ON THE NIELSEN–THURSTON–BERS TYPE OF SOME SELF-MAPS OF RIEMANN SURFACES(¹)

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§ 1. Introduction

Let S be a surface of non-excluded (see § 2) finite type, and set $\dot{S}=S\setminus\{x_0\}$ for some $x_0\in S$. Consider the very simplest self-maps of $\dot{S}$: the self-maps that are homotopic to the identity on S (in particular such maps must fix x). When is such a map parabolic, hyperbolic, or pseudohyperbolic (see § 4 for definition) in the sense of Bers [9]? When is such a map reducible (see § 2) in the sense of Thurston [36]? We give a complete answer to this question, and as a consequence obtain two interesting facts:

(I) There exist irreducible self-mappings on Riemann surfaces of every non-excluded type $(p, n)\neq(0, 3)$; that is, as long as $3p+n>3$.

(II) The Teichmüller (=Kobayashi) metric on the fibers of the Bers fiber spaces is not (a multiple of) the Poincaré metric on the fibers, unless the Bers fiber space is one dimensional.

The more exact formulation of our first important result is summarized in

THEOREM 2′. *Let S be an oriented surface of non-excluded finite type. Let $x_0\in S$ and set $\dot{S}=S\setminus\{x_0\}$. Let w be a self-map of S with $w(x_0)=x_0$ and w isotopic to the identity on S. Let J be an isotopy of w to the identity:*

$$J\colon [0, 1]\times S\rightarrow S,$$

$$J(0, x)=w(x),$$

and

$$J(1, x)=x.$$

(¹) Research partially supported by NSF Grant MCS 7801248.

Let $c(t)=J(t, x_0)$. *(Then c is a closed curve on* S.*) The map w is isotopic to the identity on* $\dot{S}$ *if and only if c is contractible in S, and w is an irreducible self map of* $\dot{S}$ *if and only if c is an essential curve on S.*

The equivalence of the contractability of the curve c to the triviality of the map w is well known. See Birman [11] and the papers quoted here. Essential curves are defined in § 9.

Since essential curves are easily found, Theorem 2′ shows the existence of irreducible maps for surfaces of most types. The types not covered by this theorem can be treated by passing to 2 or 4 sheeted covers (see § 9). It is of interest to note that the irreducible mappings are precisely the pseudo-Anosov diffeomorphisms (see § 9 of Bers [9]). Thus Theorem 2′ gives the existence of a wide class of pseudo-Anosov diffeomorphisms.

A formulation of the second result mentioned above is contained in

THEOREM 4′. *Let* π_n: $V(p, n)' \to T(p, n)$ *be the punctured Teichmüller curve for surfaces of non-excluded type* (p, n). *Then the Kobayashi–Teichmüller metric on* $V(p, n)'$ *when restricted to* $\pi_n^{-1}(\tau)$ *with* $\tau \in T(p, n)$ *does not agree with the hyperbolic metric on this surface, except if* $(p, n)=(0, 3)$.

Special cases of our results have been obtained independently by Nag [29] in his thesis. I thank Lipman Bers, Bernard Maskit, and Peter Matelski for many helpful suggestions. In particular, Bers' paper [9] is crucial to this work, and the Maskit-Matelski paper [27] stimulated much of the current investigation. I am also grateful to William Abikoff for a very careful reading of a previous draft of this manuscript.

In a series of papers, Nielsen [30] discussed automorphisms of orientable surfaces. He classified these automorphisms into various types. About 50 years later, Thurston [36] also studied automorphisms of surfaces, and introduced a different classification. Bers [9] showed that Thurston's classification can be obtained by looking at the element of the modular group induced by a self-map of a surface. Quite recently, Gilman [18] obtained the relations between the older Nielsen classification and the new Thurston–Bers classification. These few historical remarks should help explain the title of this apper.

Several interesting new problems arise as a result of this work. These will be pursued further in the future.

§ 2. Self-mappings isotopic to the identity

Let S be an oriented surface of type (p, n). Assume S is of *non-excluded* type; that is,

$$2p-2+n>0. \tag{2.1}$$

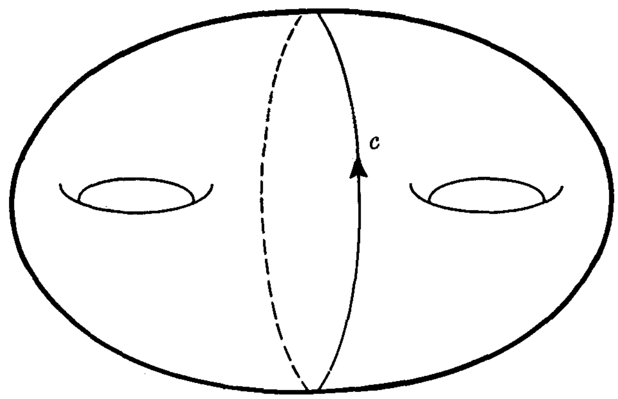

Figure 1. An admissible curve (c) on a surface of type (2, 0).

A finite non-empty set of disjoint Jordan curves $C=\{C_1, ..., C_r\}$ will be called *admissible* (see Figure 1) if no C_j can be deformed into a point, a boundary component of S, or into C_k with $k\neq j$. Following Thurston [36] and Bers [9], we say that an orientation preserving homeomorphism $f: S\to S$ is *reduced* by C if this set is admissible and if $f(C)=C$. A self-mapping f of S will be called *reducible* if it is isotopic to a reduced mapping, *irreducible* if it is not. If f is reduced by C, we let $S_1, ..., S_m$ be the components of $S\setminus C$. These will be called the (*proper*) *parts* of $S\setminus C$ or of S. Then each surface S_j is again of finite non-excluded type and f permutes the parts S_j. We let α_j be the smallest positive integer so that f^{α_j} fixes S_j. We shall denote the restriction $f^{\alpha_j}|S_j$ by the symbol f^{α_j} when the meaning is clear.

If f is reduced by C, then we say that f is *completely reduced* by C if for each j, f^{α_j} is irreducible. Bers [9] has shown that every reducible mapping is isotopic to a completely reduced mapping. If f is completely reduced, then the f^{α_j} are called (Gilman [18]) the *component maps* of f.

Let $x_0\in S$ and set $\dot{S}=S\setminus\{x_0\}$. Let $f: S\to S$ be a self-mapping of S. Assume that f is isotopic to the identity on S and that $f(x_0)=x_0$.

Problem A. *Find necessary and sufficient conditions for $f: \dot{S}\to\dot{S}$ to be reducible.*

We consider the group of orientation preserving self-mappings f of the surface S that satisfy two conditions

(1) $f(x_0)=x_0$ and

(2) f is isotopic to the identity self-map of S.

We factor this group by the normal subgroup of self-mappings that are isotopic to the identity as self-maps of $\dot{S}$. We denote the factor group by

$$\text{Isot}\,(S, x_0).$$

We are interested in describing and classifying the elements of the group Isot (S, x_0). *Throughout this paper we restrict our attention to orientable surfaces of non-excluded finite type, and maps between surfaces that are topological and orientation preserving.*

§ 3. Another extremal problem

Let S now be a Riemann surface of non-excluded finite type (p, n); that is, $S = \hat{S} \setminus \{x_1, ..., x_n\}$, where $\hat{S}$ is a compact Riemann surface of genus p, $x_1, ..., x_n$ are n distinct points on $\hat{S}$, and (2.1) is satisfied.

For a quasiconformal map $f: S \to \tilde{S}$ between Riemann surfaces, we let

$$K(f) = \text{dilatation of } f;$$

it is given by the formula

$$K(f) = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty},$$

where μ is the Beltrami coefficient of f, and $\|\mu\|_\infty$ denotes the L^∞ (= essential supremum)-norm of μ.

Problem B. *Let x and y be two distinct points on S. Among all quasiconformal self-mappings $f: S \to S$ with the properties*

(i) *f is isotopic to the identity, and*

(ii) *$f(x) = y$,*

find and characterize the extremals.

In particular, let $\tilde{\varrho}(x, y) = \frac{1}{2} \log K$, where K is the dilatation of an extremal for the problem (an extremal always exists). Then $\tilde{\varrho}$ defines a metric on S (see § 10). Is $\tilde{\varrho}$ a constant multiple of the hyperbolic (Poincaré) metric on S?

We shall rely on the classical solution of a related problem. Let D be a Jordan domain in $\mathbf{C} \cup \{\infty\}$ with hyperbolic metric $\varrho_D = \varrho$ of constant curvature -4. For $x, y \in D$, there exists a unique self-mapping (Teichmüller [35], Gehring [17]) f of D so that f is the identity on the boundary of D, $f(x) = y$, and f minimizes the dilatation among all such mappings. Let $K(x, y)$ be the dilatation of such an extremal f. We shall need the following

LEMMA 1 (Teichmüller [35], Gehring [17]). *There exists a differentiable real-valued function $\varkappa$ defined on $[0, \infty)$ such that*

(1) $\varkappa(0) = 0, \quad \lim_{t \to \infty} \varkappa(t) = \infty$,

(2) *$\varkappa(t)$ is strictly increasing,*

(3) *$\varkappa(t)/t$ is strictly decreasing, and*

(4) $\frac{1}{2} \log K(x, y) = \varkappa(\varrho(x, y))$.

The exact formula for $\varkappa$, which may be found in [17], will not be needed in the sequel. The fact that $K(x, y)$ depends only on $\varrho(x, y)$ is quite easy to verify. We will also need to know (see [17]) that

(5) $\varkappa'(0) = \frac{1}{4}$.

As a matter of fact ([35]),

$$K = 1 + \frac{\varrho}{2} + o(\varrho^2), \quad \varrho \to 0,$$

and

$$K > 1 + \frac{\varrho}{2}, \quad \text{all } \varrho.$$

It is quite easy to conclude from [17] and the methods of § 10, that the function

$$(x, y) \mapsto \varkappa(\varrho(x, y))$$

on $D \times D$ defines a complete metric on D. The metric gives rise to the usual topology on D. It is invariant under the full group of automorphisms of D, and is not a multiple of the hyperbolic metric on any segment which is part of a geodesic in the ϱ-metric.

The solutions to Problems A and B and some related problems to be stated in § 6 involve not surprisingly, the theory of Teichmüller spaces; which we now review.

§ 4. Teichmüller spaces and their modular groups

We shall follow the notation of Bers [9]. Let S be an oriented surface of finite non-excluded type (p, n). *A conformal structure of the first kind* is a topological mapping σ of S onto a Riemann surface of finite type. From now on, "conformal structure of the first kind" will by abuse of language be abbreviated by "conformal structure". Two conformal structures σ_1 and σ_2 on S are *strongly equivalent* if there exists a conformal map c of $\sigma_1(S)$ onto $\sigma_2(S)$ such that $\sigma_2^{-1} \circ c \circ \sigma_1$ is isotopic to the identity. The strong equivalence classes $[\sigma]$ of structures form the points of the *Teichmüller space* $T(p, n)$ and the (*Teichmüller*) *distance* between two points $[\sigma_1]$ and $[\sigma_2]$ is defined by

$$\langle [\sigma_1], [\sigma_2] \rangle = \tfrac{1}{2} \log K(h),$$

where h is the unique extremal isotopic to $\sigma_2 \circ \sigma_1^{-1}$. With this metric $T(p, n)$ is a complete space homeomorphic to $\mathbf{C}^{3p-3+n}$. Moreover, $T(p, n)$ is a complex manifold, and, by a result of Royden [34], the Teichmüller metric is the same as the hyperbolic Kobayashi metric on $T(p, n)$.

The *modular group* Mod (p, n) is the group of isotopy classes of self-mappings of S. It acts as a group of holomorphic isometries of $T(p, n)$ as follows. If f is a self-map of S, then the self-map f^* of $T(p, n)$ sends $[\sigma]$ into $f^*([\sigma]) = [\sigma \circ f^{-1}]$, where σ is a conformal structure on S. The modular group acts effectively on $T(p, n)$ unless the type appears in the following short list of *exceptional* types:

$$(0, 3), \ (0, 4), \ (1, 1), \ (1, 2), \quad \text{and} \quad (2, 0).$$

Let $\chi \in \operatorname{Mod}(p, n)$ and assume that χ acts non-trivially on $T(p, n)$. Bers [9] has introdused a classification of elements of Mod (p, n) by setting

$$a(\chi) = \inf_{\tau \in T(p,n)} \langle \tau, \chi(\tau) \rangle, \tag{4.1}$$

and calling χ *elliptic* if it has a fixed point in $T(p, n)$, *parabolic* if there is no fixed point and $a(\chi) = 0$, *hyperbolic* if $a(\chi) > 0$ and there is a $\tau \in T(p, n)$ with $a(\chi) = \langle \tau, \chi(\tau) \rangle$, and *pseudo-hyperbolic* if $a(\chi) > 0$ and $a(\chi) < \langle \tau, \chi(\tau) \rangle$ for all $\tau \in T(p, n)$.

