On the non-homogeneous Poisson process (I)

by

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This paper[1]) contains a complete discussion of those stochastic processes which can be treated as ordinary or composed Poisson processes without any homogeneity conditions. The theorems obtained here imply those formulated in some earlier papers[2]).

In particular, our considerations deal with all time-dependent processes for which the sample functions are arbitrary step functions of a real variable. In the first part we consider only the case of non-decreasing step functions having only jumps equal to 1.

In recent papers H. Cramér and E. Marczewski have investigated the increments in the homogeneous Poisson process not only for intervals but, more generally, for arbitrary Borel sets. It seems that this method, which treats set functions instead of functions of a real variable, is efficacious and general, *i. e.* just the right one, and therefore I do consider a process as a space of set functions, defined for subsets of an Euclidean space.

1. Let X be a fixed Borel subset of a finite dimensional Euclidean space and B_0 a denumerable field of Borel subsets of X such that the smallest σ field containing B_0 is the field $\boldsymbol{B}$ of all Borel subsets of X. By a *register* we understand every finite real valued set function $\omega(\varepsilon)$ defined and σ-additive in B_0 (in the classical Poisson process $\omega(\varepsilon)$ may be understood, for instance, as the number of telephone calls in a subset ε of the time axis).

We consider a probability-measure Pr defined on a σ field of subsets of a fixed set Ω of registers. For brevity we write $Pr(\ldots)$ instead of $Pr(\{\omega|\ldots\})$.

[1]) Presented to the Polish Mathematical Society, Wrocław Section, on June 20, 1952. Cf. my preliminary report, Colloquium Mathematicum 3.

[2]) A. Rényi [7], L. Jánossy, A. Rényi and J. Aczel [4], especially §§ 1 and 2, p. 211-217; A. Rényi [8], especially §§ 1 and 2, p. 84-90; K. Florek, E. Marczewski and C. Ryll-Nardzewski [2] and E. Marczewski [6]. In particular we shall give precise formulations of some ideas of P. Lévy [5] (Chapter VII, especially p. 173-180).

We suppose that:

1° for every ε belonging to the field B_0, the number $\omega(\varepsilon)$ treated as an ω-function is Pr-measurable,

2° $\omega(\varepsilon_1), \omega(\varepsilon_2), \ldots, \omega(\varepsilon_k)$ are stochastically independent ω functions, whenever ε_j are disjoint sets belonging to the field B_0.

Then we call Ω a *process*.

Obviously, every register ω may be extended to a σ-additive set function in $\boldsymbol{B}$ (denoted also by ω).

LEMMA 1. *The conditions* 1° *and* 2° *are fulfilled for any sets belonging to* $\boldsymbol{B}$.

Proof. It suffices to prove that if a field $\boldsymbol{M}$ satisfies conditions 1° and 2°, then the field $\boldsymbol{M}^*$, consisting of all limits of sets belonging to $\boldsymbol{M}$, also fulfills these conditions.

The condition 1° for $\boldsymbol{M}^*$ is an immediate consequence of the σ-additivity of registers.

In order to prove condition 2° for the field $\boldsymbol{M}^*$ it suffices to observe that any disjoint sets $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \in \boldsymbol{M}^*$ may be represented as follows:

$$\varepsilon_j = \lim_n \varepsilon_n^j, \quad j=1,2,\ldots,k; \qquad \varepsilon_n^j \in \boldsymbol{M},$$

$$\varepsilon_n^i \varepsilon_n^j = 0, \quad i \neq j,$$

and to apply the theorem saying that the passage to the limit preserves the independence[3]).

For given Ω and Pr we call a point $x_0 \in X$ *singular* if

$$Pr(\omega(\{x_0\}) \neq 0) > 0.$$

LEMMA 2. *The set S of all singular points is at most denumerable.*

Proof. In fact, if S were non denumerable, then there would exist a number $\delta > 0$ and a denumerable set $D = (x_1, x_2; \ldots)$ of different points such that

$$Pr(|\omega(\{x_n\})| > \delta) > \delta \qquad (n=1,2,\ldots)$$

which would contradict the convergence of the series $\sum \omega(\{x_n\}) = \omega(D)$.

