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# ON THE NONEXISTENCE OF $L^2$-SOLUTIONS OF $n$TH ORDER DIFFERENTIAL EQUATIONS

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Consider the equation

$$(D_r^{(n)}x)(t)+a(t)x(t)=0, \quad t\geqq T\geqq 0 \tag{E}$$

where $D_r^{(n)}x$ is the generalised derivative of $x$ defined as follows: for every $t\geqq T$

$$(D_r^{(0)}x)(t)=x(t)$$
$$(D_r^{(1)}x)(t)=r_1(t)x'(t)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(D_r^{(k)}x)(t)=r_k(t)(D_r^{(k-1)}x)'(t), \quad k=1,2\ldots,n \quad \text{with} \quad r_n(t)=1.$$

The functions $a:[0,\infty)\to\mathbb{R}$ and $r_i:[T,\infty)\to\mathbb{R}$ are continuous. Moreover, for every $i=1,2,\ldots,n-1$: $r_i>0$ on $[T,\infty)$ and

$$\int^{\infty}\frac{dt}{r_i(t)}=\infty. \tag{C}$$

We notice that the equation (E) can be written as a first order system of $n$ linear equations and consequently for any $c_k\in\mathbb{R}$, $k=1,2,\ldots,n-1$ and $t_0\geqq T$ there exists a unique solution $x$ of (E) defined on $[T,\infty)$ and such that

$$(D_r^{(k)}x)(t_0)=c_k, \quad k=1,2,\ldots,n-1.$$

As it is pointed out in (**2**, pages 91 and 93) the class of operators $D_r^{(n)}$ properly contains the disconjugate operator

$$L_x\equiv x^{(n)}+p_1x^{(n-1)}+\ldots+p_nx$$

where $p_i$, $i=1,2,\ldots,n$ are continuous functions. For this and other reasons there is an increasing interest in differential equations of the form (E) and the asymptotic behavior of their solutions.

Our purpose here is to study the nonexistence of $L^2(T,\infty)$ solutions of the linear differential equations of the form (E) and extend the results obtained for (E) to nonlinear cases. As far as we know, the only known results for such kind of operators are given in (**8**), but our results below differ from those given in (**8**) and generalise and extend such results obtained in (**1**) and (**3**), (**5**)–(**7**).

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We introduce the following notation: for every $i=1,2,\ldots,n-1$ and for all $t \geqq T$

$$\rho_i(t)=\int_T^t \frac{ds}{r_i(s)}$$

and we give the following result.

**Theorem.** *Let the conditions*

$$\int^{\infty} \prod_{i=1}^{n-1} \rho_i^2(t)\,|a(t)|^2\,dt<\infty \tag{C$_1$}$$

*and*

$$\int^{\infty}\left(\int_t^{\infty}\frac{1}{r_1(t_1)}\int_{t_1}^{\infty}\frac{1}{r_2(t_2)}\cdots\int_{t_{n-2}}^{\infty}\frac{1}{r_{n-1}(t_{n-1})}\right.$$
$$\left.\times\left(\int_{t_{n-1}}^{\infty}|a(s)|^2\,ds\right)^{1/2} dt_{n-1}\,dt_{n-2}\ldots dt_1\right)^2 dt<\infty \tag{C$_2$}$$

*be satisfied. Then the differential equation* (E) *has no nontrivial $L^2$-solutions.*

**Proof.** Let $x$ be a nontrivial $L^2$-solution of the equation (E). Then integrating (E) from $T_1$ to $T_2$ with $T_2>T_1\geqq T$, we obtain

$$r_{n-1}(T_2)(D_r^{(n-2)}x)'(T_2)-r_{n-1}(T_1)(D_r^{(n-2)}x)'(T_1)+\int_{T_1}^{T_2}a(s)x(s)\,ds=0 \tag{1}$$

and hence, by the Cauchy-Schwarz inequality, we derive the result

$$|r_{n-1}(T_2)(D_r^{(n-2)}x)'(T_2)-r_{n-1}(T_1)(D_r^{(n-2)}x)'(T_1)|\leqq\left(\int_{T_1}^{T_2}|a(s)|^2\,ds\right)^{1/2}\left(\int_{T_1}^{T_2}|x(s)|^2\,ds\right)^{1/2}$$