The number $a(\chi)$ is not easily computed. If χ is induced by the self-map f of S and σ is a conformal structure on S, then $\sigma \circ f^{-1} \circ \sigma^{-1}$ is a self-map of the Riemann surface $\sigma(S)$. Let h_σ be the unique extremal self-map of $\sigma(S)$ isotopic to $\sigma \circ f^{-1} \circ \sigma^{-1}$. Then

$$a(\chi) = \inf_{[\sigma] \in T(p,n)} \tfrac{1}{2} \log K_{h_\sigma}.$$

We should also observe that χ induces a quasiconformal self-mapping on every surface represented in $T(p, n)$.

§ 5. Fiber spaces over Teichmüller spaces and their modular groups

Let Γ be a finitely generated Fuchsian group of the first kind operating on the upper half plane U. We denote by $M(\Gamma)$ the space of Beltrami coefficients for Γ. For every $\mu \in M(\Gamma)$ there exists a unique homeomorphism

$$w^\mu \colon \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\},$$

(i) which is normalized to fix $0, 1, \infty$,
(ii) has Beltrami coefficient μ in U, and
(iii) is conformal in the lower half plane U^*.

Two Beltrami coefficients μ, ν are *equivalent* if $w^\mu | U^* = w^\nu | U^*$. The *trivial* Beltrami coefficients are those equivalent to zero. The Teichmüller space $T(\Gamma)$ is the set of equivalence classes $[\mu]$ of Beltrami coefficients $\mu \in M(\Gamma)$. If Γ has type (p, n), then $T(\Gamma)$ can be used as a model for $T(p, n)$.

For each $\mu \in M(\Gamma)$, there is also a unique μ-conformal normalized automorphism of U that is denoted by w_μ. The equivalence relation on $M(\Gamma)$ can also be described alternately by:

$$\mu \sim \nu \Leftrightarrow w_\mu | \mathbf{R} = w_\nu | \mathbf{R}.$$

Remark. A Beltrami coefficient, as an element of the Banach space $L^\infty(\Gamma)$ of Beltrami differentials, has L^∞-norm ($\|\cdot\|_\infty$) less than one. It will be useful to abuse language occasionally and call elements of norm one of $L^\infty(\Gamma)$, Beltrami coefficients of norm one. If μ is a Beltrami coefficient of norm one and $z \in \mathbf{C}$ with $|z| < 1$, then $z\mu$ is a Beltrami coefficient in the ordinary sense.

Every element $\mu \in M(\Gamma)$ induces an isomorphism θ^μ of Γ onto a quasi-Fuchsian group Γ^μ defined by

$$\theta^\mu(\gamma) = w^\mu \circ \gamma \circ (w^\mu)^{-1}, \quad \gamma \in \Gamma.$$

The isomorphism θ^μ depends only on $[\mu]$. Similarly,

$$\theta_\mu(\gamma) = w_\mu \circ \gamma \circ (w_\mu)^{-1}$$

defines an isomorphism of Γ onto a *quasiconformally equivalent* Fuchsian group. These mappings are called *right translations* and allow us to place an arbitrary point of $T(p, n)$ at the "origin" of Teichmüller space.

An automorphism θ of the Fuchsian group Γ is called *geometric* if there exists a quasiconformal self map w of U such that

$$\theta(\gamma) = w \circ \gamma \circ w^{-1}, \quad \text{all } \gamma \in \Gamma.$$

We let the *extended modular group*, mod Γ, denote the group of geometric automorphisms of Γ, and we define the *modular group* as

$$\operatorname{Mod} \Gamma = \operatorname{mod} \Gamma / \Gamma;$$

that is, Mod Γ is the quotient group of geometric automorphisms by the normal subgroup of inner automorphisms. The group Mod Γ acts on $T(\Gamma)$ as follows: if the element χ of Mod Γ is represented by the quasiconformal self map w of U and if $\mu \in M(\Gamma)$, then

$$\chi([\mu]) = [\text{Beltrami coefficient of } w^\mu \circ w^{-1}].$$

If Γ is torsion free of type (p, n), then Mod Γ may be taken to be a model for Mod (p, n).

The *Bers fiber space* $F(\Gamma)$ is defined by

$$F(\Gamma) = \{([\mu], z);\ [\mu] \in T(\Gamma),\ z \in w^\mu(U)\}.$$

The natural projection of $F(\Gamma)$ onto $T(\Gamma)$ will be denoted by π. The extended modular group acts on $F(\Gamma)$ as follows: if $\theta \in \operatorname{mod}\Gamma$ is represented by the quasiconformal map w, then

$$\theta([\mu], z) = ([\nu], \hat{z}),$$

where $\mu \in M(\Gamma)$, $z \in w^\mu(U)$, $\nu =$ Beltrami coefficient of $w^\mu \circ w^{-1}$, and

$$\hat{z} = w^\nu \circ w \circ (w^\mu)^{-1}(z).$$

The action of mod Γ on $F(\Gamma)$ is always effective. It follows easily from the above definitions, that if $\theta \in \operatorname{mod}\Gamma$ and χ is the image of θ in Mod Γ, then the following diagram commutes

$$\begin{array}{ccc} F(\Gamma) & \xrightarrow{\theta} & F(\Gamma) \\ \downarrow \pi & & \downarrow \pi \\ T(\Gamma) & \xrightarrow{\chi} & T(\Gamma). \end{array} \tag{5.1}$$

Observe that the action of Γ on $F(\Gamma)$ is particularly simple since $\Gamma \hookrightarrow \operatorname{mod}\Gamma$. For $\gamma \in \Gamma$, $z \in \pi^{-1}([\mu])$,

$$\gamma([\mu], z) = ([\mu], \gamma^\mu z),$$

where $\mu \in M(\Gamma)$, and

$$\gamma^\mu = w^\mu \circ \gamma \circ (w^\mu)^{-1} = \theta^\mu(\gamma).$$

The quotient spaces

$$V(\Gamma) = F(\Gamma)/\Gamma$$

provide various models for the *Teichmüller curves*. For Γ torsion free of type (p, n), $V(\Gamma) = V(p, n)'$ is a model for the *punctured Teichmüller curve* for surfaces of type (p, n). See Earle–Kra [13] as well as Bers [8] and Kra [22], [24] for more details on the concepts discussed in this section.

Let us assume now that Γ is a torsion free Fuchsian group of type (p, n), and let us choose a point $a \in U$.

Let $A = \Gamma a = \{\gamma a; \gamma \in \Gamma\}$, and let

$$h: U \to U \setminus A$$

be a holomorphic covering map. The *Fuchsian model* for the action of Γ on $U \setminus A$ is the group

$$\dot{\Gamma} = \{g \in \operatorname{Aut} U;\ \exists \gamma \in \Gamma \text{ with } h \circ g = \gamma \circ h\},$$

where Aut U is the group of complex analytic automorphisms of $U(=\mathrm{SL}(2,\mathbf{R})/\pm I)$. Then

$$U/\dot{\Gamma} \cong (U/\Gamma)\setminus\{\hat{a}\} = (U\setminus A)/\Gamma, \tag{5.2}$$

where $\hat{a}$ is the equivalence class of a in U/Γ, and $\dot{\Gamma}$ is a torsion free Fuchsian group of type $(p, n+1)$.

Throughout this paper we will follow the above notational convention: given $a \in U$; $A=\Gamma a$ will denote its Γ-orbit, $\hat{a}$ its U/Γ-equivalence class, and $\dot{\Gamma}$ a torsion free Fuchsian group defined by (5.2).

We define a surjective mapping

$$h^*\colon M(\dot{\Gamma}) \to M(\Gamma)$$

by the formula

$$(h_\mu^*)\circ h = \mu \frac{\overline{h'}}{h'}, \quad \mu \in M(\dot{\Gamma}).$$

The mapping h^* induces a mapping

$$\psi\colon T(\dot{\Gamma}) \to F(\Gamma)$$

by

$$\psi([\mu]) = ([\nu], w^\nu(a)),$$

where $\mu \in M(\dot{\Gamma})$ and $\nu = h^*\mu$. The mapping ψ is a biholomorphic surjective map, whose existence shows that $F(\Gamma)$ is complex analytically isomorphic to $T(\dot{\Gamma})$, which is a model for $T(p, n+1)$. Further, the projection map $\pi\colon F(\Gamma) \to T(\Gamma)$ may be identified with the forgetful map $T(p, n+1) \to T(p, n)$ discussed in Earle–Kra [13].

We proceed to study the action induced by the isomorphism ψ on the modular groups. Let $\theta \in \operatorname{mod}\Gamma$. Assume that θ is induced by a quasiconformal map w that conjugates Γ into itself. The mapping w can be replaced by another mapping that induces the same isomorphism θ and sends the distinguished point a onto itself (see Lemma 2 of § 9). Thus without loss of generality we assumed that $w(a)=a$ (see Bers [8], Kra [22], Riera [33]). The condition $w(a)=a$ suffices to guarantee that w is an automorphism of $U\setminus A$. Hence, there exists a quasiconformal $W\colon U \to U$ such that the following diagram commutes

$$\begin{array}{ccc} U & \xrightarrow{W} & U \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ U\setminus A & \xrightarrow{w} & U\setminus A. \end{array}$$

The mapping w induces a self-map of $(U\setminus A)/\Gamma$. Hence W induces a self-map of $U/\dot{\Gamma}$, and thus W conjugates $\dot{\Gamma}$ onto itself and defines an element χ of Mod $\dot{\Gamma}$. It is now straight forward to check that the diagram

$$\begin{tikzcd} T(\dot\Gamma) \arrow[r, "\chi"] \arrow[d, "\psi"] & T(\dot\Gamma) \arrow[d, "\psi"] \\ F(\Gamma) \arrow[r, "\theta"] & F(\Gamma) \end{tikzcd}$$

commutes. We have constructed an isomorphism

$$I: \text{mod}\ \Gamma \to \text{Mod}\ \dot\Gamma. \tag{5.3}$$

The Image of I is a subgroup of index $n+1$ in Mod $\dot\Gamma$. The image consists of those elements $\chi \in$ Mod $\dot\Gamma$ that are induced by self-mappings W of $U/\dot\Gamma$ that fix one of the (specified) punctures on $U/\dot\Gamma$. For more details see [8], [22], [33].

We are now ready to classify the elements of the group Isot (S, x_0), which was defined in § 2.

PROPOSITION 1. *There is a canonical isomorphism $I: \pi_1(S) \to$ Isot (S, x_0). In particular, the elements of* Isot (S, x_0) *are classified by the fundamental group of S.*

Proof. Represent S as U/Γ for some torsion free Fuchsian group Γ. Then, of course, $\Gamma \cong \pi_1(S)$. Choose $a \in U$, so that $\hat a$, its equivalence class in U/Γ, represents $x_0 \in S$. As above represent $(U/\Gamma)\setminus\{\hat a\}$ by a torsion free Fuchsian group $\dot\Gamma$ so that $(U/\Gamma)\setminus\{\hat a\} = U/\dot\Gamma$.

The isomorphism I is the restriction of the isomorphism (5.3) to $\Gamma \subset \text{mod}\ \Gamma$. We shall discuss this isomorphism in detail and show that indeed $I(\Gamma) = \text{Isot}\ (U/\Gamma, \hat a)$. For the convenience of the reader, we will repeat certain arguments from [8], [22], [33]. Let $\gamma \in \Gamma$ and choose a quasiconformal automorphism w_0 of U so that $w_0(a) = \gamma(a)$ and w_0 acts trivially on Γ (that is, $w_0 \circ g \circ w_0^{-1} = g$ for all $g \in \Gamma$). Let $w = w_0^{-1} \circ \gamma$. Then w conjugates Γ onto itself and $w(a) = a$. The mapping w induces an automorphism W of U/Γ (that fixes $\hat a$) so that

$$\begin{tikzcd} U \arrow[r, "w"] \arrow[d, "q"] & U \arrow[d, "q"] \\ U/\Gamma \arrow[r, "W"] & U/\Gamma \end{tikzcd} \tag{5.4}$$

commutes. We now use the observation of Ahlfors [3] that W is homotopic (or equivalently, isotopic) to the identity on U/Γ if and only if w induces an inner automorphism of Γ. We thus see that $I(w) \in \text{Isot}\ (U/\Gamma, \hat a) \subset \text{Mod}\ \dot\Gamma$. The mapping I is well defined. If $\tilde w_0$ also acts trivially on Γ and sends a onto $\gamma(a)$, then $\tilde w = \tilde w_0^{-1} \circ \gamma$ and w induce self-maps $\tilde W$ and W of $(U/\Gamma)\setminus\{\hat a\} = U/\dot\Gamma$. We must show that $\tilde W \circ W^{-1}$ is isotopic to the identity on $U/\dot\Gamma$. Now $\tilde W \circ W^{-1}$ is induced by the self map $\tilde w \circ w^{-1} = \tilde w_0^{-1} \circ w_0$ that acts trivially on Γ and fixes the point $a \in U$. Such a map is homotopic to the identity (on U/Γ) by the homotopy along Poincaré geodesics defined by Ahlfors [3]. This homotopy keeps fixed the point $\hat{a}$. It follows that $\tilde{w} \circ w^{-1}$ is isotopic to the identity on $(U/\Gamma) \setminus \{\hat{a}\}$ (Baer [4], [5], Epstein [14]; see also Bers [8], Kra [22], and Birman [11]).

