F_n) \geq 1$. Therefore $\mathcal{R}(f_0|_{F_0}, f_1|_{F_1}, \ldots, f_n|_{F_n})$ is not a monomial if and only if the critical face $F = F_0 \times F_1 \times \cdots \times F_n$ of Δ is an affine cube. □

Two TCMDs of Q which are connected by an edge on the fiber polytope $\Sigma(\Delta, Q)$ are said to be related by a ***flip***. If the corresponding critical face F is an affine cube, then we call it a ***cubical flip***, otherwise it is a ***noncubical flip***. Thus a cubical flip consists of replacing the "bottom" by the "top" in a codimension 1 projection of a regular cube (cf. [3, Section 4]).

Except for degenerate cases, there are exactly three types of flips in the plane. The flip of type (1, 1, 1) is cubical, while the flip of type (2, 1, 0) and (3, 0, 0) are not cubical. They are "prismatical" and "tetrahedrical."

The following characterization of R-equivalence is the most intuitive.

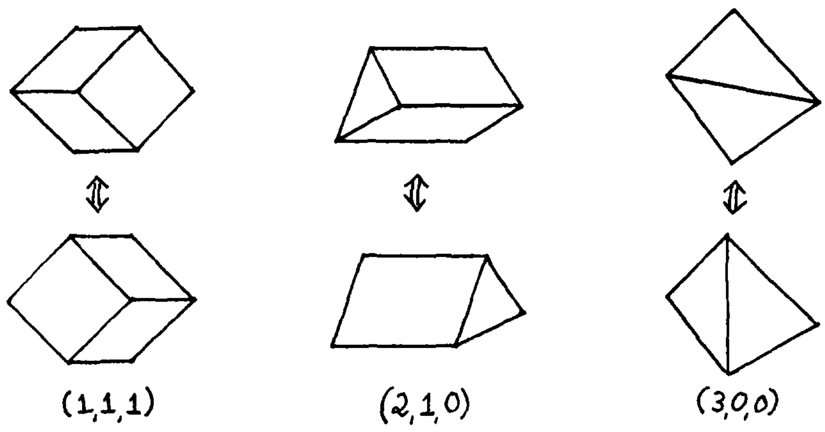


Figure 5. One cubical flip and two noncubical flips in the plane.

COROLLARY 5.2. *Two TCMDs are R-equivalent if and only if they are connected by a sequence of noncubical flips.*

As an illustration consider our (continued) Example 2.1. Corollary 5.2 implies that the TCMD in Figure 3 is obtained from the CMD is Figure 4 by a refinement followed by a sequence of noncubical flips (Figure 5).

Remark 5.1. It is an instructive exercise to verify the results in this section for the case $n = 1$. Here $\mathcal{R}$ is a Sylvester resultant, $P = conv(\mathcal{A})$ is a planar trapezoid, and $Q = Q_0 + Q_1$ is a line segment. A completely explicit description of the Newton polytope $\mathcal{N}(\mathcal{R})$ was given in [9]. There are two types of flips: the cubical flip (1, 1) corresponds to a four-element circuit of $\mathcal{A}$, while the noncubical flip (2, 0) corresponds to a three-element circuit of $\mathcal{A}$. Performing a noncubical flip on a TCMD of Q means decomposing a triangle of the corresponding triangulation of $\mathcal{A}$ into two smaller triangles, or vice versa.

Example 2.1 (continued). The configuration $\mathcal{A} \in \mathbf{Z}^5$ consists of the eight points

$$a_1 : (1, 0, 0, 0, 0) \quad a_2 : (1, 0, 0, 2, 2) \quad a_3 : (1, 0, 0, 1, 3) \quad b_1 : (0, 1, 0, 0, 0)$$
$$b_2 : (0, 1, 0, 2, 0) \quad b_3 : (0, 1, 0, 1, 2) \quad c_1 : (0, 0, 1, 1, 1) \quad c_2 : (0, 0, 1, 3, 0).$$