which, in view of (C), ($C_1$) and the fact that $x$ is a nontrivial $L^2$-solution, implies that

$$\lim_{t\to\infty} r_{n-1}(t)\,|(D_r^{(n-2)}x)'(t)|=c, \quad c\in\mathbb{R}.$$

We prove that $c=0$. Indeed, if $c\neq 0$, then we obviously have

$$r_{n-1}(t)\,|(D_r^{(n-2)}x)'(t)|\geqq\frac{c}{2} \quad \text{for all large } t$$

which, by integration, gives

$$\lim_{t\to\infty} r_{n-2}(t)\,|(D_r^{(n-3)}x)'(t)|=\infty.$$

Proceeding in the same way, finally we find that

$$\lim_{t\to\infty}|x(t)|=\infty$$

which is an immediate contradiction to the fact that $x\in L^2$.

So, from (1), we have that for every $t \geqq T$

$$r_{n-1}(t)(D_r^{(n-2)}x)'(t) = \int_t^\infty a(s)x(s)\,ds$$

and consequently

$$(D_r^{(n-2)}x)'(t) = \frac{1}{r_{n-1}(t)} \int_t^\infty a(s)x(s)\,ds \quad \text{for every } t \geqq T.$$

Integrating (2) from $T_1$ to $T_2$ with $T_2 > T_1 \geqq T$, we get

$$r_{n-2}(T_2)(D_r^{(n-3)}x)'(T_2) - r_{n-2}(T_1)(D_r^{(n-3)}x)'(T_1)$$
$$= \int_{T_1}^{T_2} \frac{1}{r_{n-1}(t_{n-1})} \int_{t_{n-1}}^\infty a(s)x(s)\,ds\,dt_{n-1} \tag{3}$$

from which, by partial integration and the Cauchy–Schwarz inequality, we obtain

$$|r_{n-2}(T_2)(D_r^{(n-3)}x)'(T_2) - r_{n-2}(T_1)(D_r^{(n-3)}x)'(T_1)|$$
$$\leqq \left(\int_{T_1}^{T_2} \frac{d\tau}{r_{n-1}(\tau)}\right)\left(\int_{T_2}^\infty |a(s)|\,|x(s)|\,ds\right) + \int_{T_1}^{T_2}\left(\int_{T_1}^{t_{n-1}} \frac{d\tau}{r_{n-1}(\tau)}\right)|a(t_{n-1})|\,|x(t_{n-1})|\,dt_{n-1}$$
$$\leqq 2\int_{T_1}^\infty \left(\int_{T_1}^s \frac{d\tau}{r_{n-1}(\tau)}\right)|a(s)|\,|x(s)|\,ds \leqq 2\left(\int_{T_1}^\infty \left(\int_{T_1}^s \frac{d\tau}{r_{n-1}(\tau)}\right)^2 |a(s)|^2\,ds\right)^{1/2}\left(\int_{T_1}^\infty |x(s)|^2\,ds\right)^{1/2}.$$

Using ($C_1$) and taking into account (C) and the fact that $x \in L^2$, from the above relation we get

$$\lim_{t\to\infty} r_{n-2}(t)(D_r^{(n-3)}x)'(t) = 0.$$

Hence, from (3) we derive, for every $t \geqq T$,

$$r_{n-2}(t)(D_r^{(n-3)}x)'(t) = -\int_t^\infty \frac{1}{r_{n-1}(t_{n-1})} \int_{t_{n-1}}^\infty a(s)x(s)\,ds\,dt_{n-1};$$

consequently

$$(D_r^{(n-3)}x)'(t) = -\frac{1}{r_{n-2}(t)} \int_t^\infty \frac{1}{r_{n-1}(t_{n-1})} \int_{t_{n-1}}^\infty a(s)x(s)\,ds\,dt_{n-1}.$$