2. A σ-measure ω will be called *simple*, if it is purely atomic, and if $\omega(\{x_0\}) = 1$ for every atom x_0 of ω.

In this section we assume that

(A_1) Every $\omega \in \Omega$ is a simple measure.

For every $\varepsilon \in \boldsymbol{B}$ let us denote by $m(\varepsilon)$ the expected value of $\omega(\varepsilon)$:

$$m(\varepsilon) = \int_\Omega \omega(\varepsilon)\, d Pr(\omega).$$

[3]) H. Cramér [1] and Marczewski [6].

Theorem 1. *The set function $m(\varepsilon)$ is a finite σ-measure in $\boldsymbol{B}$. If $x_0 \epsilon X$ is an atom of m (or, in other words, if x_0 is singular), then obviously $\omega((x_0))$ assumes the value 1 or 0 with probability $m((x_0))$ or $1-m((x_0))$ respectively. If ε contains no singular point, then $\omega(\varepsilon)$ has the Poisson distribution*

$$P_k(\varepsilon)=Pr(\omega(\varepsilon)=k)=\frac{[m(\varepsilon)]^k}{k!}e^{-m(\varepsilon)} \quad (k=0,1,2,\ldots). \tag{1}$$

Proof. Let us assume that

(S) There are no singular points.

We shall prove that

$$(*) \qquad P_0(\varepsilon)>0 \quad \text{for} \quad \varepsilon \epsilon \boldsymbol{B}.$$

Let us denote by ε_{in} a double sequence of sets such that

$$(\sharp) \qquad \varepsilon=\varepsilon_{1n}+\varepsilon_{2n}+\ldots+\varepsilon_{nn}, \qquad \varepsilon_{in}\varepsilon_{jn}=0 \quad \text{for} \quad i\neq j,$$

and that for any points $x\neq y$ belonging to ε there is a subscript N such that, for $n>N$, x and y belong to different sets in the decomposition ($\sharp$).

If $P_0(\varepsilon)=0$, then in view of 2^0, there is a sequence $\{i_n\}$ such that

$$P_0(\varepsilon_{i_1 1}\cdot\varepsilon_{i_2 2}\cdot\ldots\cdot\varepsilon_{i_n n})=0,$$

whence

$$P_0(\varepsilon_{i_1 1}\cdot\varepsilon_{i_2 2}\cdot\ldots)=0,$$

which is imposible in view of (S), since the intersection $\varepsilon_{i_1 1}\cdot\varepsilon_{i_2 2}\cdot\ldots$ contains at most one point.

The inequality (*) is thus proved.

The set function defined by the formula.

$$m(\varepsilon)=-\log P_0(\varepsilon)$$

is a finite σ-measure in $\boldsymbol{B}$ vanishing for one point sets. We shall prove (1), which implies that $m(\varepsilon)$ is indeed equal to the expected value of $\omega(\varepsilon)$.

In view of (S) the measure m vanishes for one point sets and consequently there are decompositions ($\sharp$) such that $m(\varepsilon_{in})=m(\varepsilon)/n$. Let us set

$$\Gamma_n=\{\omega|\omega(\varepsilon_{in})\leqslant 1 \quad \text{for} \quad i=1,2,\ldots,n\}.$$

Since the registers ω admit the value 1 for their atoms and since the atoms belonging to ε are separated by the sets ε_{in} for large n, we have $Pr(\Gamma_n)\to 1$, whence

$$P_k(\varepsilon)=\lim_n Pr(\omega\epsilon\Gamma_n \text{ and } \omega(\varepsilon)=k)=\lim_n \binom{n}{k}e^{-\frac{m(\varepsilon)}{n}(n-k)}\left(1-e^{-\frac{m(\varepsilon)}{n}}\right)^k.$$

Since

$$\lim_n n\left(1-e^{-\frac{m(\varepsilon)}{n}}\right)=m(\varepsilon),$$

the classical Poisson theorem implies (1).