The polytope $P = conv(\mathcal{A})$ is 4-dimensional and has 11 facets, 26 2-faces, 23 edges, and 8 vertices. It is not simple. For each vertex of P we list the number of adjacent vertices:

a_1	a_2	a_3	b_1	b_2	b_3	c_1	c_2
6	5	6	6	5	6	5	7

The fiber polytope $\Sigma(\mathcal{A}) = \Sigma(\Delta, Q)$ is a 3-dimensional polytope with 23 facets, 57 edges, and 36 vertices. Hence the heptagon $Q = Q_0 + Q_1 + Q_2$ has precisely 36 TCMDs. Each of these (for instance, the one in Figure 3) corresponds to a unique coherent triangulation of P. The 36 TCMDs are grouped into six R-equivalence classes, one for each extreme monomial of the resultant $\mathcal{R}$. The cardinalities of these classes are 8 for $a_1^5 b_3^7 c_1^6 c_2$, 6 for $a_1^2 a_2^3 b_2^6 b_3 c_2^7$, 5 for $a_1^2 a_3^3 b_2^7 c_1 c_2^6$, 5 for $a_2^5 b_1^6 b_3 c_1^4 c_2^3$, 1 for $a_2^2 a_3^3 b_1^6 b_2 c_1^5 c_2^2$, and 11 for $a_3^5 b_1^7 c_1^7$.

6. Combinatorics of resultant polytopes

We continue our study of the resultant polytope $\mathcal{N}(\mathcal{R})$. The next theorem concerns its dimension. Throughout Section 6 we assume that $\{\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n\}$ is essential. By (43) and Corollary 5.1, we have $dim(\mathcal{N}(\mathcal{R})) \leq m - 2n - 1$, where $m = \sum_{i=0}^{n} card(\mathcal{A}_i)$.

THEOREM 6.1. *The dimension of the resultant polytope* $\mathcal{N}(\mathcal{R})$ *equals* $m - 2n - 1$.

A *mixed dependency* of the family $\{\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n\}$ is a vector $\lambda = (\ldots, \lambda_{i,\mathbf{a}}, \ldots) \in \mathbf{R}^m$ which satisfies the following system of linear equations:

(i) $\sum_{\mathbf{a}\in\mathcal{A}_i} \lambda_{i,\mathbf{a}} = 0$ for $i = 0, 1, \ldots, n$, and
(ii) $\sum_{i=0}^{n} \sum_{\mathbf{a}\in\mathcal{A}_i} \lambda_{i,\mathbf{a}} \cdot \mathbf{a} = \mathbf{0}$ (the zero vector in $\mathbf{R}^n$).

Let $V \subset \mathbf{R}^m$ denote the linear subspace of all mixed dependencies on $\{\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n\}$. The total number of equations (i) and (ii) equals $2n + 1$, hence $dim(V) \geq m - 2n - 1$. Clearly the space V is invariant under translations of each of the set $\mathcal{A}_i$. Hence we may assume that each sets $\mathcal{A}_i$ contains the origin $\mathbf{0}$. We set $\mathcal{B}_i := \mathcal{A}_i \backslash \{\mathbf{0}\}$ and $\mathcal{B} := \mathcal{B}_0 \cup \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_n$. Then $\mathcal{B}$ is a spanning configuration of $m - n - 1$ vectors in the vector space $\mathbf{R}^n$. The space V is naturally identified with the space of linear dependencies on $\mathcal{B}$. This implies

$$dim(V) = m - 2n - 1. \qquad (47)$$

Proof of Theorem 6.1. We will show that the space V is the translate of the affine span of $\mathcal{N}(\mathcal{R})$. Since $\{\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n\}$ is essential, we have $dim(\sum_{i\in I} \mathcal{B}_i) \geq |I|$ for each proper subset I of $\{0, 1, \ldots, n\}$. We can select vectors $\mathbf{b}^{(i)} \in \mathcal{B}_i$ such that each proper subset of $\{\mathbf{b}^{(0)}, \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(n)}\}$ is linearly independent in $\mathbf{R}^n$. Let $\lambda_0 \in V$ be the unique linear relation among these $n + 1$ vectors. For each $\mathbf{b}$ in one of the sets $\mathcal{B}_i \backslash \{\mathbf{b}^{(i)}\}$, let $\lambda_b \in V$ be the unique linear dependency on $\{\mathbf{b}^{(0)}, \ldots, \mathbf{b}^{(n)}\} \backslash \{\mathbf{b}^{(i)}\} \cup \{\mathbf{b}\}$. Let Λ denote the collection of all λ_b, augmented by the vector λ_0. Thus $card(\Lambda) = m - 2n - 1$. It is easy to see that Λ is linearly independent, because the corresponding $m \times (m - 2n - 1)$-matrix has a nonsingular upper triangular maximal minor. Hence Λ is a basis for V.

