Tôhoku Math. Journ.
28 (1976), 373-379.

# ON THE NON-EXISTENCE OF FLAT CONTACT METRIC STRUCTURES

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(Received March 29, 1975)

1. **Introduction.** It is well known that a contact manifold admits a Riemannian metric compatible with the contact structure. While such a metric is not unique, the contact structure imposes some restriction on the curvature. For example, if the characteristic vector field $\xi$ of the contact structure generates a 1-parameter group of isometries, then the sectional curvature of all plane sections containing $\xi$ is equal to 1 [1] (1/4 in their normalization). This is a restrictive class, however as the tangent sphere bundles are usually not of this type (Tashiro [3]). Our purpose here is to show that the metric cannot in general be flat. Precisely we prove the following theorem.

THEOREM. *Let $M$ be a contact manifold of dimension $\geqq 5$. Then $M$ cannot admit a contact metric structure of vanishing curvature.*

The author expresses his appreciation to Professor J. Martinet for many valuable conversations.

2. **Preliminaries.** Let $M$ be a $(2n+1)$-dimensional $C^\infty$ manifold. We say that $M$ has an *almost contact structure* if it admits a tensor field $\varphi$ of type (1, 1), a vector field $\xi$ and a 1-form $\eta$ such that $\eta(\xi)=1$ and $\varphi^2=-I+\eta\otimes\xi$. From these conditions one can easily obtain $\varphi\xi=0$ and $\eta\circ\varphi=0$. Moreover on a $C^\infty$ manifold with an almost contact structure $(\varphi, \xi, \eta)$ there exists a Riemannian metric $g$ satisfying

$$g(\varphi X, \varphi Y)=g(X, Y)-\eta(X)\eta(Y)$$

for any two vector fields $X$ and $Y$ on $M$. Note that $\eta$ is the covariant form of $\xi$ and we call $(\varphi, \xi, \eta, g)$ an *almost contact metric structure*. We also define a 2-form $\Phi$ by $\Phi(X, Y)=g(X, \varphi Y)$.

On the other hand we say $M$ has a *contact structure* if it admits a global 1-form $\eta$ such that $\eta\wedge(d\eta)^n\neq 0$. It is well known that a manifold with a contact structure $\eta$ admits an almost contact metric structure such that

$$d\eta(X, Y)=g(X, \varphi Y)\ .$$

We then say that $(\varphi, \xi, \eta, g)$ is a *contact metric structure.*

On a manifold with an almost contact structure $(\varphi, \xi, \eta)$, S. Sasaki and Y. Hatakeyama [2] defined four tensors $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ by

$$\begin{aligned} N^{(1)}(X, Y) &= [\varphi, \varphi](X, Y) + 2d\eta(X, Y)\xi , \\ N^{(2)}(X, Y) &= (\mathscr{L}_{\varphi X}\eta)(Y) - (\mathscr{L}_{\varphi Y}\eta)(X) , \\ N^{(3)}(X) &= (\mathscr{L}_{\xi}\varphi)X , \\ N^{(4)}(X) &= (\mathscr{L}_{\xi}\eta)(X) , \end{aligned}$$

where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of $\varphi$ and $\mathscr{L}$ denotes Lie differentiation.

It is easy to show that for a contact metric structure $N^{(2)}$ and $N^{(4)}$ vanish [2]. Recall that the Riemannian connection $\nabla$ of $g$ is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = &\, Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &+ g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X) . \end{aligned}$$

Using this and the coboundary formula for $d$ one can straightforwardly obtain a general formula for the covariant derivative of $\varphi$ for an almost contact metric structure $(\varphi, \xi, \eta, g)$, namely

$$\begin{aligned} 2g((\nabla_X\varphi)Y, Z) = &\, 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z) \\ &+ g(N^{(1)}(Y, Z), \varphi X) + N^{(2)}(Y, Z)\eta(X) \\ &+ 2d\eta(\varphi Y, X)\eta(Z) - 2d\eta(\varphi Z, X)\eta(Y) . \end{aligned} \tag{2.1}$$

We close this section with the following lemma.