In the case of the set of singular points being non void, $S=(s_1,s_2,\ldots)$ (see Lemma 2), it is to be proved that the expected value of S is finite. Obviously $\sum_n \omega((s_n))=\omega(S)<\infty$, whence, by the Borel-Cantelli lemma,

$$\int_\Omega \omega(S)\,dPr(\omega)=\sum_n Pr(\omega((s_n))=1)<\infty.$$

Theorem 1 is thus proved.

This theorem, together with Lemma 1, permits us to determine effectively the distribution of $\omega(\varepsilon)$ for every $\varepsilon\epsilon\boldsymbol{B}$.

Of course, if X is a time interval, then every atom of m is such a moment t_0 for which the probability of a call is positive. If the process is homogeneous in time, then there is no atom and $m(\varepsilon)$ is proportional to the Lebesgue measure.

3. In this section we deal with a generalization of processes considered in Section 2. This generalization includes, for instance, the case in which X is the infinite time-axis T or a subset of T with infinite Lebesgue measure. In this particular case, for time homogeneous processes, the results of the present section are actually the same as those of Marczewski[4]).

We shall use the term "register" in a more general sense than in Section 1: instead of the finiteness of ω in X, we assume that there is a fixed ascending sequence $\{X_n\}$ of Borel sets with $X=X_1+X_2+\ldots$ and such that, for every n, every register ω restricted to subsets of X_n is finite. (In the case of the ordinary Poisson process, we may choose as $\{X_n\}$ a sequence of intervals).

Moreover we assume that the conditions 1^0, 2^0 and (A_1) are fulfilled by $\omega(\varepsilon)$, when ε runs over the class of Borel subsets of X_n. For the sake of simplicity, we assume also the condition (S).

Setting, as in the preceding section,

$$m(\varepsilon)=\int_\varepsilon \omega(\varepsilon)\,dPr(\omega) \quad \text{for} \quad \varepsilon\epsilon\boldsymbol{B}$$

we shall prove the following

Theorem 2. *The set function $m(\varepsilon)$ is a σ-finite non-atomic σ-measure in $\boldsymbol{B}$. For every $\varepsilon\epsilon\boldsymbol{B}$ we have $Pr\{\omega(\varepsilon)<\infty\}=0$ or 1, according to whether $m(\varepsilon)$ is finite or not. The formula* (1) *is valid for every ε with $m(\varepsilon)<\infty$.*

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[4]) S. Hartman and E. Marczewski [3], especially p. 130, Theorem 4.

Proof. For every $\varepsilon \epsilon \boldsymbol{B}$ we have

$$(\overset{**}{*}) \qquad \omega(\varepsilon) = \sum_{n=0}^{\infty} \omega\big(\varepsilon(X_{n+1}-X_n)\big), \quad \text{where} \quad X_0 = 0.$$

In view of theorem 1 the terms of this series are independent ω-functions with expected values $m\big(\varepsilon(X_{n+1}-X_n)\big)$. It follows easily from the properties of the Poisson distribution that the sum of such a series is finite with probability 1 if and only if

$$\sum_{n=0}^{\infty} m\big(\varepsilon(X_{n+1}-X_n)\big) < \infty.$$

Then, the random variable $(\overset{**}{*})$ also has the Poisson distribution with the expected value $m(\varepsilon)$.

If $m(\varepsilon)=\infty$, then $\omega(\varepsilon)=\infty$ with probability 1.

REMARK. Condition 2^0 is fulfilled for any sets $\varepsilon \epsilon \boldsymbol{B}$ with $m(\varepsilon)<\infty$. That is an immediate consequence of $(\overset{**}{*})$ and of a theorem mentioned above[5]).

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[5]) Marczewski [6] p. 000

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