The mapping I is a homomorphism. Let $\gamma_j \in \Gamma$, $j=1,2$. Choose w_{0_j} such that $w_{0_j} \circ \gamma = \gamma \circ w_{0_j}$ all $\gamma \in \Gamma$, and $w_{0_j}(a) = \gamma_j(a)$. Let $w_j = w_{0_j}^{-1} \circ \gamma_j$, and construct W_j: $U/\Gamma \to U/\Gamma$ so that $W_j \circ q = w_j \circ q$. Then clearly

$$W_1 \circ W_2 \circ q = W_1 \circ q \circ w_2 = q \circ w_1 \circ w_2,$$

showing that $I(\gamma_1 \circ \gamma_2) = I(\gamma_1) \circ I(\gamma_2)$.

To show that I is injective, let $\gamma \in \Gamma$ and assume that $I(\gamma) = 1$. Using the notation introduced at the beginning of this proof, we see that W is isotopic to the identity on $(U/\Gamma) \setminus \{\hat{a}\}$. In particular, w must act trivially on Γ (the inner automorphism is trivial because w fixes a). Since $w = w_0^{-1} \circ \gamma$, and both w and w_0 commute with the elements of Γ, we conclude that so does γ. Hence γ is the identity.

It remains to show that I is surjective. Let W be a self-map of U/Γ that fixes the point $\hat{a}$ and is isotopic to the identity as a self map of U/Γ. Lift W to a self-mapping w of U so that (5.4) commutes. Then $w(a) = \gamma(a)$ for some $\gamma \in \Gamma$, and it is quite easy to see that $I(\gamma)$ is the class of W in Isot $(U/\Gamma, \hat{a})$.

Remark. Proposition 1 is well known. It has been proven explicitly or implicity many times (in [8], [11], [22], for example). It is a special case of more general results. See Birman [11] and the literature quoted there.

§ 6. Teichmüller discs in $T(p, n)$

A point τ in the Teichmüller space $T(p, n)$ represents a Riemann surface S of type (p, n). Choose a torsion free Fuchsian group Γ of type (p, n) so that $S = U/\Gamma$. Take $T(\Gamma)$ as a model for $T(p, n)$.

A *formal Techmüller* (*Beltrami*) coefficient for Γ is of the form

$$\mu = k\bar{\varphi}/|\varphi|, \tag{6.1}$$

where $0 < k < 1$, and φ is a *meromorphic integrable automorphic form for* Γ *of weight* -4 and norm 1 (that is,

$$\varphi\colon U \to \mathbf{C} \cup \{\infty\}$$

is a meromorphic function satisfying

$$\varphi(\gamma z)\gamma'(z)^2 = \varphi(z), \quad \text{all } z \in U, \text{ all } \gamma \in \Gamma,$$

and

$$\|\varphi\| = \tfrac{1}{2} \iint_{U/\Gamma} |\varphi(z)\, dz \wedge d\bar{z}| = 1).$$

The automorphic form φ as above projects to a meromorphic quadratic differential Φ which has at worst simple poles at points of U/Γ *and* at the punctures of U/Γ—hence Φ has only finitely many poles. A formal Teichmüller coefficient is called a *Teichmüller coefficient* for Γ if φ is also holomorphic (in addition to being integrable) on U/Γ. The Banach space of integrable holomorphic quadratic differentials on the Riemann surface S will be denoted by $Q(S)$. Note that every element of $Q(U/\Gamma)$ comes from an automorphic form. We shall henceforth identify automorphic forms of weight -4 and quadratic differentials.

A quasiconformal mapping w whose Beltrami coefficient is a (formal) Teichmüller coefficient μ given by (6.1) will be called *a (formal) Teichmüller mapping*. The quadratic differential φ will be called the *initial differential* of w. The *terminal differential* of w is the negative of the initial differential of w^{-1}. For more information on the connection between formal Teichmüller mappings and the problems considered in this paper, the reader is refered to Bers [9].

Assume now that $\varphi \in Q(U/\Gamma)$ with $\|\varphi\| = 1$. Consider the map of the unit disc Δ into $T(\Gamma)$ given by

$$\Delta \ni z \mapsto [z\bar{\varphi}/|\varphi|] \in T(\Gamma).$$

This mapping is an isometry with respect to the hyperbolic metric on Δ and the Teichmüller metric on $T(\Gamma)$. The image of this mapping is a totally geodesic submanifold of $T(\Gamma) = T(p, n)$ called the *Teichmüller disc through τ corresponding to the differential φ*. Techmüller's theorem asserts that the Teichmüller discs through τ fill out $T(p, n)$. A *Teichmüller line through τ* consists of the points $[k\bar{\varphi}/|\varphi|]$, $-1 < k < 1$, in $T(\Gamma)$.

Problem C. *Let π: $F(\Gamma) \to T(\Gamma)$ denote the Bers fiber space. Recall that $F(\Gamma) = T(\dot{\Gamma})$. Thus we can define Teichmüller discs in $F(\Gamma)$. Let $[\mu] \in T(\Gamma)$. Is $\pi^{-1}([\mu])$ a Teichmüller disc?*

The domain $\pi^{-1}([\mu]) = w^{\mu}(U)$, $\mu \in M(\Gamma)$, has the canonical hyperbolic (Poincaré) metric on it. Does the metric that $w^{\mu}(U)$ gets from its imbedding in $T(\dot{\Gamma})$ coincide with the Poincaré metric?

It will be convenient to characterize the Teichmüller metric on $F(\Gamma)$ more directly. The Teichmüller metric on $T(\Gamma)$ arises from the solution of an extremal problem. The same is true for $F(\Gamma)$. We proceed to describe the extremal problem for $F(\Gamma)$.

Choose any point $a \in U$. We say that two Beltrami coefficients μ and $\nu \in M(\Gamma)$ are *equivalent* with respect to $\{\Gamma; a\}$ provided $w^{\mu}|U^* = w^{\nu}|U^*$ and $w^{\mu}(a) = w^{\nu}(a)$. Call the space $T(\Gamma; a)$ of such equivalence classes the *Teichmüller space of the pointed Fuchsian group* $\{\Gamma; a\}$ (see Kra [22]). It is quite easy to define a Teichmüller metric on $T(\Gamma; a)$, and to show that

$$\mu \mapsto ([\mu], w^{\mu}(a))$$

defines an isometric surjective mapping of $T(\Gamma; a)$ onto $F(\Gamma)$. Thus a Teichmüller disc in $F(\Gamma)$ through the point $([0], a)$ is of the form

$$([z\mu], w^{z\mu}(a)),$$

where $z \in \mathbf{C}$, $|z| < 1$, $\mu = \bar{\varphi}/|\varphi|$ is a formal Teichmüller coefficient of norm one with φ holomorphic on $U \setminus \{\Gamma a\}$ (that is, $\varphi \in Q((U/\Gamma) \setminus \{\hat{a}\})$). It is an easy exercise to show that the isomorphism of $T(\dot{\Gamma})$ onto $F(\Gamma) \cong T(\Gamma; a)$ preserves Teichmüller discs.

An element of $\chi \in \operatorname{Mod}(p, n)$ takes a Teichmüller disc D through $\tau \in T(p, n)$ onto the Teichmüller disc $\chi(D)$ through $\chi(\tau)$.

Problem D. *Which elements of* Mod (p, n) *leave invariant* (*as a set*) *some Teichmüller disc in* $T(p, n)$?

A necessary and sufficient condition for $\chi(D) = D$ is that D and $\chi(D)$ have (at least) two points in common (Marden–Masur [26]). This question is closely related to understanding the action of Mod (p, n) on the various boundaries of $T(p, n)$. See Abikoff [1], Bers [10], Kerckhoff [19], Marden–Masur [26].

§ 7. The Thurston-Bers classifications of elements of Mod (p, n)

Let f be an automorphism of a surface of type (p, n) and χ the corresponding element of Mod (p, n). Bers' [9] important result on the classification of elements of Mod (p, n) has several parts. For an exposition of this and several related topics see Abikoff's monograph [2].

(A) The map χ is elliptic if and only if f is (isotopic to) a periodic mapping.

(B) If f is not (isotopic to) a periodic mapping, then χ is hyperbolic if and only if f is irreducible. The reducible (non-periodic) f correspond to parabolic or pseudohyperbolic χ.

(C) A non-periodic element $\chi \in \operatorname{Mod}(p, n)$ is hyperbolic if and only if it leaves invariant a Teichmüller line in $T(p, n)$. In this case we can choose a model $T(\Gamma)$ for $T(p, n)$ so that $\chi \in \operatorname{Mod}\Gamma$ is induced by a Teichmüller mapping w that conjugates Γ into itself and satisfies $K(w^2) = K(w)^2$. Equivalently, the initial and terminal quadratic differentials of w coincide. Such a w is called ([9]) an *absolutely extremal* (non-conformal) mapping.

(D) A parabolic or pseudohyperbolic χ (non-periodic) can always be induced by a completely reduced f. The component maps of f induce elements of modular groups of the parts of S which we will call the *restrictions* of χ. The map χ is parabolic if all the restrictions are periodic (or trivial) and pseudohyperbolic if at least one restriction is hyperbolic.

PROPOSITION 2. *Let $\chi \in \operatorname{Mod}(p, n)$ and let α be a positive integer. Then χ and χ^α are both of the same type.*

Remark. The identity element of the modular group will be considered to be of elliptic type.

Proof. If χ is elliptic, then so is χ^α because a fixed point of χ is certainly a fixed point of χ^α. If χ is hyperbolic, it has an invariant line l, then l is also invariant under χ^α. So χ^α is also hyperbolic. If χ is parabolic, then all the restrictions are elliptic (or trivial). Hence all the restrictions of χ^α are also elliptic, and χ^α is parabolic. If χ is pseudohyperbolic, it is reducible and at least one of the restrictions is hyperbolic. The same is true for χ^α. Since type is preserved by raising to power, the converse is also true.

Remark. As we have seen, mod Γ is a subgroup of Mod $\dot{\Gamma}$. Hence the above classification carries over to elements of mod Γ.

An amplification of a result (Theorem 6) of Bers [9] is contained in

THEOREM 1. *An element $\chi \in \operatorname{Mod}(p, n)$ leaves invariant a Teichmüller disc in $T(p, n)$ if and only if it can be induced by a conformal self-map or a Teichmüller self-mapping w of a Riemann surface S of type (p, n) satisfying the following two equivalent conditions:*

(a) *the mapping w^2 is also a Teichmüller mapping,*

(b) *the terminal differential of w is a multiple η of the initial differential with $|\eta| = 1$.*

A mapping χ that leaves invariant a disc is either elliptic (if w can be chosen conformal), hyperbolic (if $\eta = 1$, for some choice of w), or parabolic. Conversely, every elliptic or hyperbolic χ leaves invariant a disc, while no pseudohyperbolic χ can leave a disc invariant.

Proof. An observation of Kravetz [25] (see also Earle [12]) implies that if w_1 and w_2

$$S_2 \xleftarrow{w_2} S \xrightarrow{w_1} S_1$$

are Teichmüller mappings, then $w_1 \circ w_2^{-1}$ is also a Teichmüller mapping if and only if the Beltrami coefficient μ_1 of w_1 is a constant multiple of the Beltrami coefficient μ_2 of w_2. The case where either μ_1 or μ_2 is zero is trivial. Hence write

$$\mu_j = r_j \bar{\varphi}_j / |\varphi_j|, \quad j = 1, 2,$$

where $0<r_j<1$, $\varphi_j \in Q(S)$, and $\|\varphi_j\|=1$. It is easy to see that μ_1 is a multiple of μ_2 if and only if φ_1 is a multiple of φ_2. Recall that φ_j is the initial differential of w_j.

If w is a Teichmüller mapping, then w^2 is a Teichmüller mapping if and only if the initial differential of w is a multiple of the initial differential of w^{-1} (= the negative of the terminal differential of w). Thus we have established the equivalence of (a) and (b). The constant η must have absolute value 1 because w and w^{-1} have the same dilatation, and the quadratic differentials under consideration have norm one.

Assume that an element χ of Mod (p, n) leaves invariant a disc through $\tau \in T(p, n)$. Hence χ must restrict to a Möbius transformation on this disc. Choose a Fuchsian group Γ so that $T(\Gamma)$ becomes a model for $T(p, n)$ with the origin in $T(\Gamma)$ corresponding to τ. Choose $\varphi \in Q(U/\Gamma)$ with $\|\varphi\|=1$ so that the disc corresponding to φ is invariant under χ and set $\mu=\bar{\varphi}/|\varphi|$. Choose a quasiconformal w that induces χ. Without loss of generality w is a Teichmüller mapping. Then there exists a Möbius transformation A that fixes the unit disc Δ so that

$$\chi([t\mu]) = [\text{Beltrami coefficient of } w^{t\mu} \circ w^{-1}] = [A(t)\mu], \quad \text{for all } t \in \Delta.$$

In particular,

$$\chi([0]) = [\text{Beltrami coefficient of } w^{-1}] = [A(0)\mu],$$

and we conclude that the terminal differential of w is a multiple of φ, by the uniqueness part of Teichmüller's theorem. Similarly χ^{-1} leaves invariant the same disc in $T(\Gamma)$ and χ^{-1} is induced by w^{-1}. Hence the initial differential of w is also a multiple of φ.

