ON THE NONLINEAR EQUATIONS $\Delta u+e^u=0$ AND $\partial v/\partial t=\Delta v+e^v$

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1. **Introduction.** This note is concerned with the boundary value problem BVP for the equation

$$\Delta u + e^u = 0 \qquad (x \in \Omega) \tag{1}$$

under the boundary condition $u|_{\partial\Omega}=0$, and also with the initial value problem IVP for the equation

$$\partial v/\partial t = \Delta v + e^v \qquad (t \geqq 0,\ x \in \Omega) \tag{2}$$

under the initial condition $v|_{t=0}=a(x)$ and the boundary condition $v|_{\partial\Omega}=0$. Here Ω is a bounded domain in R^m whose boundary $\partial\Omega$ is assumed to be sufficiently smooth.

We assume that $a=a(x)$ continuous in $\bar{\Omega}$. As I. M. Gel'fand [2] pointed out, these problems arise in the theory of thermal self-ignition of a chemically active mixture of gases in a vessel. He showed, in the special case where Ω is an m-dimensional ball of radius r, that for $m=1$ or 2 there exists a critical radius r_c such that BVP has two solutions, one solution or no solution, according as $0<r<r_c$, $r=r_c$ or $r>r_c$. If $m=3$, then the number of solutions of BVP can be 0, 1, 2, $\cdots$, or ∞, depending on r. It would be quite interesting to extend Gel'fand's result to the case of general Ω. However, we do not try to proceed in this direction. Instead, our main objective is to prove a certain relationship among solutions of BVP when they exist and to study the asymptotic stability of these solutions, i.e., the convergence of solutions of IVP to solutions of BVP as $t\to+\infty$. Below we describe some of our results. The other results and technical details will be published elsewhere together with generalizations including replacement of the function e^u by a general function $f=f(u)$; $R^1\to R^1$ which is smooth, positive, increasing, and strictly convex. The author wishes to thank Professor Melvyn Berger who brought the author's attention to the present problem and provided the author with helpful preliminary information.

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2. **Notation and results.** By a solution of BVP we mean a smooth classical solution. Denote the totality of solutions of BVP by $S = S(\Omega)$. Let F be a nonlinear integral operator acting on bounded functions defined by

$$(Fu)(x) = \int_\Omega G(x, y)e^{u(y)}dy \quad (x \in \bar{\Omega}),$$

where G is the Green function of $-\Delta$ in Ω under the Dirichlet boundary condition. A bounded function u belongs to S if and only if it satisfies the integral equation $u = Fu$ in Ω. We need some symbols concerning order relations between two functions v and w defined in $\bar{\Omega}$. We write $v \leqq w(x \in \bar{\Omega})$ or simply $v \leqq w$ if $v(x) \leqq w(x)$ for all $x \in \bar{\Omega}$. $v \ll w(x \in \bar{\Omega})$ or simply $v \ll w$ means that $\gamma\rho \leqq w - v(x \in \bar{\Omega})$ for some positive number γ, $\rho(x)$ being the distance from x to $\partial\Omega$. Finally, we write $v \not\equiv w$ if $v(x_0) \neq w(x_0)$ for some $x_0 \in \bar{\Omega}$. For instance, if $v \leqq w$, $v \not\equiv w$ and $v, w \in S$, then we have $v \ll w$ by the maximum principle of E. Hopf [3] (cf. [1, p. 328]). A function $u \in S$ is called the minimum solution of BVP if $u \leqq v$ for any $v \in S$. The minimum solution is unique if it exists. We are ready to state theorems.

THEOREM 1. (i) *Let λ_0 be the smallest eigenvalue of $-\Delta$ under the Dirichlet boundary condition. If $\lambda_0 < e$, then S is empty. That is, there is no solution of BVP if Ω is sufficiently large.*

(ii) *There exists a positive number l_m depending on the dimension m such that S is not empty if the diameter of Ω is less than l_m.*

THEOREM 2. *If S is not empty, then there exists the minimum solution.*

THEOREM 3. *If S is empty, then the solution v of IVP blows up in a finite time or diverges to $+\infty$ as $t \to +\infty$ in the sense that $\|v(t, \cdot)\|_C \to +\infty$ $(t \to +\infty)$, where $\| \ \|_C$ denotes the maximum norm over $\bar{\Omega}$.*

THEOREM 4. *Suppose that S is not empty and let u be the minimum solution. If $a \leqq u$, then the solution v of IVP converges uniformly to u as $t \to +\infty$.*

THEOREM 5. *Let u and v be in S. If $u \leqq v$ and $u \not\equiv v$ (hence, automatically $u \ll v$), then u is the minimum solution. In other words, there cannot be a triple u_i $(i = 1, 2, 3)$ of solutions $\in S$ with $u_1 \ll u_2 \ll u_3$.*

THEOREM 6. *Suppose that BVP has solutions more than one. Let u be the minimum solution and let ϕ be any solution different from u. Then* (i) *and* (ii) *hold.*

(i) *If $\phi \leqq a$ and $\phi \not\equiv a$, then the solution v of IVP blows up in a finite* *time or diverges to* $+\infty$ *as* $t\to+\infty$ *in the sense mentioned in Theorem* 3.