LEMMA. *On a manifold with a contact metric structure* $\mathscr{L}_{\xi}\varphi$ *is a symmetric operator.*

PROOF. Note that for a contact metric structure $(\varphi, \xi, \eta, g)$, $\nabla_{\xi}\xi = 0$ and $\nabla_{\xi}\varphi = 0$. Now

$$\begin{aligned} g((\mathscr{L}_{\xi}\varphi)X, Y) &= g(\nabla_{\xi}\varphi X - \nabla_{\varphi X}\xi - \varphi\nabla_{\xi}X + \varphi\nabla_X\xi, Y) \\ &= g(-\nabla_{\varphi X}\xi + \varphi\nabla_X\xi, Y) \end{aligned}$$

which vanishes if either $X$ or $Y$ is $\xi$. For $X$ and $Y$ orthogonal to $\xi$, $N^{(2)} = 0$ becomes $\eta([\varphi X, Y]) + \eta([X, \varphi Y]) = 0$; continuing the computation we have

$$\begin{aligned} g((\mathscr{L}_{\xi}\varphi)X, Y) &= \eta(\nabla_{\varphi X} Y) + \eta(\nabla_X\varphi Y) \\ &= \eta(\nabla_Y\varphi X) + \eta(\nabla_{\varphi Y}X) \\ &= g((\mathscr{L}_{\xi}\varphi)Y, X) . \end{aligned}$$

**3. Proof of the theorem.** For a contact metric structure $(\varphi, \xi, \eta, g)$, $N^{(2)} = 0$ and $\Phi = d\eta$, so equation (2.1) becomes

$$2g((\nabla_X\varphi)Y, Z) = g(N^{(1)}(Y, Z), \varphi X) + 2d\eta(\varphi Y, X)\eta(Z) - 2d\eta(\varphi Z, X)\eta(Y) .$$

Setting $Y = \xi$ and using the Lemma of Section 2 we obtain

$$\begin{aligned} -2g(\varphi\nabla_X\xi, Z) &= g(\varphi^2[\xi, Z] - \varphi[\xi, \varphi Z], \varphi X) - 2d\eta(\varphi Z, X) \\ &= -g(\varphi(\mathcal{L}_\xi\varphi)Z, \varphi X) - 2g(\varphi Z, \varphi X) \\ &= -g((\mathcal{L}_\xi\varphi)Z, X) - 2g(Z, X) + 2\eta(Z)\eta(X) \\ &= -g((\mathcal{L}_\xi\varphi)X, Z) - 2g(X, Z) + 2g(\eta(X)\xi, Z) , \end{aligned}$$

that is

$$-\varphi\nabla_X\xi = -\frac{1}{2}(\mathcal{L}_\xi\varphi)X - X + \eta(X)\xi .$$

Applying $\varphi$ we have

$$\nabla_X\xi = -\frac{1}{2}\varphi(\mathcal{L}_\xi\varphi)X - \varphi X . \tag{3.1}$$

We denote by $R_{XY}$ the curvature transformation $\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$. Then using (3.1)

$$\begin{aligned} R_{\xi X}\xi &= \nabla_\xi\nabla_X\xi - \nabla_{[\xi,X]}\xi \\ &= -\frac{1}{2}\varphi\nabla_\xi([\xi, \varphi X] - \varphi[\xi, X]) - \varphi\nabla_\xi X + \frac{1}{2}\varphi(\mathcal{L}_\xi\varphi)[\xi, X] + \varphi[\xi, X] \\ &= \frac{1}{2}\varphi\nabla_\xi\nabla_{\varphi X}\xi + \frac{1}{2}\nabla_\xi\nabla_X\xi - \frac{1}{2}\varphi\nabla_{\varphi[\xi,X]}\xi - \frac{1}{2}\nabla_{[\xi,X]}\xi - \varphi\nabla_X\xi . \end{aligned}$$

Therefore

$$\frac{1}{2}R_{\xi X}\xi = \frac{1}{2}\varphi(\nabla_\xi\nabla_{\varphi X}\xi - \nabla_{[\xi,\varphi X]}\xi) + \frac{1}{2}\varphi\nabla_{(\mathcal{L}_\xi\varphi)X}\xi - \varphi\nabla_X\xi \tag{3.2}$$

and hence since $M$ is flat

$$\begin{aligned} 0 &= \frac{1}{2}\varphi\Big(-\frac{1}{2}\varphi(\mathcal{L}_\xi\varphi)^2X - \varphi(\mathcal{L}_\xi\varphi)X\Big) - \frac{1}{2}(\mathcal{L}_\xi\varphi)X - X + \eta(X)\xi \\ &= \frac{1}{4}(\mathcal{L}_\xi\varphi)^2X - X + \eta(X)\xi . \end{aligned}$$

Thus we define a symmetric operator $h$ by $h = (1/2)\mathcal{L}_\xi\varphi$ and we have shown that $h^2 = -\varphi^2$; in particular note that $h$ has rank $2n$. Clearly we also have $h\xi = 0$, eigenvectors corresponding to non-zero eigenvalues are orthogonal to $\xi$ and the non-zero eigenvalues are $\pm 1$.