The converse is established by direct computation. If w is a Teichmüller self-mapping of U that conjugates Γ into itself and satisfies condition (b), then we let φ be the initial differential of w and show that the disc through 0 corresponding to φ is invariant under χ, the element of Mod Γ induced by w. Write $\mu=\bar{\varphi}/|\varphi|$, and assume that the Beltrami coefficient of w is $k\mu$, for some k with $0<k<1$. The Beltrami coefficient of w^{-1} is then $\tilde{\eta}\mu$ with $\tilde{\eta} \in \mathbf{C}$, $|\tilde{\eta}|=k$. The chain rule shows that

$$\tilde{\eta}(\mu \circ w) = -\frac{w_z}{\overline{w_z}} k\mu, \tag{7.1}$$

and that the Beltrami coefficient of $w^{t\mu} \circ w^{-1}$ is

$$\left(\frac{(t-k)\mu}{1-kt}\right)\frac{w_z}{\overline{w_z}} \circ w^{-1} = -\frac{\tilde{\eta}}{k}\frac{t-k}{1-kt}\mu = A(t)\mu,$$

where the next to the last equality is a consequence of (7.1). Since $k>0$ and $|\tilde{\eta}/k|=1$, A is a Möbius transformation fixing the unit disc.

The last part of the theorem is essentially a consequence of the Bers classification of elements of Mod (p, n).

If $\chi \in \text{Mod}\,(p, n)$ is elliptic, then the fixed point set

$$T(p, n)^{\chi} = \{t \in T(p, n);\ \chi t = t\}$$

is again a Teichmüller space of type (p', n'). If χ is induced by the conformal self-map h of S, then p' is the genus of $\tilde{S} = S/\langle h \rangle$ and n' is the number of distinguished points on $\tilde{S}$ (which is the sum of the number of punctures on $\tilde{S}$ and the number of points over which the projection $S \to \tilde{S}$ is ramified). Thus as long as $3p' - 3 + n' > 0$, $T(p, n)^{\chi}$ has positive dimension, and every point of $T(p, n)^{\chi}$ is clearly contained in a Teichmüller disc invariant under χ. In general, view h as a conformal map fixing U, the universal covering space of $S = U/\Gamma$. The mapping h acts on $M(\Gamma)$ and $Q(S)$ by the rules

$$h^*\mu \circ h = \mu h'/\bar{h}', \quad \mu \in M(\Gamma),$$

$$h^*\varphi \circ h = \varphi (h')^2, \quad \varphi \in Q(S).$$

The first formula gives, of course, the action of h on $T(\Gamma)$. Since h conjugates Γ onto itself, h^* fixes $M(\Gamma)$ and $Q(S)$. From the above two formulae, we see that for the Teichmüller coefficient $\mu = \bar{\varphi}/|\varphi|$, we have

$$h^*\mu = \bar{\psi}/|\psi|, \quad \text{where } \psi = h^*\varphi.$$

It is quite easy to see that the linear mapping $h^*\colon Q(S) \to Q(S)$ has an eigenvalue (see, for example, Farkas–Kra [15, p. 256]); which must be of absolute value one. Say $h^*\varphi = \lambda\varphi$ with $\lambda \in \mathbf{C}$, $|\lambda| = 1$. Then $h^*\mu = \bar{\lambda}\mu$. Hence the Teichmüller disc determined by φ is invariant under χ.

If χ is hyperbolic, then, by Bers' [9] theorem, χ has an invariant line, and hence an invariant Teichmüller disc.

Finally, a pseudohyperbolic χ cannot have an invariant disc, since a Möbius transformation that fixes a disc must be elliptic, parabolic, or hyperbolic. In the first two cases $a(\chi) = 0$. In the last case, by Bers' [9] theorem, χ is hyperbolic.

COROLLARY 1 (of the proof). *Let $f\colon S \to S$ be a conformal mapping of a Riemann surface of non-excluded finite type. Let (p, n) be the type of the Riemann surface $S/\langle f \rangle$. (Thus n is the number of $\langle f \rangle$-equivalence classes of punctures on S.) The mapping f is reduced if $3p - 3 + n > 0$.*

Proof. The mapping f is reduced whenever we can find an admissible curve on $S/\langle f \rangle$.

Remark. Theorem 1 provides a partial solution to Problem D. The theorem does not treat, however, the case of parabolic elements. As we shall see, in § 9, we will exhibit a class of parabolic elements of Mod (p, n) induced by products of Dehn twists about admissible curves and their inverses. If no inverses appear (that is, if we consider a product of Dehn twists), then the corresponding element of the modular group has an invariant disc. This was proven by Marden–Masur [26]. If inverses of Dehn twists are also used, then one can produce parabolic elements of Mod (p, n) that do not have any invariant discs (H. Masur, private communication). Necessary and sufficient conditions for a parabolic element to have an invariant disc are not known.

PROPOSITION 3. *Let π: $F(\Gamma)\to T(\Gamma)$ be the Bers fiber space, where Γ is a torsion free Fuchsian group of type (p, n). Let $\theta\in\operatorname{mod}\Gamma$ and $\chi\in\operatorname{Mod}\Gamma$ be induced by the same quasiconformal automorphism w of U (so that* (5.1) *commutes). Then*

$$a(\theta)\geqslant a(\chi), \tag{7.2}$$

where $a(\ )$ is defined by (4.1).

In particular: If χ is hyperbolic or pseudohyperbolic, then so is θ. If θ is elliptic or parabolic, then so is χ (it could be the idenitity).

Moreover: If χ is hyperbolic, then θ is hyperbolic whenever we can replace Γ by a quasiconformally equivalent group so that χ and θ are induced by an absolutely extremal w (for the surface U/Γ) that fixes some point $z_0\in U$.

Proof. The Teichmüller metrics on $T(\Gamma)$ and $F(\Gamma)$ are the Kobayashi metrics on these spaces (Royden [34], see also Earle–Kra [13]) and hence all maps involved are distance nonincreasing. Then for $x\in F(\Gamma)$,

$$\langle x, \theta x\rangle \geqslant \langle \pi x, \pi\theta x\rangle = \langle \pi x, \chi\pi x\rangle \geqslant a(\chi);$$

from which (7.2) follows. Assume now that χ is hyperbolic. It therefore can be induced by an abolutely extremal w (after passing to a quasiconformally equivalent Γ). Let φ be the initial (and terminal) differential of w and set $\mu=\bar\varphi/|\varphi|$. Then the Teichmüller line corresponding to φ is invariant under χ; that is, there exists a Möbius transformation A that fixes the unit disc Δ (and has the open interval $(-1, 1)$ as its invariant axis) so that

$$\chi([t\mu]) = [A(t)\mu],\quad t\in\Delta. \tag{7.3}$$

Write

$$w = B\circ w_{t_0\mu},$$

for some t_0 with $0<t_0<1$ and B a Möbius transformation. Then for any $z_0\in U$,

$$\theta([t\mu], w^{t\mu}(z_0)) = ([(At)\mu], w^{(At)\mu}\circ w(z_0)).$$

Hence we can produce an invariant line in $F(\Gamma)$ if we can find a $z_0 \in U$ with $w(z_0) = z_0$.

Let

$$\mathcal{D}: \operatorname{mod} \Gamma \to \operatorname{Mod} \Gamma$$

be the canonical projection.

COROLLARY 2. *For every hyperbolic* $\chi \in \operatorname{Mod} \Gamma$, *we can find a hyperbolic* $\theta \in \operatorname{mod} \Gamma$ *and a positive integer* α *so that* $\mathcal{D}(\theta) = \chi^\alpha$, *provided* Γ *is of type* (p, n) *with* $p \geq 2$.

Proof. Without loss of generality χ is induced by an absolutely extremal w that is a Teichmüller mapping conjugating Γ into itself. Let φ be the initial differential of w. Let W be the self-map of U/Γ induced by w. Then W permutes the zeros of the projection of φ to U/Γ. Hence a power of W fixes these zeros. We conclude that for some $z_0 \in U$ and some positive integer α, there is a $g \in \Gamma$ with

$$w^\alpha(z_0) = g(z_0).$$

(The assumption $p \geq 2$ is needed to guarantee a non-empty zero set for φ.) Let θ be the element of $\operatorname{mod} \Gamma$ induced by $g^{-1} \circ w^\alpha$. Then θ is hyperbolic by Proposition 3, and $\mathcal{D}(\theta) = \chi^\alpha$.

§ 8. Metrics on the fibers of $F(\Gamma)$

Let $\pi: F(\Gamma) \to T(\Gamma)$ be the Bers fiber space, where Γ is a torsion free Fuchsian group of type (p, n) satisfying (2.1). For every $\tau \in T(\Gamma)$, $\pi^{-1}(\tau)$ is a domain in $\mathbf{C} \cup \{\infty\}$ bounded by a quasicircle passing through $0, 1, \infty$. The domain $\pi^{-1}(\tau)$ has two canonically defined metrics on it: the Teichmüller metric $\langle \cdot, \cdot \rangle$, and the non-Euclidean metric ϱ of constant negative curvature -4. Both metrics are invariant under $\operatorname{mod} \Gamma$.

PROPOSITION 4. (a) *Let* $x, y \in \pi^{-1}(\tau)$. *Then for* $x \neq y$,

$$\varkappa(\varrho(x, y)) < \langle x, y \rangle \leq \varrho(x, y), \tag{8.1}$$

where $\varkappa$ *is the function of Lemma* 1. *Further, if* Γ *has type* $(0, 3)$, *then*

$$\langle x, y \rangle = \varrho(x, y). \tag{8.2}$$

(b) *In general, the Teichmüller metric on* $\pi^{-1}(\tau)$ *is complete.*

(c) *For* $x_j = ([\mu_j], z_j) \in F(\Gamma)$, $j = 1, 2$,

$$d(z_1, z_2) \leq \langle x_1, x_2 \rangle,$$

where d *is the non-Euclidean metric on* $\mathbf{C} \setminus \{0, 1\}$.

Proof. Assertion (b) is obvious since the Teichmüller metric is complete on $F(\Gamma)$ and $\pi^{-1}(\tau)$ is closed in $F(\Gamma)$. Alternately, (b) follows form (a) since both ϱ and $\varkappa\circ\varrho$ are complete metrics on the domain $\pi^{-1}(\tau)$.

To obtain the inequalities in (8.1), it suffices to assume that $\tau=0$, and thus that $\pi^{-1}(0)=U$. We identify $F(\Gamma)$ with $T(\Gamma;a)$ for some $a\in U$ and take $x=a$. There is now a unique formal Teichmüller Beltrami coefficient

$$\mu_0=t\bar{\varphi}/|\varphi|\in M(\Gamma),$$

where $\varphi\in Q((U\setminus A)/\Gamma)$, φ has a simple pole at a, $\|\varphi\|=1$, and $t>0$, such that $[\mu_0]=[0]$, and

$$w^{\mu_0}(a)=y$$

($A=\Gamma a$, and we abreviate $\bar{\varphi}/|\varphi|$ by μ). It follows that

$$\langle x,y\rangle=\tfrac{1}{2}\log\frac{1+t}{1-t}.$$

Now the map $z\mapsto([0],z)$ of U into $F(\Gamma)$ is holomorphic. Since the Teichmüller metric on $F(\Gamma)$ is the Kobayashi metric, this map is distance non-increasing. Hence

$$\langle x,y\rangle\leqslant\varrho(x,y).$$

If $\pi^{-1}(0)$ were a Teichmüller disc, then the map

$$\Delta\ni z\mapsto([0],w^{z\mu}(a))\in F(\Gamma) \tag{8.3}$$

would be an isometric mapping of the unit disc Δ onto $\pi^{-1}(0)$ in $F(\Gamma)$. This would mean that the Beltrami coefficient $z\mu$ would be trivial for all $z\in\mathbf{C}$ with $|z|<1$. In particular, $w^{z\mu}(x)=x$ for all $x\in\mathbf{R}$, and hence

$$\frac{d}{dt}w^{t\mu}(x)\bigg|_{t=0}=0,\quad\text{all } x\in\mathbf{R};$$

that is (see Bers [7]),

$$\frac{x(x-1)}{2\pi i}\iint_U\frac{\mu(\zeta)\,d\zeta\wedge d\bar{\zeta}}{\zeta(\zeta-1)(\zeta-x)}=0,\quad\text{all } x\in\mathbf{R},$$

or μ would also be *locally trivial*. Now if Γ is of type (0, 3), then $T(\Gamma)$ is a point and $F(\Gamma)=U$. In this case, the holomorphic mapping (8.3) is distance non-increasing establishing the equality (8.2).