To complete the proof, it suffices to show that for each vector λ in Λ there exists an edge of $\mathcal{N}(\mathcal{R})$ parallel to that edge. Note that the support of λ intersects each set $\mathcal{B}_i$ at most once. Lift λ to a vector of length m which satisfies (i) and (ii), and abbreviate $\mathcal{A}'_i := supp(\lambda) \cap \mathcal{A}_i$. Then either $\mathcal{A}'_i$ is empty or contains two elements. The resultant polytope of $\mathcal{A}'_i$ has dimension 1; it is an edge parallel to λ, by Proposition 1.1. By Corollary 4.2, there exists an edge of $\mathcal{N}(\mathcal{R})$ which is parallel to λ. □

The *support* of a vector $\lambda \in \mathbf{R}^m$ is the set of points $\mathbf{a} \in \cup_i \mathcal{A}_i$ for which $\lambda_{i,\mathbf{a}} \neq 0$. It is a denoted $supp(\lambda)$. A nonzero vector λ in V is a *circuit* if $supp(\lambda)$ is minimal with respect to inclusion. We call $\lambda \in V$ a *mixed circuit* if $card(supp(\lambda) \cap \mathcal{A}_i) \leq 2$ for all $i = 0, 1, \ldots, n$. Note that $card(supp(\lambda) \cap \mathcal{A}_i)$ can only be 0 or ≥ 2, not 1, by the condition (i) above.

COROLLARY 6.1. *The edge directions of the resultant polytope $\mathcal{N}(\mathcal{R})$ are precisely the mixed circuits in V. Their number is bounded above by $\binom{m_0}{2}\binom{m_1}{2}\cdots\binom{m_n}{2}$.*

This implies the following bound for the number of vertices.

COROLLARY 6.2. *The number of vertices of $\mathcal{N}(\mathcal{R})$ is bounded above by*

$$2 \cdot \sum_{j=0}^{m-2n-2} \binom{\binom{m_0}{2}\binom{m_1}{2}\cdots\binom{m_n}{2}}{j}. \tag{48}$$

Proof. We need to give an upper bound on the number of open cells in the normal fan of $\mathcal{N}(\mathcal{R})$. Consider the hyperplane arrangement $\mathcal{H}$ whose hyperplanes are the spans of the codimension 1 cells in the normal fan of $\mathcal{N}(\mathcal{R})$. Now, by Theorem 6.1 and Corollary 6.1, $\mathcal{H}$ is a $(m-2n-1)$-dimensional arrangement of at most $\binom{m_0}{2}\binom{m_1}{2}\cdots\binom{m_n}{2}$ hyperplanes. Using Buck's formula [4], the number of its open cells is bounded above by (48). □

In each fixed dimension there are only finitely many combinatorial types of resultant polytopes. The following estimate is rather weak and can undoubtedly be improved. It would be interesting to find a more tight upper bound, as well as a matching lower bound.

PROPOSITION 6.1. *A resultant polytope of dimension d has at most $(3d-3)^{2d^2}$ vertices.*

Proof. By Theorem 6.1 and Corollary 6.2, the number of vertices is bounded above by $(2d-2)(\prod m_i)^{2d-2}$. In the subsequent Theorem 6.2 we will show that every resultant polytope is isomorphic to a resultant polytope with $m_0, m_1, \ldots, m_n \geq 3$. So, we may assume these inequalities. They imply $3n+3 \leq m = d+2n+1$, and therefore $m_i \leq m \leq 3d-3$ and $n \leq d-2$, which implies the stated bound. □

THEOREM 6.2. *Every resultant polytope is affinely isomorphic to a resultant polytope $\mathcal{N}(\mathcal{R})$ of an essential family $\{\mathcal{A}_0, \ldots, \mathcal{A}_n\}$ with $m_i = card(\mathcal{A}_i) \geq 3$ for $i = 0, 1, \ldots, n$.*