Recall that $d\eta(X, Y) = (1/2)(g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X))$ as can be easily deduced from the coboundary formula for $d$. Thus

$$2g(X, \varphi Y) = g(-\varphi hX - \varphi X, Y) - g(-\varphi hY - \varphi Y, X)$$

giving

$$g(\varphi hX, Y) = g(\varphi hY, X) = -g(hY, \varphi X) = -g(h\varphi X, Y)\ ,$$

that is $h$ and $\varphi$ anti-commute. In particular then, if $X$ is an eigenvector of the eigenvalue $+1$, $\varphi X$ is an eigenvector of $-1$ and vice-versa. Thus the contact distribution $D$ defined by $\eta = 0$ is decomposed into the orthogonal eigenspaces of $\pm 1$ which we denote by $[+1]$ and $[-1]$.

We now show that the distribution $[-1]$ is integrable. If $X$ and $Y$ are vector fields belonging to $[-1]$, (3.1) gives $\nabla_X \xi = 0$, $\nabla_Y \xi = 0$. Thus since $M$ is flat $0 = R_{XY}\xi = -\nabla_{[X,Y]}\xi = \varphi h[X, Y] + \varphi[X, Y]$. But $\eta([X, Y]) = -2d\eta(X, Y) = 0$, thus applying $\varphi$ we have $h[X, Y] = -[X, Y]$.

We denote by $[-1] \oplus [\xi]$ the distribution spanned by $[-1]$ and $\xi$, it is also integrable. For, any vector field belonging to $[-1]$ can be written as $\varphi X$ for some $X \in [+1]$. Thus (3.2) becomes $0 = (1/2)R_{\xi X}\xi = -(1/2)\varphi\nabla_{[\xi, \varphi X]}\xi$ and (3.1) shows that $[\xi, \varphi X] \in [-1]$.

Since $[-1] \oplus [\xi]$ is integrable, we can choose local coordinates $(u^0, \cdots, u^{2n})$ such that $\partial/\partial u^0, \cdots, \partial/\partial u^n \in [-1] \oplus [\xi]$ and we define local vector fields $X_i$, $i = 1, \cdots, n$ by $X_i = \partial/\partial u^{n+i} + \sum_{j=0}^{n} f_i^j \partial/\partial u^j$ where the $f_i^j$'s are functions chosen so that $X_i \in [+1]$. Thus $X_1, \cdots, X_n$ are $n$ linearly independent vector fields spanning $[+1]$. Clearly $[\partial/\partial u^k, X_i] \in [-1] \oplus [\xi]$ for $k = 0, \cdots, n$ and hence $\xi$ is parallel along $[\partial/\partial u^k, X_i]$. Therefore using (3.1)

$$0 = \nabla_{[\partial/\partial u^k, X_i]}\xi = \nabla_{\partial/\partial u^k}\nabla_{X_i}\xi - \nabla_{X_i}\nabla_{\partial/\partial u^k}\xi = -2\nabla_{\partial/\partial u^k}\varphi X_i$$

from which we have that

$$\nabla_{\varphi X_j}\varphi X_i = 0\ . \tag{3.3}$$

Similarly, noting that $[X_i, X_j] \in [-1]$,

$$0 = R_{X_i X_j}\xi = -2\nabla_{X_i}\varphi X_j + 2\nabla_{X_j}\varphi X_i$$

giving

$$\nabla_{X_i}\varphi X_j = \nabla_{X_j}\varphi X_i \tag{3.4}$$

or equivalently

$$\varphi[X_i, X_j] = -(\nabla_{X_i}\varphi)X_j + (\nabla_{X_j}\varphi)X_i\ . \tag{3.5}$$