To establish the (first) strict equality in (8.1), note that w^{μ_0} is the identity on $\mathbf{R}$ and sends x into y. Hence it is a competing function for the extremal problem associated with Lemma 1. It is clearly not the extremal function for this problem (see Teichmüller [35]). Hence

$$\varkappa(\varrho(x, y)) < \langle x, y \rangle,$$

establishing part (a).

Part (c) follows from the fact that the mapping (projection onto the second coordinate)

$$F(\Gamma) \ni ([\mu], z) \mapsto z \in \mathbf{C} \setminus \{0, 1\}$$

is distance non-increasing with respect to $\langle \cdot, \cdot \rangle$ and d (see Kra [23]).

COROLLARY 3 (of the proof). *The fiber $\pi^{-1}(0)$ is a Teichmüller disc if and only if there exists a formal Teichmüller Beltrami coefficient μ of norm* 1, $\mu = \bar{\varphi}/|\varphi|$, *where φ has a simple pole at $\hat{a} \in U/\Gamma$ with $\varphi \in Q((U/\Gamma) \setminus \{\hat{a}\})$, $\|\varphi\| = 1$, such that $z\mu$ is a trivial coefficient for all $z \in \Delta$.*

COROLLARY 4 (of the proof). *There exist Beltrami coefficients $\mu \neq 0$ such that $z\mu$ is trivial for each $z \in \mathbf{C}$, $|z| < 1/\|\mu\|$.*

Proof. Take any Beltrami coefficient for a triangle group (a group Γ of type (0, 3)—torsion is permitted), and use the fact that the corresponding Teichmüller space is zero dimensional.

Remark. Some years ago this author asked the following question. If μ is a trivial Beltrami coefficient and $0 < t < 1$, is $t\mu$ also trivial? Gehring [17] produced an example to show that the answer is no. The following simple example due to Edgar Reich also shows that in general the answer is no. Reich works with the unit disc Δ. Fix $\alpha \in \mathbf{C}$, $|\alpha| < 1$, and put

$$w(z) = z + \alpha(1 - z\bar{z}), \quad z \in \Delta.$$

Then w is a quasiconformal automorphism of the unit disc fixing the unit circle. The Beltrami coefficient of w is

$$\mu(z) = \frac{-\alpha z}{1 - \alpha \bar{z}}.$$

If $\alpha \neq 0$, then μ is not locally trivial since for any L^1-holomorphic function f on Δ,

$$\iint_\Delta \mu(z) f(z) \left| \frac{dz \wedge d\bar{z}}{2} \right| = -\pi \int_{z=0}^{\alpha} z f(z)\, dz;$$

which is non-zero for $f = 1$, for example.

In § 10 we will give more interesting examples involving formal Teichmüller differentials.

For $g\in\Gamma$, we define two functions on $T(\Gamma)$:

$$f_1(\tau)=\inf_{x\in\pi^{-1}(\tau)}\langle x, gx\rangle,$$

$$f_2(\tau)=\inf_{x\in\pi^{-1}(\tau)}\varrho(x, gx).$$

It is well known that $f_2(\tau)$ is the length of the geodesic corresponding to the element g of the fundamental group of the Riemann surface represented by τ. For parabolic g,

$$f_1(\tau)=0=f_2(\tau),\quad \text{all } \tau\in T(\Gamma).$$

If g is hyperbolic, then for $\tau=[\mu]$, $\mu\in M(\Gamma)$,

$$f_2(\tau)=\tfrac{1}{2}\left|\log\lambda_\mu\right|,$$

where λ_μ is the multiplier of the hyperbolic element

$$g_\mu=w_\mu\circ g\circ w_\mu^{-1},$$

and

$$f_2(\tau)=\varrho(z, g_\mu z),\quad \text{for all } z \text{ on the axis of } g_\mu.$$

(See § 9.) Thus we also have

PROPOSITION 5. *For hyperbolic* $g\in\Gamma$, f_2: $T(\Gamma)\to(0,\infty)$ *is a continuous (real analytic) function, and*

$$\varkappa(f_2(\tau))\leqslant f_1(\tau)\leqslant f_2(\tau),\quad \textit{all } \tau\in T(\Gamma). \tag{8.4}$$

§ 9. Solution to Problem A

We need a slight generalization of a well known fact about (Teichmüller) trivial automorphisms.

LEMMA 2. *Let* Γ *be a finitely generated torsion free Fuchsian group of the first kind. Let* l_j, $j=1, \ldots, k$ *be the axis of a hyperbolic element* $\gamma_j\in\Gamma$. *Assume that these axes project to disjoint Jordan curves on* U/Γ *under the canonical projection* q: $U\to U/\Gamma$. *Let* c_j *be a closed collar about* $q(\gamma_j)$, *with these collars pair-wise disjoint. Let* $C=q^{-1}(c_1\cup\ldots\cup c_k)$. *Then* $U'=U\setminus C$ *is open in* U, *and for each pair of points* x, y *in the same component of* U', *there exists a quasi-*

conformal automorphism w of U such that $w(x)=y$,

$$w\circ g = g\circ w, \quad \textit{for all } g\in\Gamma \tag{9.1}$$

(*hence w is the identity on* $\mathbf{R}$), *and*

$$w \textit{ is the identity on } C. \tag{9.2}$$

Proof. Let $c=c_1\cup\ldots\cup c_k$. Since $U'=q^{-1}(S\setminus c)$, U' is open in U. Let D be a component of U', and let $a\in D$. Define

$D'=\{y\in D;\ \exists$ a quasiconformal automorphism w of U satisfying $w(a)=y$, (9.1), and (9.2)$\}$.

The set D' is not empty since $a\in D'$. Assume that $y_0\in D'$, and assume that w_0 is the corresponding map. Choose a small disc D_1 around y_0 so that $D_1\subset D$ and $q|D_1$ is injective. For $y\in D_1$, there exists a quasiconformal automorphism w_1 of D_1 such that w_1 is the identity on the boundary of D_1 and $w_1(y_0)=y$. We extend w_1 to ΓD_1(= the image in U of D_1 under Γ) by invariance, and to be the identity on $U\setminus\Gamma D_1$. Then $w=w_1\circ w_0$ sends a to y and satisfies (9.1) and (9.2). Thus $D_1\subset D'$ and D' is open in D. Precisely the same argument shows that D' is closed in D. Hence $D=D'$.

We now return to Problem A and use the notation of § 2. We represent the surface S with some conformal structure by a torsion free finitely generated Fuchsian group Γ of the first kind so that $S=U/\Gamma$. We choose $a\in U$ so that $\dot{S}=(U/\Gamma)\setminus\{\hat{a}\}$. Finally, we choose another Fuchsian group $\dot{\Gamma}$ so that $\dot{S}=U/\dot{\Gamma}$.

We have seen (Proposition 1 of § 5), that the elements of Isot $(S,\hat{a})$ are classified by $\pi_1(S)\cong\Gamma$. We shall call a hyperbolic element $g\in\Gamma$ *simple* if g is a power of an element whose axis projects to an admissible Jordan curve on S. If g is not simple, it is called *essential* (according to Maskit–Matelski [27]) if the axis of g projects to a curve that intersects every admissible curve on S. The element $g\in\Gamma$ is essential if and only if the projection of the axis of g is a *filling* curve on S, as defined by Thurston [36]. See also [32] and Figures 2 and 3.

It is easy to verify that g is essential if and only if the complement in S of the projection of the axis of g consists of a union of discs and punctured discs.

THEOREM 2. *Let $1\neq f\in$ Isot $(S,\hat{a})$ and let $g\in\Gamma$ be the corresponding elements of $\Gamma\cong\pi_1(S)$.*

(a) *If S has type $(0,3)$ then g is a parabolic (hyperbolic) element of* mod Γ *if and only if g is a parabolic (hyperbolic) element of Γ. In particular, f is reducible if and only if g is parabolic.*

(b) *Assume that S is not of type $(0,3)$, then:*

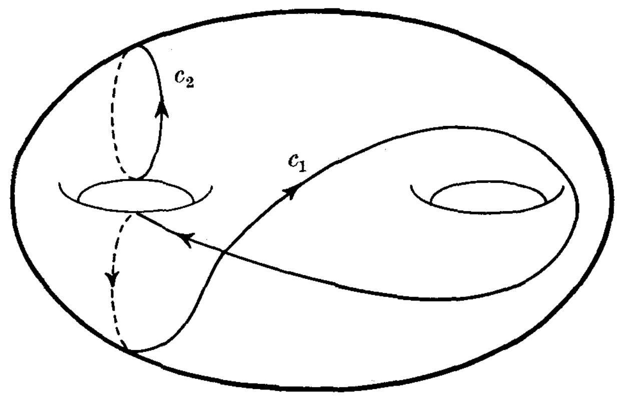

Figure 2. A non-essential, non-simple curve (c_1) on a surface of type (2,0). c_2 is a reducing curve for the corresponding self-map of a surface of type (2,1).

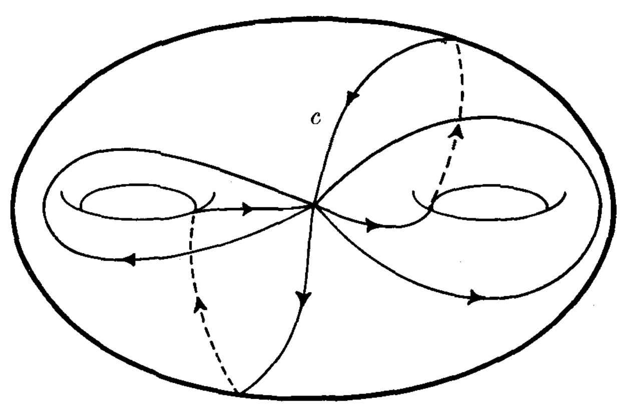

Figure 3. An essential curve on a surface of type (2,0).

(i) *g is a parabolic element of* $\operatorname{mod}\Gamma$ *if and only if g is either a parabolic or a simple hyperbolic element of* Γ,

(ii) *g is a hyperbolic element of* $\operatorname{mod}\Gamma$ *if and only if g is an essential hyperbolic element of* Γ*, and*

(iii) *g is pseudohyperbolic element of* $\operatorname{mod}\Gamma$ *if and only if g is a non-simple non-essential element of* Γ.

In particular, f is reducible if and only if g is not a hyperbolic essential element of Γ.

Remark. For type (0, 3) every hyperbolic element of Γ is essential. Hence (a) is a special case of (b).

Proof of the theorem. Part (a) is completely obvious since the Teichmüller metric on $F(\Gamma) \cong U$ agrees with the Poincaré metric on U (by Proposition 4 of § 8). To prove part (b) we examine the action of g on $F(\Gamma)$. Recall that

$$g([\mu], z) = ([\mu], g^{\mu}(z)),$$

where

$$z \in w^{\mu}(U), \quad \mu \in M(\Gamma), \quad \text{and} \quad g^{\mu} = w^{\mu} \circ g \circ (w^{\mu})^{-1}.$$

By Proposition 4, for every $x \in F(\Gamma)$,

$$\langle x, gx \rangle \leqslant \varrho(x, gx).$$

Note that $\varrho(x, gx)$ is well defined since x and gx are always in the same fiber. It is clear that for a parabolic element g of Γ, $a(g) = 0$ (see § 8). Assume that g is a hyperbolic element of Γ. By Proposition 2 of § 7, it suffices to assume that g is primitive (not a power of another element of Γ). We will compute for certain $z \in w^{\mu}(U)$, the Poincaré distance $\varrho(z, g^{\mu}(z))$.

Choose a normalized Riemann map $h: w^{\mu}(U) \to U$. Note that $h \circ w^{\mu}$ is a normalized μ-conformal automorphism of U and hence equal to w_{μ}, and that

$$g_{\mu} = w_{\mu} \circ g \circ w_{\mu}^{-1} = h \circ g^{\mu} \circ h^{-1} \quad (\text{all } g \in \Gamma).$$

Hence $\Gamma_{\mu} = w_{\mu} \Gamma w_{\mu}^{-1}$ is the Fuchsian model for the quasi-Fuchsian group Γ^{μ}. Invariance of the Poincaré distance under conformal maps shows that

$$\varrho_{w^{\mu}(U)}(z, g^{\mu}(z)) = \varrho_U(h(z), g_{\mu} \circ h(z)),$$

for all $z \in w^{\mu}(T)$. Thus

$$\inf_{z \in w^{\mu}(U)} \varrho_{w^{\mu}(U)}(z, g^{\mu}(z)) = \inf_{z \in U} \varrho_U(z, g_{\mu}(z)) = \tfrac{1}{2} |\log \lambda_{\mu}|, \tag{9.3}$$

where λ_{μ} is the multiplier of g_{μ}. The axis of g_{μ} projects to a curve on U/Γ_{μ}. Assume now that g is simple and primitive. Bers [6] has shown how to construct a sequence $\mu_j \in M(\Gamma)$ so that

$$\lim_{j \to \infty} \lambda_{\mu_j} = 1.$$

Hence the infimum in (9.3) must be zero, and inequality (8.1) shows that $a(g) = 0$, or that g is a parabolic element of mod Γ.