Proof and algorithm. Suppose that $m_0, \ldots, m_k \geq 3$ and $m_{k+1} = \cdots = m_n = 2$. We give an algebraic procedure which expresses our n-variate resultant $\mathcal{R}$ in terms of a k-variate resultant $\mathcal{R}'$. Consider any linear transformation in $SL(n, \mathbf{Z})$ which maps the directions $\mathcal{A}_i$ to multiples $\nu_i \cdot \mathbf{e}_i$ of the unit vectors $\mathbf{e}_i$ for $i = k+1, \ldots, n$. The corresponding monoidal change of coordinates on the torus $(\mathbf{C}^*)^n$ transforms (1) into a system

$$f_0(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = \cdots = f_k(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = c_{k+1,1} \cdot x_{k+1}^{\nu_{k+1}} - c_{k+1,2} = \cdots = c_{n,1} \cdot x_n^{\nu_n} - c_{n,2} = 0. \tag{49}$$

For each of the $\prod_{i=k+1}^{n} \nu_i$ many choices of roots of unity, consider the k-variate system

$$f_i(x_1, \ldots, x_k, (c_{k+1,2}/c_{k+1,1})^{1/\nu_{k+1}}, \ldots, (c_{n,2}/c_{n,1})^{1/\nu_n}) = 0, \quad (i = 0, \ldots, k) \tag{50}$$

and let $\mathcal{R}'$ denotes its sparse mixed resultant. This is a polynomial in the coefficients of $f_0, \ldots, f_k$ and an algebraic function in the $c_{i,j}$, $i > k$. Clearly (49) is solvable if and only if (50) is solvable. The resultant $\mathcal{R}$ of (49) equals, up to a monomial factor, the product of the $\mathcal{R}'$ over all $\prod_{i=k+1}^{n} \nu_i$ choices of roots of unity. Therefore the Newton polytope $\mathcal{N}(\mathcal{R})$ is affinely isomorphic to the Newton polytope $\mathcal{N}(\mathcal{R}')$. □

Example 2.3 (continued). To compute the resultant of (19) we perform the change of variables $y \mapsto x^2 z^{-1}$, and we solve $z = -c_1^{-1} c_2$ to get the univariate system

$$a_3 x^7 - a_2 c_1^{-1} c_2 x^6 - a_1 c_1^{-3} c_2^3 = b_3 x^5 + b_2 c_1^{-2} c_2^2 x^2 + b_1 c_1^{-2} c_2^2 = 0. \tag{51}$$

Our resultant $\mathcal{R}$ equals $c_1^{21} c_2^{-14}$ times the Sylvester resultant of (51). In particular, $\mathcal{N}(\mathcal{R})$ is affinely isomorphic to the resultant polytope for $\mathcal{A}_0 = \{0, 6, 7\}$ and $\mathcal{A}_1 = \{0, 2, 5\}$ in $\mathbf{Z}^1$. Thus $\mathcal{N}(\mathcal{R})$ is combinatorially isomorphic to the polytope $N_{2,2}$ in [9, Figure 2].

Remark 6.1. The algorithm in the proof of Theorem 6.2 is particularly interesting in the case $k = 0$. In this case (49) consists of one equation f_0 with three or more terms and n "binomials" $c_{i,1} x_i^{\nu_i} - c_{i,2}$. In (50) this system is reduced to a single equation with no variables at all !!! Such an equation is its own sparse mixed resultant, and its Newton polytope is a regular simplex of dimension $m_0 - 1$. We conclude that $\mathcal{N}(\mathcal{R})$ is an $(m_0 - 1)$-simplex whenever $m_1 = \cdots = m_n = 2$.

We now come to the classification of all resultant polytopes of dimension ≤ 3. Let us first recall the results of Gelfand, Kapranov, and Zelevinsky in [9] in the univariate case ($n = 1$). For two univariate equations, having m_0 and m_1 terms respectively, the resultant polytope is combinatorially isomorphic to a certain polytope N_{m_0-1,m_1-1} of dimension $m_0 + m_1 - 3$. The polytope N_{m_0-1,m_1-1} has $\binom{m_0+m_1-2}{m_0-1}$ vertices. See [9, §3] for an explicit description of the face lattice of this polytope.