Using (3.3) and (3.1)

$$0 = R_{X_i \varphi X_j}\xi = -\nabla_{[X_i, \varphi X_j]}\xi = \varphi h[X_i, \varphi X_j] + \varphi[X_i, \varphi X_j]$$

from which

$$g([X_i, \varphi X_j], X_k) = -g(h[X_i, \varphi X_j], X_k) = -g([X_i, \varphi X_j], X_k)$$

and hence

$$g([X_i, \varphi X_j], X_k) = 0 \, . \tag{3.6}$$

We now compute $(\nabla_{X_i}\varphi)X_j$ explicitly. Using (3.3), (3.6) and (3.5)

$$\begin{aligned} 2g((\nabla_{X_i}\varphi)X_j, X_k) &= g([\varphi, \varphi](X_j, X_k), \varphi X_i) \\ &= -g([X_j, X_k], \varphi X_i) \\ &= g(-(\nabla_{X_j}\varphi)X_k + (\nabla_{X_k}\varphi)X_j, X_i) \, . \end{aligned}$$

Since $\Phi = d\eta$, the sum of the cyclic permutations of $i$, $j$, $k$ in $g((\nabla_{X_i}\varphi)X_j, X_k)$ is zero. Thus our computation yields $g((\nabla_{X_i}\varphi)X_j, X_k) = 0$. Similarly

$$\begin{aligned} 2g((\nabla_{X_i}\varphi)X_j, \varphi X_k) &= g(-[X_j, \varphi X_k] - [\varphi X_j, X_k], \varphi X_i) \\ &= g(-\nabla_{X_j}\varphi X_k + \nabla_{\varphi X_k}X_j - \nabla_{\varphi X_j}X_k + \nabla_{X_k}\varphi X_j, \varphi X_i) \end{aligned}$$

which vanishes by (3.3) and (3.4). Finally

$$\begin{aligned} 2g((\nabla_{X_i}\varphi)X_j, \xi) &= g(\varphi^2[X_j, \xi] - \varphi[\varphi X_j, \xi], \varphi X_i) + 2d\eta(\varphi X_j, X_i) \\ &= 2g(\varphi h X_j, \varphi X_i) + 2g(X_j, X_i) \\ &= 4g(X_j, X_i) \, . \end{aligned}$$

Thus for any vector fields $X$ and $Y$ in $[+1]$,

$$(\nabla_X\varphi)Y = 2g(X, Y)\xi \, . \tag{3.7}$$

Note that (3.5) now gives $[X_i, X_j] = 0$.

Before differentiating (3.7) we show that $\nabla_{X_i}X_j \in [+1]$. First note that

$$-2g(\nabla_{\varphi X_i}X_j, X_k) = 2g((\nabla_{\varphi X_i}\varphi)X_j, \varphi X_k) \, ,$$

but the right side vanishes by a computation of the type we have been doing. Therefore

$$g(\nabla_{X_i}X_j, \varphi X_k) = -g(X_j, \nabla_{X_i}\varphi X_k) = -g(X_j, [X_i, \varphi X_k]) = 0$$

by (3.6). That $g(\nabla_{X_i}X_j, \xi) = 0$ is trivial.

Now to show the non-existence of flat contact metric structures for $\dim M \geqq 5$, we shall contradict the linear independence of the $X_i$'s. Note also that we have so far used only the vanishing of $R_{XY}\xi$. Equation (3.7) can be written as

$$\nabla_{X_i}\varphi X_j - \varphi\nabla_{X_i}X_j = 2g(X_i, X_j)\xi \, .$$

Differentiating this we have

$$\nabla_{X_k}\nabla_{X_i}\varphi X_j - (\nabla_{X_k}\varphi)\nabla_{X_i}X_j - \varphi\nabla_{X_k}\nabla_{X_i}X_j = 2(X_k g(X_i, X_j))\xi - 4g(X_i, X_j)\varphi X_k .$$

Taking the inner product with $\varphi X_l$, remembering (3.7) and that $\nabla_{X_i}X_j \in [+1]$, we have

$$g(\nabla_{X_k}\nabla_{X_i}\varphi X_j, \varphi X_l) - g(\nabla_{X_k}\nabla_{X_i}X_j, X_l) = -4g(X_i, X_j)g(X_k, X_l) .$$

Interchanging $i$ and $k$, $i \neq k$ and subtracting we have

$$0 = g(X_i, X_j)g(X_k, X_l) - g(X_k, X_j)g(X_i, X_l)$$

by virtue of the flatness and $[X_i, X_k] = 0$. Setting $i = j$ and $k = l$ we have $0 = g(X_i, X_i)g(X_k, X_k) - g(X_i, X_k)^2$ contradicting the linear independence of $X_i$ and $X_k$.