By the same arguments, as before, we get for every $i = 1, 2, \ldots, n-1$ and all $u \geqq T$

$$(D_r^{(i-1)}x)'(u) = \frac{(-1)^{n-i-1}}{r_i(u)} \int_u^\infty \frac{1}{r_{i+1}(t_{i+1})} \int_{t_{i+1}}^\infty \frac{1}{r_{i+2}(t_{i+2})} \cdots \int_{t_{n-1}}^\infty a(s)x(s)\,ds\,dt_{n-1}\cdots dt_{i+1}. \tag{4}$$

From (4) for $i = 1$, we have for every $u \geqq T$

$$x'(u) = \frac{(-1)^{n-2}}{r_1(u)} \int_u^\infty \frac{1}{r_2(t_2)} \int_{t_2}^\infty \frac{1}{r_3(t_3)} \cdots \int_{t_{n-1}}^\infty a(s)x(s)\,ds\,dt_{n-1}\ldots dt_2$$

which, by integration from $T_1$ to $T_2$ with $T_2 > T_1 \geqq T$ and by similar arguments as before, gives for every $t \geqq T$

$$x(t) = (-1)^{n-1} \int_t^\infty \frac{1}{r_1(t_1)} \int_{t_1}^\infty \frac{1}{r_2(t_2)} \cdots \int_{t_{n-1}}^\infty a(s)x(s)\, ds\, dt_{n-1} \ldots dt_1. \tag{5}$$

Taking the square of (5), integrating the obtained result from $T_0$ to $\infty$ with $T_0 \geqq T$ and by the application of the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
&\int_{T_0}^\infty |x(s)|^2\, ds \\
&\leqq \int_{T_0}^\infty \left( \int_t^\infty \frac{1}{r_1(t_1)} \int_{t_1}^\infty \frac{1}{r_2(t_2)} \cdots \int_{t_{n-2}}^\infty \frac{1}{r_{n-1}(t_{n-1})} \int_{t_{n-1}}^\infty |a(s)|\, |x(s)|\, ds\, dt_{n-1} \ldots dt_1 \right)^2 dt \\
&\leqq \int_{T_0}^\infty \left( \int_t^\infty \frac{1}{r_1(t)} \int_{t_1}^\infty \frac{1}{r_2(t_2)} \cdots \int_{t_{n-2}}^\infty \frac{1}{r_{n-1}(t_{n-1})} \left( \int_{t_{n-1}}^\infty |a(s)|^2\, ds \right)^{1/2} \right. \\
&\quad \left. \times \left( \int_{t_{n-1}}^\infty |x(s)|^2\, ds \right)^{1/2} dt_{n-1}\, dt_{n-2} \ldots dt_2\, dt_1 \right)^2 dt \\
&\leqq \int_{T_0}^\infty |x(s)|^2\, ds \cdot \int_{T_0}^\infty \left( \int_t^\infty \frac{1}{r_1(t)} \int_{t_1}^\infty \frac{1}{r_2(t_2)} \cdots \int_{t_{n-2}}^\infty \frac{1}{r_{n-1}(t_{n-1})} \right. \\
&\quad \left. \times \left( \int_{t_{n-1}}^\infty |a(s)|^2\, ds \right)^{1/2} dt_{n-1}\, dt_{n-2} \ldots dt_2\, dt_1 \right)^2 dt
\end{aligned}$$

which in view of ($C_2$) and the fact that $x$ is a nontrivial $L^2$-solution, is an immediate contradiction.

**Remark 1.** In the case of the nonlinear differential equation

$$(D_r^{(n)}x)(t) + a(t)f(x(t)) = 0, \quad t \geqq T \tag{$\tilde{E}$}$$

where in addition to the above assumptions on the functions $a$ and $r_i$, $i = 1, 2, \ldots, n-1$, we suppose that the function $f$ is continuous on $\mathbb{R}$ and such that:

(i) $u \neq 0 \Rightarrow f(u) \neq 0$;

(ii) $\liminf_{u \to \infty} f(u) > 0$;

(iii) For every $L^2$-solution of ($\tilde{E}$)

$$\limsup_{t \to \infty} \int_t^\infty |x(s)|^2\, ds \Big/ \int_t^\infty |f(x)(s)|^2\, ds > 0.$$

Following the same arguments, one can see that under the conditions ($C_1$) and ($C_2$) the equation ($\tilde{E}$) has no nontrivial solutions $x$ with the property

$$\int^\infty |f(x(t))|^2\, dt < \infty. \tag{6}$$

In the case where $r_i \equiv 1$ on $[T, \infty)$, ($i = 1, 2, \ldots, n-1$) we have a generalisation of the results obtained in (5)–(7).