(ii) *If* $a \leqq \phi$ *and* $a \not\equiv \phi$, *then* v *converges uniformly to* u *as* $t\to+\infty$.

REMARK 7. From Theorems 4 and 6 we see the following. Any solution of BVP other than the minimum one is unstable. The minimum solution is always stable from below and is stable from above provided that there exists another solution. As a matter of fact, the stability of the minimum solution from above when it is the only solution depends on circumstances which are too complicated to be given here.

3. **Outline of the proofs.** We give brief indications of proof for preceding theorems.

Theorem 1. (i) Let ϕ_0 be the eigenfunction of $-\Delta$ associated with the eigenvalue λ_0. We may assume that $\phi_0(x) > 0 (x \in \Omega)$ and $(\phi_0, 1) = 1$, where $(\,,\,)$ means the inner product in $L_2(\Omega)$. Form the inner product of ϕ_0 and each side of (1). Then by means of Jensen's inequality we obtain $-\lambda_0 J + e^J \leqq 0$ for $J = (\phi_0, u)$, which is impossible if $\lambda_0 < e$.

(ii) Application of Schauder's fixed point theorem.

Theorem 2. Functions $\{u_n\}$ defined by the iteration $u_{n+1} = Fu_n$ $(n=0, 1, \cdots)$ and $u_0 \equiv 0$ forms an increasing sequence. u_n does not exceed any $\phi \in S$ if S is not empty. Thus u_n converges and the limit is the minimum solution.

Theorem 3. For simplicity we assume $0 \leqq a$. Then it is enough to consider the special case $a=0$. In this case, $v(t, x)$ is increasing in t for each x, for we can prove $w \equiv v_t$ is positive everywhere. In fact, w is equal to $\Delta 0 + e^0 = 1$ at $t=0$, $w|_{\partial\Omega} = 0$ for $t > 0$, and the equation $w_t = \Delta w + e^v w$ is satisfied. Thus $v(t, x)$ converges to some $u(x)$ as $t\to+\infty$ for each x if v is bounded by a constant. This limit u satisfies $u = Fu$ which follows from

$$v(t, x) = \int_0^t ds \int_\Omega U(s, x, y) e^{v(t-\sigma, y)} dy$$

by making $t\to+\infty$, where U is the Green function of the heat equation.

Theorem 4. For simplicity we assume $0 \leqq a$. Then $0 \leqq a \leqq \mathfrak{u}$. Let v_0 be the solution of IVP for $a \equiv 0$. It is enough to show v_0 tends to $\mathfrak{u}$. As above, v_0 increases to some $u \in S$ as $t\to\infty$. However, u must coincide with $\mathfrak{u}$ since $\mathfrak{u}$ is the minimum solution.

Theorem 5. First we note the following

LEMMA 8. *Let* $\phi, \psi \in S$ *and suppose that* $\phi \ll \psi$. *Then* (i) *and* (ii) *hold.*

(i) *Let α be a number with $0<\alpha<1$. Then $\{v_n\}$ defined by $v_{n+1} = Fv_n$ $(n=0, 1, \cdots)$ and $v_0=(1-\alpha)\phi+\alpha\psi$ forms a decreasing sequence.*

(ii) *Let β be a positive number. Then $\{w_n\}$ defined by $w_{n+1}=Fw_n$ $(n=0, 1, \cdots)$ and $w_0=(1+\beta)\psi-\beta\phi$ forms an increasing sequence.*

This lemma can be established as follows.

(i) Since F is monotone with respect to the order relatlon $\leqq$, it is enough to show $v_1 \leqq v_0$. By the convexity of e^u we have $e^{(1-\alpha)\phi+\alpha\psi} \leqq (1-\alpha)e^{\phi}+\alpha e^{\psi}$. Applying the integral operator with the kernel $G(x, y)$ to both sides of this inequality, we obtain $v_1 \leqq v_0$ noting $\phi=F\phi$ and $\psi=F\psi$.

(ii) can be established similarly.

Now coming back to Theorem 5, suppose that u_i $(i=1, 2, 3)$ are those mentioned at the end of the statement. Then by means of Lemma 8 we can prove that there exist an infinite number of solutions $\phi_j \in S$ between u_2 and u_3 such that ϕ_j lie densely in an *upper neighborhood* of u_2 in a certain way. On the other hand, existence of such ϕ_j is shown to be impossible with the aid of some theorems concerning perturbation of eigenvalues (cf. [**4**]).

Theorem 6. We just say the following as a hint for proof of (i). If a is equal to $(1+\beta)\phi-\beta u$ for some $\beta>0$, then by the convexity of e^u we have $v_t=\Delta a+e^a \geqq 0$ at $t=0$ since ϕ, $u \in S$. From this we can prove $v_t \geqq 0$ everywhere. Thus v is increasing in t and converges to some $\bar{u} \in S$ if v is bounded. Then we should have $u \ll \phi \ll \bar{u}$ which contradicts Theorem 5.

References

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