4. **Remarks.** In dimension 3 it is easy to construct flat contact metric structures. For example, consider $\boldsymbol{R}^3$ with coordinates $(X^1, X^2, X^3)$ and define a contact structure $\eta$ by $\eta = (1/2)(\cos X^3 dX^1 + \sin X^3 dX^2)$. Then $\xi$ is $2(\cos X^3\partial/\partial X^1 + \sin X^3\partial/\partial X^2)$ and the metric $g$ whose components are $g_{ij} = (1/4)\delta_{ij}$ gives a flat contact metric structure. Geometrically we see that $\partial/\partial X^3$ spans the $[+1]$ distribution and $\sin X^3\partial/\partial X^1 - \cos X^3\partial/\partial X^2$ spans $[-1]$, i.e. $\xi$ is parallel along $[-1]$ and rotates as we move parallel to the $X^3$-axis. Note also that $\eta$ is invariant under the group of translations generated by $\{X^A \rightarrow X^A + 2\pi,\ A = 1, 2, 3\}$ and therefore the 3-dimensional torus $T^3$ also carries this structure. It is still an open question whether or not $T^5$ carries a contact structure, but if it does it can not have a flat associated metric.

Constructing the diffeomorphism of $\boldsymbol{R}^3$ that maps this $\eta$ to the standard contact form $\eta_0 = (1/2)(dZ - YdX)$ we see that the metric $g_0$ whose components are given by

$$\frac{1}{4}\begin{pmatrix} 1 + Y^2 + Z^2 & Z & -Y \\ Z & 1 & 0 \\ -Y & 0 & 1 \end{pmatrix}$$

makes $(\eta_0, g_0)$ a flat contact metric structure.

Note that in the proof of our theorem, the vanishing of $R_{\xi X}\xi$ is enough to obtain the decomposition of the contact distribution into the $\pm 1$ eigenspaces of the operator $h = (1/2)\mathscr{L}_\xi\varphi$. Moreover $R_{XY}\xi = 0$ for $X$ and $Y$ in $[-1]$ is sufficient for the integrability of $[-1]$. Thus we have the following result.

THEOREM. *Let $M$ be a contact manifold of dimension $2n + 1$ with contact metric structure $(\varphi, \xi, \eta, g)$. If the sectional curvature of all* *plane sections containing $\xi$ vanish, then the operator $h = (1/2)\mathscr{L}_\xi \varphi$ has rank $2n$ and the contact distribution is decomposed into $\pm 1$ eigenspaces of $h$. Moreover if $R_{XY}\xi = 0$ for $X, Y \in [-1]$, $M$ admits a foliation by $n$-dimensional integral submanifolds of the contact distribution.*

We close with an example of such a structure. Consider on $\boldsymbol{R}^5$ with coordinates $(X^1, \cdots, X^5)$, the standard contact structure

$$\eta = \frac{1}{2}(dX^5 - X^3 dX^1 - X^4 dX^2) .$$

Then $\eta$ together with the metric $g$ whose components are given by

$$\frac{1}{4}\begin{pmatrix} 1 + (X^3)^2 + (X^5)^2 & X^3X^4 & X^5 & 0 & -X^3 \\ X^3X^4 & 1 + (X^4)^2 + (X^5)^2 & 0 & X^5 & -X^4 \\ X^5 & 0 & 1 & 0 & 0 \\ 0 & X^5 & 0 & 1 & 0 \\ -X^3 & -X^4 & 0 & 0 & 1 \end{pmatrix}$$

is a contact metric structure. $g$ is not flat, but $R_{\xi X}\xi = 0$ and $R_{\partial/\partial X^3 \partial/\partial X^4}\xi = 0$. Defining $h$ by $h = (1/2)\mathscr{L}_\xi\varphi$, one can easily check that $h$ determines a decomposition of the contact distribution into $\pm 1$ eigenspaces of $h$. $[-1]$ is spanned by $\partial/\partial X^3$ and $\partial/\partial X^4$ and $[+1]$ is spanned by $\partial/\partial X^1 - X^5\partial/\partial X^3 + X^3\partial/\partial X^5$ and $\partial/\partial X^2 - X^5\partial/\partial X^4 + X^4\partial/\partial X^5$.

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