If g is a hyperbolic and non-simple element of Γ, then the axis of g projects to a closed geodesic on S with a non-trivial self intersection. By the Keen-Halpern collar lemma (see Matelski [28]), there is a constant $C>0$ such that

$$f_2(\tau) \geqslant C, \quad \text{all } \tau \in T(\Gamma)$$

(see Proposition 5). Hence $a(g)>0$ and g must be a hyperbolic or pseudohyperbolic element of mod Γ.

Assume that g is essential. (See Figure 3.) Choose a minimizing sequence

$$\{x_j\} \subset F(\Gamma),$$
$$\lim_{j\to\infty} \langle x_j, gx_j\rangle = a(g). \tag{9.4}$$

Recall that $F(\Gamma) \cong T(\dot{\Gamma})$ and thus each poipt $x_j \in F(\Gamma)$ represents a surface of type $(p, n+1)$ in $T(\dot{\Gamma})$, where Γ has type (p, n). There are two possibilities:

(I) There exist a constant $C_1>0$ such that all simple closed geodesics (admissible curves) on all x_j have length $\geqslant C_1$, or

(II) By passing to a subsequence we may assume x_j carries an admissible curve l_j of length ε_j with

$$\lim_{j\to\infty} \varepsilon_j = 0.$$

In case (I), we modify an argument of Bers [9] to show that the minimum $a(g)$ is achieved. By Lemma 4 of [9], we may assume, by passing to a subsequence if necessary, that there exists a $\theta_j \in \operatorname{Mod} \dot{\Gamma}$ such that $y_j = \theta_j x_j$ converges to an element $y \in F(\Gamma)$. Since each θ_j is an isometry,

$$\langle y_j, \theta_j \circ g \circ \theta_j^{-1}(y_j)\rangle = \langle x_j, gx_j\rangle.$$

Thus by (9.4)

$$\lim_{j\to\infty} \langle y_j, \theta_j \circ g \circ \theta_j^{-1}(y_j)\rangle = a(g). \tag{9.5}$$

Since $y = \lim_{j\to\infty} y_j$, we may assume (by selecting a subsequence if necessary) that the sequence $\{\theta_j \circ g \circ \theta_j^{-1}(y_j)\}$ converges to some point $z \in F(\Gamma)$. We claim that

$$\lim_{j\to\infty} \theta_j \circ g \circ \theta_j^{-1}(y) = z. \tag{9.6}$$

This follows from the inequalities,

$$\begin{aligned}\langle \theta_j \circ g \circ \theta_j^{-1}(y), z\rangle &\leqslant \langle \theta_j \circ g \circ \theta_j^{-1}(y), \theta_j \circ g \circ \theta_j^{-1}(y_j)\rangle + \langle \theta_j \circ g \circ \theta_j^{-1}(y_j), z\rangle \\ &= \langle y, y_j\rangle + \langle \theta_j \circ g \circ \theta_j^{-1}(y_j), z\rangle.\end{aligned}$$

The limit (9.6) implies that for any $\varepsilon > 0$,

$$\langle(\theta_j \circ g \circ \theta_j^{-1})^{-1} \circ (\theta_i \circ g \circ \theta_i^{-1})(y), y\rangle \leqslant \varepsilon$$

for large i and j. Since Mod $\dot{\Gamma}$ acts properly discontinuously on $F(\Gamma)$, we may assume that $\theta_j \circ g \circ \theta_j^{-1}$ is constant (by passing to a subsequence). Setting

$$\theta_j \circ g \circ \theta_j^{-1} = \tilde{g} \in \operatorname{Mod} \dot{\Gamma},$$

we conclude that $\langle y, \tilde{g}y\rangle = a(g)$, by (9.6), or that

$$\langle x, gx\rangle = a(g),$$

where $x = \theta_j^{-1} y$ (for any j). Hence g is a hyperbolic element of mod Γ.

In case (II), we modify an argument of Maskit–Matelski [27]. The curve l_j is also an admissible curve on the surface $\tau_j = \pi(x_j) \in T(\Gamma)$. The length of this curve l_j on τ_j is less than ε_j. Since the axis of g projects to a curve that crosses l_j, the Keen–Halpern collar lemma implies that the length of the axis of g on τ_j must go to infinity. Thus

$$\lim_{j\to\infty} \inf_{z \in \pi^{-1}(\tau_j)} \varrho(z, gz) = \lim_{j\to\infty} \tfrac{1}{2}|\log \lambda_j| = +\infty,$$

where λ_j is the multiplier of g_{μ_j}, with $\mu_j \in M(\Gamma)$, $[\mu_j] = \tau_j$. Thus

$$\lim_{j\to\infty} \varrho(x_j, gx_j) = +\infty,$$

and by Proposition 4 of § 8, the sequence $\{x_j\}$ could not have been a minimizing sequence for (9.4).

Before considering the case of non-simple non-essential hyperbolic $g \in \Gamma$, we investigate the question of finding reducing curves for reducible elements of Isot $(S, \hat{a})$.

Consider first a primitive parabolic element $g \in \Gamma$. We want to determine the action of the corresponding element $f \in$ Isot $(S, \hat{a})$ on $\dot{S}$. Without loss of generality $g(z) = z + 1$ and a satisfies Im $a > 1$. Thus, by a well known lemma of Shimizu–Leutbecher (see, for example, Kra [21, pp. 58–62]), we conclude that $\gamma(U_1) \cap U_1$ is empty for all $\gamma \in \Gamma$ such that γ is not a power of g, where

$$U_1 = \{z \in \mathbf{C};\ \operatorname{Im} z > 1\}.$$

We must construct a quasiconformal automorphism w_0 that commutes with Γ and satisfies $w_0(a) = a + 1$. Choose $\varepsilon > 0$ so that

$$\bar{U}_2 = \{z \in \mathbf{C};\ \operatorname{Im} a - \varepsilon \leqslant \operatorname{Im} z \leqslant \operatorname{Im} a + \varepsilon\} \subset U_1.$$

Define

$$w_0(z) = \operatorname{Re} z + 1 - \frac{1}{\varepsilon} |\operatorname{Im} (z-a)| + i \operatorname{Im} z, \quad z \in \bar{U}_2.$$

Note that $w_0(z+1) = w_0(z)+1$, and that w_0 is the identity on the boundary of $\bar{U}_2$. We extend w_0 to $\Gamma \bar{U}_2$ by invariance:

$$w_0(\gamma z) = \gamma \circ w_0(z), \quad z \in \bar{U}_2, \gamma \in \Gamma,$$

and set it to be the identity on $U \setminus \Gamma \bar{U}_2$. We compute the action of $w = w_0^{-1} \circ g$ on $\dot{S} = (U \setminus A)/\Gamma$, where $A = \Gamma a$. Note that $q(\bar{U}_2) \setminus \{\hat{a}\}$ is a punctured annulus $\mathfrak{a}$ on $\dot{S}$, and that the projected map $W: \dot{S} \to \dot{S}$ is the identity outside this annulus. (Here q is the projection of U to U/Γ.) By modifying our construction slightly we may assume that W is the identity also on a smaller punctured annulus around $\hat{a}$. We can draw two conclusions: W is the inverse of a Dehn twist about the curve $q(\{\operatorname{Im} z = a - \varepsilon\})$ (see Marden–Masur [26]), and W is completely reduced by a family of admissible curves in $\dot{S} \setminus \mathfrak{a}$. See Figures 4 and 5.

We now assume that g is a primitive simple hyperbolic element of Γ. Without loss of generality we assume that $g(z) = \lambda z$, $\lambda \in \mathbf{R}$, $\lambda > 1$. The axis of g projects to a geodesic on S. We take a collar neighborhood of this geodesic on S and lift it to U. Without loss of generality we may assume that one component of this set is of the form

$$\bar{U}_2 = \left\{ z \in U;\ z = re^{i\theta},\ r > 0, \frac{\pi}{2} - \varepsilon \leqslant \theta \leqslant \frac{\pi}{2} + \varepsilon \right\}.$$

We take $\boldsymbol{a}$ to be on the imaginary axis, and we define w_0 (in polar coordinates) by

$$w_0(r, \theta) = \left(r \exp \left[\left(1 - \frac{1}{\varepsilon} \left| \theta - \frac{\pi}{2} \right| \right) \log \lambda \right], \theta \right), \ (r, \theta) \in \bar{U}_2.$$

Again, $w_0(\lambda z) = \lambda z$, $z \in \bar{U}_2$ and w_0 is the identity on the boundary of $\bar{U}_2$. Hence we continue exactly as in the case of a parabolic element of Γ. Here W is a product of the inverse of a Dehn twist about the the curve

$$q \left(\left\{ \operatorname{Arg} z = \frac{\pi}{2} - \varepsilon \right\} \right),$$

followed by a Dehn twist about the curve

$$q \left(\left\{ \operatorname{Arg} z = \frac{\pi}{2} + \varepsilon \right\} \right).$$

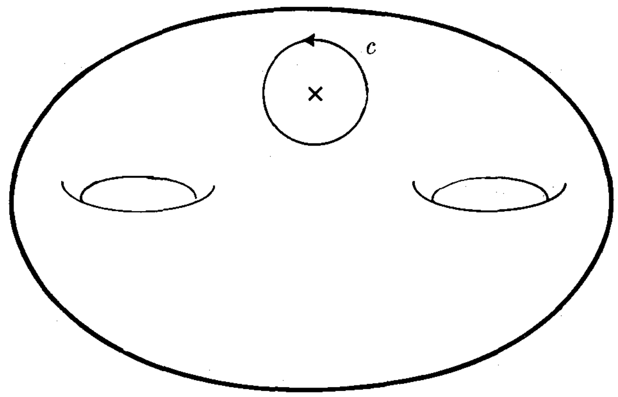

Figure 4. A curve on a surface of type $(2, 1)$ corresponding to a parabolic element of the covering group.

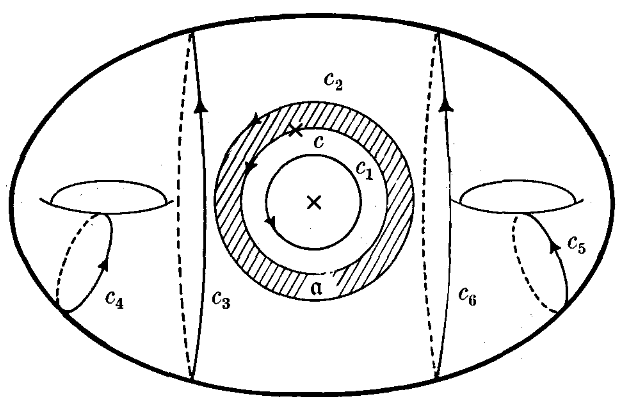

Figure 5. The inverse of the Dehn twist about c_2 on a surface of type $(2, 2)$ corresponding to the parabolic element of Figure 4. $\mathfrak{a}$ = annulus where "action" of Dehn twist takes place. $c_2, \ldots, c_6$ are reducing curves for Dehn twist. Note that the restriction of this map to $\mathfrak{a}$ may be identified with the restriction of the spin in Figure 6 to the corresponding $\mathfrak{a}$, since the Dehn twist about c_1 can be "unwound".

Such a map is known as a *spin* (see Birman [11] and the literature quoted there) on $\dot{S}$ about the curve

$$q\left(\left\{\operatorname{Arg} z = \frac{\pi}{2}\right\}\right).$$

See Figures 1, 6, and 7.

Remark. The case of simple parabolic $g \in \Gamma$ is similar to the case of a simple hyperbolic g. In the first case, the second Dehn twist can always be unwound to become the trivial map.

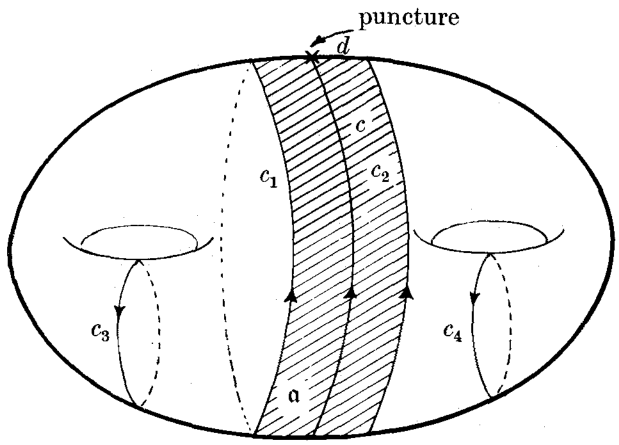

Figure 6. The spin about c on a surface of type $(2, 1)$ corresponding to the admissible element of Figure 1. $\mathfrak{a}$ is the punctured annulus where "action" of spin takes place. The spin about c is the inverse of the Dehn twist about c_2 followed by a Dehn twist about c_1. The admissible curves c_3 and c_4 along with c_1 and c_2 are reducing curves for the spin.