**Remark 2.** Suppose that in ($\tilde{\text{E}}$) $r_i \equiv 1$ on $[T, \infty)$, $(i = 1, 2, \ldots, n-1)$ and the function $f$ satisfies a Lipschitz condition uniformly on every bounded interval and

$$f(u) = 0(|u|) \quad \text{as} \quad u \to \infty;$$

then under the conditions (iii) and

$$\int^{\infty} t^{2n-1} |a(t)|^2 \, dt < \infty \tag{C$_3$}$$

the equation ($\tilde{\text{E}}$) has no nontrivial solution $x$ with the property (6).

This result extends the previous results given in (**1**) and (**3**).

**Note.** In this case conditions ($C_1$) and ($C_2$) become respectively

$$\int^{\infty} t^{2n-2} |a(t)|^2 \, dt < \infty \tag{C$_2'$}$$

and

$$\int_T^{\infty} \left( \int_t^{\infty} \int_{t_1}^{\infty} \cdots \int_{t_{n-2}}^{\infty} \left( \int_{t_{n-1}}^{\infty} |a(s)|^2 \, ds \right)^{1/2} dt_{n-1} \, dt_{n-2} \ldots dt_1 \right)^2 dt < \infty \tag{C$_2'$}$$

By the application of an inequality of Hardy-Littleweed-Pólya (**4**, Theorem 238, p. 244), for some positive constant $K$ ($C_2'$) gives

$$\int_T^{\infty} \left( \int_t^{\infty} \int_{t_1}^{\infty} \cdots \int_{t_{n-2}}^{\infty} \left( \int_{t_{n-1}}^{\infty} |a(s)|^2 \, ds \right)^{1/2} dt_{n-1} \, dt_{n-2} \ldots dt_1 \right)^2 dt$$

$$\leqq \int_T^{\infty} \left( \int_t^{\infty} \int_{t_1}^{\infty} \cdots \int_{t_{n-3}}^{\infty} t_{n-1} \left( \int_{t_{n-1}}^{\infty} |a(s)|^2 \, ds \right)^{1/2} dt_{n-1} \, dt_{n-3} \ldots dt_1 \right)^2 dt \leqq \ldots$$

$$\leqq \int_T^{\infty} \left( \int_t^{\infty} u^{n-2} \left( \int_u^{\infty} |a(s)|^2 \, ds \right)^{1/2} \right)^2 dt < K \int_T^{\infty} u^2 \cdot u^{2n-4} \int_u^{\infty} |a(s)|^2 \, ds \, du$$

$$\leqq K \int_T^{\infty} \int_u^{\infty} s^{2n-2} |a(s)|^2 \, ds \, du \leqq K \int_T^{\infty} s^{2n-1} |a(s)|^2 \, ds.$$

Thus, one can see that ($C_3$) is stronger than ($C_1$) and ($C_2$) but because of its form it is more useful in applications.

**Remark 3.** The following example shows the sharpness of the condition ($C_3$). Indeed, in the case of the equation

$$x^{(n)}(t) + \frac{c}{t^n} x(t) = 0, \quad t > 0 \tag{7}$$

$x(t) = t^{\alpha}$ with $\alpha < -\frac{1}{2}$ is a solution of (7), with $c = \alpha(\alpha-1)\ldots(\alpha-n+1)$, which is a $L^2$-solution. But in this case ($C_3$) fails, while for all $\beta < 2n-1$ the condition

$$\int^{\infty} t^{\beta} |a(t)|^2 \, dt < \infty$$

is satisfied.

**Remark 4.** Finally, we remark that, by using the Hölder instead of the Cauchy–Schwarz inequality, we can obtain analogous results for the nonexistence of $L^p$-solutions of differential equations of the form (E) and ($\tilde{\text{E}}$).

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