The essential family constructed in our proof of Theorem 6.2 may consist of multisets. Therefore we need to extend the list of polytopes in [9] to the degenerate case when $\mathcal{A}_0$ and $\mathcal{A}_1$ are multisets, say

$$\mathcal{A}_0 = \{i_0^{d_0} < i_1^{d_1} < \cdots < i_r^{d_r}\} \text{ and } \mathcal{A}_1 = \{j_0^{e_0} < j_1^{e_1} < \cdots < j_s^{e_s}\}, \tag{52}$$

with cardinalities $d_0 + d_1 + \cdots + d_r = m_0$ and $e_0 + e_1 + \cdots + e_s = m_1$. The combinatorial type of the resultant polytope of (52) depends only on the multiplicity vectors $(d_0, \ldots, d_r)$ and $(e_0, \ldots, e_s)$, and we denote it by $\mathcal{N}_{(d_0,\ldots,d_r),(e_0,\ldots,e_s)}$. This follows from our results in Section 5 because the R-equivalence classes of TCMDs depend only on the multiplicity vectors. The polytope $\mathcal{N}_{(d_0,\ldots,d_r),(e_0,\ldots,e_r)}$ is always a degeneration of the Gelfand-Kapranov-Zelevinsky polytope N_{m_0-1,m_1-1}; in particular it has less vertices.

We list all three-dimensional polytopes in this class.

(a) The polytope $\mathcal{N}_{(1,1,1),(1,1,1)}$ equals the Gelfand-Kapranov-Zelevinsky polytope $N_{2,2}$. It equals the resultant polytope in Figure 2.

(b) The polytope $\mathcal{N}_{(1,2),(1,1,1)}$ is a square-based pyramid. It is the Newton polytope of

$$\mathcal{R} = a_0^2 b_2 - a_0 b_1 a_1 - a_0 b_1 a_2 + b_0 a_1^2 + 2 b_0 a_1 a_2 + b_0 a_2^2, \tag{53}$$

which is sparse mixed resultant of the digenerate system

$$a_0 + a_1 x + a_2 x = b_0 + b_1 x + b_2 x^2 = 0. \tag{54}$$

Also the polytope $\mathcal{N}_{(1,2),(2,1)}$ is a square-based pyramid. To see this, replace $b_1 x$ by b_1 in (54) and recompute (53).

(c) The polytope $\mathcal{N}_{(1,2),(1,2)}$ is a tetrahedron. Indeed, for $\mathcal{A}_0 = \mathcal{A}_1 = \{0, 1, 1\}$ the sparse mixed resultant equals $det\begin{pmatrix} a_0 & a_1+a_2 \\ b_0 & b_1+b_2 \end{pmatrix}$. Its Newton polytope is a tetrahedron.

COROLLARY 6.3

(a) The only resultant polytope of dimension 2 is the triangle.

(b) The only resultant polytopes of dimension 3 are the tetrahedron, the square-based pyramid, and the polytope $N_{2,2}$ in Figure 2.

Proof. By Theorem 6.2 we may assume that each of the given support sets $\mathcal{A}_i$ has cardinality $m_i \geq 3$. By Theorem 6.1, the dimension of the resultant polytope equals $d = \sum_{i=0}^{n} m_i - 2n - 1$, hence $d \geq n + 2$.

If $d = 2$ then $n = 0$ and $m_0 = 3$, i.e., the system (1) consists of one equation with three distinct terms in zero variables. The resultant $\mathcal{R}$ of such a system is equal to that three-term equation, and $\mathcal{N}(\mathcal{R})$ is a triangle (cf. Remark 6.6).

If $d = 3$ then there are two cases. Either $n = 0$ and $m_0 = 4$, in which case $\mathcal{N}(\mathcal{R})$ is a tetrahedron (cf. Remark 6.1), or $n = 1$ and $m_0 = m_1 = 3$, in which case $\mathcal{N}(\mathcal{R})$ is one of the three polytopes $\mathcal{N}_{(\dots),(\dots)}$ listed above. □

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