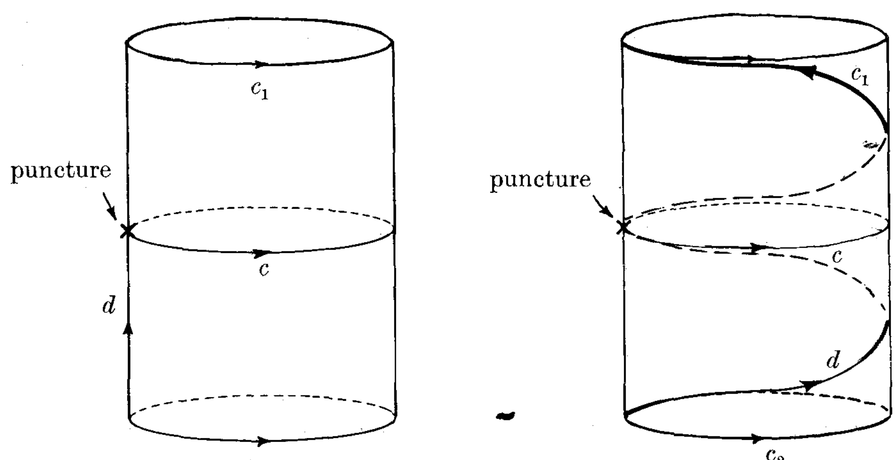

Figure 7. The action of the spin of Figure 6 on the curves d, c, c_2, c_1 of the punctured annulus $\mathfrak{a}$.

Finally, continuing with the proof of Theorem 2, we are ready to consider hyperbolic non-simple, non-essential elements $g \in \Gamma$. (See Figure 2.) In this case there exists an admissible curve c on $S = U/\Gamma$ such that c is disjoint from the projection of the axis of g to U/Γ. The curve c is also admissible on $\dot{S}$. By Lemma 2, we can choose w to be the identity on the preimage of c in U. Hence c is a reducing curve for W (defined as above). Since W is reducible, g must be parabolic or pseudohyperbolic. Since it cannot be parabolic, we have completed the proof of Theorem 2.

Remarks. (1) We have shown that for non-essential g, there is always an admissible curve on U/Γ that is disjoint from the image of the axis of g. By taking a maximal set of disjoint and homotopically distinct curves of this type, we clearly can obtain an admissible set C of Jordan curves so that the self map f of $\dot{S}$ corresponding to g is completely reduced by C.

(2) Maskit–Matelski [27] have showed that for essential elements $g \in \Gamma$, the function f_2 (of Proposition 5 of § 8) achieves a minimum on $T(\Gamma)$. Theorem 2 shows that these are also the elements g of Γ for which $f(x) = \langle x, gx \rangle$ achieves a minimum somewhere on $F(\Gamma)$. The relation between these two extremal problems is not completely clear. See § 12.

S. Wolpert has informed the author (oral communication) that for essential $g \in \Gamma$ the function f_2 is a Morse function on $T(\Gamma)$, as a consequence of Kerckhoff's work [20]. It has a unique minimum on $T(\Gamma)$ and every critical point of f_2 is an absolute minimum.

(3) The fact that parabolic and simple hyperbolic elements of Γ act as parabolic elements of mod Γ was also obtained by Nag [29].

COROLLARY 5. *Let S be a surface of non-exluded type $(p, n) \neq (0, 3)$. There exist non-conformal absolutely extremal self maps of S (with respect to some conformal structure).*

Proof. If $n > 0$, the result follows immediately from our theorem and the (easily verified) existence of essential curves on a surface of type $(p, n-1)$, since $T(p, n)$ is isomorphic to the Bers fiber space of a surface of type $(p, n-1)$. (Except if $(p, n) = (1, 1)$, where $T(1, 1)$ is isomorphic to the fiber space over a point.) It thus remains to consider type $(p, 0)$, $p \geqslant 2$. The case $p = 2$ follows trivially from the isomorphism $T(2, 0) \cong T(0, 6)$. In general, we know that surfaces of type $(0, 2n+2)$ admit absolutely extremal maps that are not holomorphic. Let $x_1, ..., x_{2n+2}$ be $2n+2$ distinct points on $\mathbf{C} \cup \{\infty\}$. Let $\Sigma = \mathbf{C} \cup \{\infty\} \setminus \{x_1, ..., x_{2n+2}\}$. Let w be an absolutely extremal non-holomorphic self-map of Σ. By our theorem such a map exists for some choice of x_j, $j = 1, ..., 2n+2$. Let S be a two sheeted cover of $\mathbf{C} \cup \{\infty\}$ that is branched over x_j, $j = 1, ..., 2n+2$. The surface S is, of course, a hyperelliptic Riemann surface. Lift w to a self map W of S. The maps w and w^2 are Teichmüller mappings, with $K(w^2) = K(w)^2$. Hence W and W^2 are Teichmüller mappings with $K(W^2) = K(W)^2$. Thus W is aboslutely extremal by Theorem 6 of Bers [9].

In the proof of the above corollary, we have encountered an interesting

Open Problem. Let $\Sigma = \mathbf{C} \setminus \{0, 1, z_1, ..., z_{n-3}\}$ with $n \geqslant 4$ and $\{z_1, ..., z_{n-3}\}$ distinct in $\mathbf{C} \setminus \{0, 1\}$. Find necessary and sufficient conditions for the existence of non-holomorphic absolutely extremal self-maps of Σ.

It is instructive to reformulate Theorem 2 in purely topological terms. This reformulation is contained in Theorem 2′ of the introduction. It should be noted that we can actually distinguish topologically the four types (elliptic, parabolic, hyperbolic, pseudo-hyperbolic) of Bers, rather than just the two types (reducible, irreducible) of Thurston. Similarly, the concept of a parabolic element of Γ can be described completely in terms of $\pi_1(S)$: a parabolic element of Γ corresponds to a loop that is contractible to a puncture on S. Thus we also have

THEOREM 2′ (addendum). *Furthermore,*

(1) *c is a power of a Jordan curve on S if and only if (w is reducible and) all the component maps of $w\colon \dot{S}\to\dot{S}$ are isotopic to periodic maps, and*

(2) *c is a non-essential non-simple curve on S if and only if some component map of $w\colon \dot{S}\to\dot{S}$ is irreducible.*

§ 10. Solutions to Problems B and C

THEOREM 3. *Let Γ be a torsion free Fuchsian group of type $(p, n)\neq(0, 3)$. Then for all $x, y\in F(\Gamma)$ with $x\neq y$, and $\pi(x)=\pi(y)$, we have*

$$\langle x, y\rangle<\varrho(x, y).$$

In particular, for every $\tau\in T(\Gamma)$, the fiber $\pi^{-1}(\tau)$ is not a Teichmüller disc.

Proof. Assume there exists a $\tau\in T(\Gamma)$ and x and $y\in\pi^{-1}(\tau)$ such that

$$\langle x, y\rangle=\varrho(x, y). \tag{10.1}$$

Without loss of generality we may assume that $\tau=0$ and $x=a$ is used as base point for identifying $F(\Gamma)$ with a Teichmüller space $T(\dot{\Gamma})\cong T(\Gamma; a)$. Thus we may write

$$y=w^{t_0\mu}(a),$$

where $\mu=\bar{\varphi}/|\varphi|$, φ is an integrable meromorphic form ($\varphi\in Q((U/\Gamma)\setminus\{\hat{a}\})$) of the type described in § 8, $0<t_0<1$, and

$$\varrho_U(x, y)=\tfrac{1}{2}\log\frac{1+t_0}{1-t_0}=\varrho_\Delta(0, t_0).$$

Consider any point z on the geodesic line segment in the ϱ metric between $x=a$ and y. We know that

$$\langle x, z\rangle\leqslant\varrho(x, z),\quad \langle z, y\rangle\leqslant\varrho(z, y),$$

by Proposition 4, and that

$$\varrho(x, z) + \varrho(z, y) = \varrho(x, y) = \langle x, y \rangle,$$

by the assumption (10.1) and the choices made. By the triangle inequality

$$\langle x, z \rangle = \varrho(x, z).$$

We conclude that the segment in U (between x and y) is also a geodesic ray in the Teichmüller metric. Thus for any z on this segment

$$z = w^{t\mu}(a) \quad \text{with } \varrho(x, z) = \tfrac{1}{2} \log \frac{1+t}{1-t}.$$

In particular, $0 \leqslant t \leqslant t_0$ implies that $t\mu$ is trivial; that is,

$$w^{t\mu}(\xi) = \xi, \quad \text{all } \xi \in \mathbf{R}. \tag{10.2}$$

Since for fixed $\xi \in \mathbf{R}$,

$$t \mapsto w^{t\mu}(\xi)$$

is a holomorphic function from the disc Δ into $\mathbf{C}$, we conclude by the identity theorem for holomorphic functions that (10.2) holds for all $t \in \Delta$, and that $t\mu$ is trivial for all $t \in \Delta$. Thus the assumption (10.1) for a single pair of points implies that $\pi^{-1}(0)$ is a Teichmüller disc in $F(\Gamma)$. In particular, this would imply that the axis of a hyperbolic element $g \in \Gamma$ corresponding to a simple loop on U/Γ would be an invariant line in the Teichmüller metric for the element g in mod Γ. Thus g would be a hyperbolic element of mod Γ (Bers [9]). This contradicts Theorem 2.

Remark. The fact that the fibers of π: $F(\Gamma) \to T(\Gamma)$ are not Teichmüller discs has also been obtained by Nag [29].

COROLLARY 6 (of the proof). *The Teichmüller metric on any fiber $\pi^{-1}(\tau)$ is not a constant multiple of the Poincaré metric. As a matter of fact, if we restrict attention to any segment of a Poincaré geodesic in $\pi^{-1}(\tau)$, then the two metrics are not constant multiples of each other on this segment.*

COROLLARY 7. *Let Γ be a torsion free Fuchsian group of type $(p, n) \neq (0, 3)$. Let $\varphi \in Q((U \setminus A)/\Gamma)$, where $A = \Gamma a$, $a \in U$, with $\|\varphi\| = 1$. Let*

$$B = \{t \in \Delta;\ t\bar{\varphi}/|\varphi| \text{ is a trivial Beltrami coefficient}\}.$$

Then B is a discrete (possibly empty) subset of the unit disc Δ.

Remarks. (1) In particular, the Beltrami coefficient $\bar{\varphi}/|\varphi|$ can not be locally trivial.

(2) If $g \in \Gamma$ yields a hyperbolic element of mod Γ, then the axis of g is never in a single fiber in $F(\Gamma)$. It would be interesting to describe the image of this axis in $T(\Gamma)$.

The above results have solved Problem C.

We turn now to Problem B. Consider an arbitrary surface S of finite type (p, n) satisfying (2.1). Represent S as U/Γ with Γ a torsion free Fuchsian group. The matric $\tilde{\varrho}$ defined in § 3 coincides with the projection to U/Γ of the Teichmüller metric on $\pi^{-1}(0) \subset F(\Gamma)$. Hence we have obtained

COROLLARY 8. *The metric $\tilde{\varrho}$ on U/Γ is complete and never a multiple of the Poincaré metric on U/Γ, except if U/Γ is of type* $(0, 3)$.

We formulate the above corollary as follows:

THEOREM 4. *Let S be a Riemann surface of non-excluded finite type $(p, n) \neq (0, 3)$. Then S carries two canonically defined metrics on it: the Poincaré metric ϱ of constant negative curvature -4 and the Teichmüller metric $\tilde{\varrho}$ (which is the restriction to S of the Kobayashi metric on the punctured Teichmüller curve). These metrics are not constant multiples of one another. However, there exists a universal constant $c > 0$ such that*

$$c\varrho < \tilde{\varrho} < \varrho,$$

on $S \times S \setminus \{\text{diagonal}\}$. *The metrics ϱ and $\tilde{\varrho}$ are invariant under conformal maps.*

Remarks. (1) If S is of type $(0, 3)$, then $\varrho = \tilde{\varrho}$.

(2) We have also established Theorem 4′ of the introduction. We have also obtained

PROPOSITION 6. *Let S be a Riemann surface of finite type, and let $x_0 \in S$. There exists a constant $c_1 > 0$ such that for all $y \in S$ with $\varrho(x_0, y) < c_1$, there exists a unique (extremal) quasiconformal mapping w with the properties*

(i) *w is homotopic to the identity (on S),*

(ii) *$w(x_0) = y$, and*

(iii) *whenever $\tilde{w}$ satisfies* (i) *and* (ii), *then* $K(w) \leq K(\tilde{w})$.

Remarks. (1) The mapping w satisfies

$$\tfrac{1}{2} \log K(w) = \tilde{\varrho}(x_0, y).$$

(2) In connection with Proposition 6, consider the Bers fiber space π: $F(\Gamma) \to T(\Gamma)$.

Fix $x_0 \in \pi^{-1}(0)$. It would be interesting to describe the shape of the sets

$$\{y \in \pi^{-1}(0);\ \langle y, x_0\rangle = \varepsilon\},$$

and

$$\{y \in \pi^{-1}(0);\ \langle y, x_0\rangle = \langle y, gx_0\rangle\},$$

where

$$\varepsilon > 0, \text{ and } g \in \Gamma,\ g \neq 1.$$

In particular, how does the constant c_1 depend on x_0 (and S)?

§ 11. Infinitesimal forms of the metrics on the fibers

Let Γ be a finitely generated torsion free Fuchsian group of the first kind. Let π: $F(\Gamma) \to T(\Gamma)$ be the associated Bers fiber space. We are interested in computing the infinitesimal form of the Teichmüller metric on $\pi^{-1}(0)$ and comparing it with the Poincaré metric. We know that for $z_0 \in U$,

$$\varrho(z_0, z_0 + t) = \frac{|t|}{2 \operatorname{Im} z_0} + o(t^2), \quad t \to 0.$$

To compute $\langle z_0, z_0 + t\rangle$, we choose $\varphi \in Q((U/\Gamma) \setminus \langle \hat{z}_0\rangle)$ such that $\mu = \bar{\varphi}/|\varphi|$ is locally trivial with respect to the group Γ. Then, of course,

$$\langle z_0, w^{t\mu}(z_0)\rangle = |t| + o(t^2), \quad t \to 0.$$

Since (see, for example, Bers [7])

$$w^{t\mu}(z_0) = z_0 + t\,\frac{z_0(z_0-1)}{2\pi i} \iint_U \frac{\mu(\zeta)\, d\zeta \wedge d\bar{\zeta}}{\zeta(\zeta-1)\,\zeta - z_0)} + o(t^2), \quad t \to 0,$$

we see that

$$\varrho(z_0, w^{t\mu}(z_0)) = \frac{|t|}{2 \operatorname{Im} z_0} \left| \frac{z_0(z_0-1)}{2\pi i} \iint_U \frac{\mu(\zeta)\, d\zeta \wedge d\bar{\zeta}}{\zeta(\zeta-1)(\zeta - z_0)} \right| + o(t^2), \quad t \to 0,$$

and hence (as a consequence of (8.1) and (5) of Lemma 1) that

$$2 \operatorname{Im} z_0 \leqslant \left| \frac{z_0(z_0-1)}{2\pi i} \iint_U \frac{\mu(\zeta)\, d\zeta \wedge d\bar{\zeta}}{\zeta(\zeta-1)(\zeta - z_0)} \right| \leqslant 8 \operatorname{Im} z_0,$$

with equality in the first inequality for groups of type (0, 3).

§ 12. Absolutely extremal self-mappings

The following theorem is due to Thurston [36]; see also Poénaru [31]. A proof appears in [32]. The proof given below is a slight modification of an argument of Bers (oral communication). Our proof does not rely on any ideas from the theory of measured foliations.

THEOREM 5. *Let $w: S\to S$ be an absolutely extremal mapping with dilatation $K>1$. Then K is an algebraic integer.*

Proof. First assume that S is a compact Riemann surface of genus $p\geqslant 2$, and that the initial (and hence terminal) differential of w is the square of an abelian differential, φ^2. At a non-critical point P of w, there is a natural parameter z vanishing at P so that (see Bers [9])

$$\varphi = dz \quad \text{near } P. \tag{12.1}$$

If ζ is such a natural parameter at $w(P)$, then w is represented at P by

$$\operatorname{Re}\zeta = K^{\frac{1}{2}}\operatorname{Re} z, \quad \operatorname{Im}\zeta = K^{-\frac{1}{2}}\operatorname{Im} z. \tag{12.2}$$

The mapping w induces an automorphism T of the first homology group on S with integral coefficients:

$$T: H_1(S)\to H_1(S).$$

If c is a closed curve on S, T sends the homology class of c into the homology class of the closed curve $w(c)$. If we choose a canonical homology basis for $H_1(S)$, then with respect to this basis T is represented by a symplectic matrix—in particular, by a matrix with integral entries. There is a canonical pairing between $H_1(S)$ and $\mathcal{H}$, the vector space of harmonic one forms on S:

$$\langle c, \omega\rangle = \int_c \omega, \qquad c\in H_1(S), \quad \omega\in\mathcal{H}.$$

Let T^* be dual map to T:

$$T^*: \mathcal{H}\to\mathcal{H};$$

it satisfies

$$\langle Tc, \omega\rangle = \langle c, T^*\omega\rangle;$$

that is

$$\int_{Tc}\omega = \int_c T^*\omega. \tag{12.3}$$

In particular, with respect to an appropriate basis for $\mathcal{H}$ (the basis dual to the one used for the symplectic representation for T), T^* is represented by the adjoint of T.

Now $\operatorname{Re}\varphi$ and $\operatorname{Im}\varphi \in \mathcal{H}$, and (12.1) together with (12.2) show that for every closed curve c on S,

$$\int_{Tc} \operatorname{Re}\varphi = K^{1/2} \int_c \operatorname{Re}\varphi. \tag{12.4}$$

Hence (12.3) shows that

$$T^* (\operatorname{Re}\varphi) = K^{\frac{1}{2}} (\operatorname{Re}\varphi).$$

Similarly,

$$T^* (\operatorname{Im}\varphi) = K^{-\frac{1}{2}} (\operatorname{Im}\varphi).$$

In particular, $K^{\frac{1}{2}}$ is an eigenvalue of an integral matrix, and hence must be an algebraic integer.

Now for the general case. Assume that S has type (p, n). Let $\tilde{S}$ be the compactification of S. The mapping w extends to a formal Teichmüller self-mapping of $\tilde{S}$ (with the same dilatation). Let $x_1, ..., x_m$ consist of the poles and zeros of odd order of η, the initial differential of w. The mapping w permutes these points, and by pasing to a power of w, we may assume that

$$w(x_j) = x_j, \quad j = 1, ..., m.$$

As in Ahlfors [3], one constructs a four sheeted or two sheeted cover M of $\tilde{S}$ which is ramified precisely over $\{x_1, ..., x_m\}$. We lift w to a self-mapping W of M so that

$$\begin{array}{ccc} M & \xrightarrow{W} & M \\ \downarrow & & \downarrow \\ \tilde{S} & \xrightarrow{w} & \tilde{S} \end{array}$$

commutes. The easiest way to see that W is absolutely extremal is to observe that W^k is a Teichmüller mapping if and only if w^k is a formal Teichmüller mapping whose initial differential is permitted to have poles only at $x_1, ..., x_m$, and $K(W^2) = K(w^2) = K(w)^2 = K(W)^2$. Hence by Theorem 6 of Bers [9], W is absolutely extremal. The initial differential of w lifts to M. On M, it is holomorphic and has zeros of even orders only. Lifting it to another two sheeted cover, if necessary, this quadratic differential becomes a square of an abelian. This finishes the proof.

Let us consider a more general situation. Let w be a Teichmüller self-mapping of a surface S. Assume that the dilatation of w is K, and that the terminal quadratic differential of w is a multiple (of absolute value one) of the initial differential. Assume that one (and hence) both are squares of holomorphic abelian differentials (on a compact surface). Calling these abelian differentials φ and ψ respectively, we have

$$\psi = e^{i\theta/2}\varphi, \quad \theta \in \mathbf{R}.$$

The analysis of the previous section goes through with the following replacing (12.4):

$$\int_{Tc} \mathrm{Re}\ \psi = K^{1/2} \int_c \mathrm{Re}\ \varphi.$$

Thus we conclude that

$$T^* (\mathrm{Re}\ e^{i\theta/2}\varphi) = K^{\frac{1}{2}} (\mathrm{Re}\ \varphi),$$

and similarly

$$T^* (\mathrm{Im}\ e^{i\theta/2}\varphi) = K^{-\frac{1}{2}} (\mathrm{Im}\ \varphi).$$

The generalization of Thurston's (Theorem 5) result becomes

THEOREM 5′. *Let w: $S \to S$ be a Teichmüller mapping with dilatation $K > 1$. Assume that the terminal differential ψ of w is a multiple ($e^{i\theta}$, $\theta \in \mathbf{R}$) of the initial differential φ of w:*

$$\psi = e^{i\theta}\varphi.$$

Then

$$\left(\tfrac{1}{2} \cos \frac{\theta}{2}\right) (K^{1/2} + K^{-1/2}) \pm \frac{1}{2} \sqrt{\left(\cos^2 \frac{\theta}{2}\right) (K + 2 + K^{-1}) - 4}$$

are algebraic integers.

Remark. For $\theta = 0$, we obtain as before that $\pm K^{\frac{1}{2}}$ are algebraic integers. The same is true when $\theta = 2\pi$ (as is to be expected); while for $\theta = \pi$, we obtain no information ($\pm i$ are algebraic integers). By studying the action that w induces on an invariant Teichmüller disc, one can conclude that

$$\tfrac{1}{2}(K + 2 + K^{-1})(1 + \cos \theta)$$

is an algebraic integer. This statement is equivalent to Theorem 5′. By interpreting Theorem 5′ in terms of the action of w as an element of the modular group, we obtain

THEOREM 6. *Let $\chi \in \mathrm{Mod}\,(p, n)$ leave invariant the Teichmüller disc $D \subset T(p, n)$. Let T be the trace of the Möbius transformation $\chi \mid D$. Then T is a real algebraic integer, and*

$$\chi \textit{ is elliptic} \Leftrightarrow \chi \mid D \textit{ is the identity or } |T| < 2,$$

$$\chi \textit{ is parabolic} \Leftrightarrow \chi \mid D \textit{ is not the identity and } |T| = 2,$$

and

$$\chi \textit{ is hyperbolic} \Leftrightarrow |T| > 2.$$

Let c be an essential curve on a surface S of non-excluded finite type (p, n). We have associated two real numbers to this curve:

$$K = \text{dilatation of absolutely extremal mapping corresponding to } c,$$

and

$$\lambda = \exp 2l,$$

where l is the length of shortest geodesic determined by c. To be more explicit, represent S by U/Γ with Γ torsion free of type (p, n). Then c corresponds to an element $g \in \Gamma$, with g a hyperbolic element of mod Γ. Without loss of generality g has fixed points at 0, ∞ and multiplier >1. We have seen that (Theorem 2 of § 9) that there exists a point $x_0 \in F(\Gamma)$ such that

$$a(g) = \langle x_0, gx_0 \rangle.$$

Of course, the dilatation of the corresponding absolutely extremal self-mapping (of the surface of type $(p, n+1)$) is

$$K = \exp\,(2\langle x_0, gx_0 \rangle).$$

Maskit–Matelski [27] have shown that there is a $\tau \in T(\Gamma)$ with $\tau = [\mu_0]$, $\mu_0 \in M(\Gamma)$, so that

$$\tfrac{1}{2} \log \lambda_\mu \geqslant \tfrac{1}{2} \log \lambda_{\mu_0}, \quad \text{all } \mu \in M(\Gamma), \tag{12.5}$$

where for $\mu \in M(\Gamma)$, λ_μ is the multiplier of $g_\mu = w_\mu \circ g \circ w_\mu^{-1}$. Recall the functions f_1 and f_2 introduced in § 8. The number λ is, of course, λ_{μ_0}.

PROPOSITION 7. (a) *If* $(p, n) = (0, 3)$, *then*

$$K = \lambda.$$

(b) *For* $(p, n) \neq (0,3)$, *we have*

$$K < \lambda.$$

Proof. Part (a) is obvious. For part (b), recall that for all x_1 on the Poincaré axis of g_{μ_0}, we have

$$\tfrac{1}{2} \log K = a(g) \leqslant \langle x_1, gx_1 \rangle < \varrho(x_1, gx_1) = \tfrac{1}{2} \log \lambda.$$

We have already remarked that K is an algebraic integer. The number λ also satisfies an equation—a transcendental equation. Translate so that $\mu_0=0$. Then equation (12.5) together with the fact that

$$\lambda_\mu = w_\mu(\lambda), \quad \text{all } \mu \in M(\Gamma),$$

shows that (see, for example, Bers [7])

$$\frac{\lambda(\lambda-1)}{2\pi} \iint_U \mu(\zeta) \frac{1}{\zeta(\zeta-1)(\zeta-\lambda)} |d\zeta \wedge d\bar{\zeta}| = 0, \quad \text{all } \mu \in M(\Gamma).$$

This is equivalent to the condition that the Poincaré series giving the variation of the length of the geodesic corresponding to g,

$$\sum_{\gamma \in \Gamma} \frac{\gamma'(\zeta)^2}{(\gamma\zeta)(\gamma\zeta-1)(\gamma\zeta-\lambda)},$$

vanishes identically (see Gardiner [16]).

It would be of interest to determine if, in the above situation, $\pi^{-1}(0)$ contains a point where $a(g)$ is assumed. The precise relation between the two invariants K and λ should be clarified by future investigations.

Note added in proof (March 12, 1981). Proposition 3 can be strengthened. The element $\theta \in \operatorname{mod} \Gamma$ is hyperbolic whenever $\chi \in \operatorname{Mod} \Gamma$ is hyperbolic. The arguments used to prove Proposition 3 yield, in certain cases, a relation between the axis of θ and the axis of χ. Details will appear elsewhere.

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Received June 24, 1980
Received in revised form December 15, 1980