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On the nonlocal Cauchy problem for semilinear fractional order evolution equations

Research Article

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Abstract: In this paper, we develop the approach and techniques of [Boucherif A., Precup R., Semilinear evolution equations with nonlocal initial conditions, Dynam. Systems Appl., 2007, 16(3), 507–516], [Zhou Y., Jiao F., Nonlocal Cauchy problem for fractional evolution equations, Nonlinar Anal. Real World Appl., 2010, 11(5), 4465–4475] to deal with nonlocal Cauchy problem for semilinear fractional order evolution equations. We present two new sufficient conditions on existence of mild solutions. The first result relies on a growth condition on the whole time interval via Schaefer fixed point theorem. The second result relies on a growth condition splitted into two parts, one for the subinterval containing the points associated with the nonlocal conditions, and the other for the rest of the interval via O'Regan fixed point theorem.

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1. Introduction

The nonlocal condition has a better effect on the solution and is more precise for physical measurements than the classical condition alone. For the contribution to the nonlocal Cauchy problem for nonlinear evolution equations we refer the reader to Byszewski [5, 6], Jackson [14], Deng [8], Liang et al. [17], Ntouyas and Tsamatos [25] and other papers (see for instance [4, 7, 11–13, 18, 30, 35] and references therein).

Boucherif and Precup [2] explored a new approach and conditions to study existence of solutions to the following initial value problem for first order differential equations with nonlocal conditions:

$$\begin{aligned} &x'(t) = F(t, x(t)), && \text{for a.e.} \quad t \in J = [0, 1],\\ &x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0, && k = 1, 2, \dots, m, \end{aligned}$$

where $F\colon J \times \mathbb{R} \to \mathbb{R}$ is a given function and a_k are real numbers with $\sum_{k=1}^{m} a_k \neq -1$ and t_k, $k = 1, 2, \dots, m$, are given points satisfying $0 < t_1 \le t_2 \le \dots \le t_m < 1$. The idea was to put less restrictive conditions on F by splitting the growth condition on F into two parts, one for $t \in [0, t_m]$ and the other for $t \in [t_m, 1]$.

In [3] Boucherif and Precup adopted the idea of [2] via fixed point methods and presented existence results for mild solutions to the following nonlocal Cauchy problem for first order evolution equations:

$$\begin{aligned} &x'(t) + Ax(t) = f(t, x(t)), && t \in J,\\ &x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0, && k = 1, 2, \dots, m, \end{aligned}$$

where $A\colon D(A) \subseteq X \to X$ is the generator of a C_0-semigroup $\{T(t) : t \ge 0\}$ on a Banach space X and $f\colon J \times X \to X$ is a given function.

In [23, 24] Nica and Precup developed further the approach and techniques of [2] and applied them in order to study the nonlocal Cauchy problem for first order nonlinear differential systems.

Recently, fractional order differential equations found application in studies related with viscoelasticity, electrical circuits, nonlinear oscillation of earthquake and etc. There appeared a number monographs which provide with the main theoretical tools for the qualitative analysis of fractional order differential equations, and at the same time show the interconnection as well as the contrast between integer order differential models and fractional order differential models [1, 9, 15, 16, 19, 20, 27, 29].

A pioneering work on existence of solutions to the following initial value problem for fractional order differential equations with nonlocal conditions:

$$\begin{aligned} &{}^{C}D_{0,t}^{\alpha} x(t) = f(t, x(t)), && \alpha \in (0, 1), \quad t \in J,\\ &x(0) + G(u) = x_0, && x_0 \in X, \end{aligned}$$

where the symbol ${}^{C}D_{0,t}^{\alpha}$ denotes the Caputo fractional derivative of order α with the lower limit zero, $f\colon J \times X \to X$ and the nonlocal term $G\colon C(J, X) \to X$, is due to N'Guérékata [21]. In [22] N'Guérékata noted that the results from [21] hold only in finite dimensional spaces. Dong et al. [10] revisited the above problem and presented some new existence results under certain suitable conditions, extending the results of [21] to infinite dimensional spaces.

Zhou and Jiao [36] studied the following nonlocal Cauchy problem for fractional order evolution equations:

$$\begin{aligned} &{}^{C}D_{0,t}^{\alpha} x(t) = Ax(t) + f(t, x(t)), && \alpha \in (0, 1), \quad t \in J,\\ &x(0) + G(x) = x_0, && x_0 \in X. \end{aligned}$$

They gave a suitable definition of a mild solution associated with characteristic solution operators of this problem and established existence results in the case when f and G satisfy Lipschitz continuous and growth conditions on J via Banach and Krasnoselskii fixed point theorems.

Motivated by [2, 3, 33, 34, 36] we investigate existence of mild solutions to the following Cauchy problem for fractional order evolution equations with nonlocal conditions:

$$\begin{aligned} {}^{C}D^{\alpha}_{0,t}x(t) &= Ax(t) + f(t, x(t)), \qquad \alpha \in (0,1), \quad t \in J, \\ x(0) &= \sum_{k=1}^{m} a_k x(t_k), \qquad k = 1, 2, \dots, m. \end{aligned} \tag{1}$$

We develop the approach and techniques from the above papers and establish two new existence results under general and weak assumptions on f by utilizing fractional calculus and Schaefer and O'Regan fixed point theorems. We give a suitable definition of a mild solution to equation (1) by introducing a bounded operator $B = \left[I - \sum_{k=1}^{m} a_k \mathcal{T}(t_k)\right]^{-1}$. Our first existence result relies on a growth condition on J and the second one relies on a growth condition involving two parts, one for $[0, t_m]$, and the other for $[t_m, 1]$. Our assumptions on f are more general and less restrictive than those imposed in [34, 36].

2. Preliminaries

Let $C(J, X)$ be the Banach space of all X-valued continuous functions from J into X endowed with the norm $\|x\|_{C(J,X)} = \sup_{t\in J}\|x(t)\|$. For brevity, we denote $\|x\|_C = \|x\|_{C(J,X)}$.

Definition 2.1 ([15]).

The fractional integral of order γ with the lower limit $a \in \mathbb{R}$ for a function $f\colon [a, \infty) \to \mathbb{R}$ is

$$I^{\gamma}_{a,t}f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{f(s)}{(t-s)^{1-\gamma}}\, ds, \qquad t > a, \quad \gamma > 0,$$

provided that the righthand side is point-wise defined on $[a, \infty)$, where $\Gamma(\cdot)$ is the gamma function. The Riemann–Liouville derivative of order γ with the lower limit zero for a function $f\colon [0, \infty) \to \mathbb{R}$ is

$${}^{L}D^{\gamma}_{0,t}f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}}\, ds, \qquad t > 0, \quad n-1 < \gamma < n.$$

The Caputo derivative of order γ for a function $f\colon [0, \infty) \to \mathbb{R}$ is

$${}^{C}D^{\gamma}_{0,t}f(t) = {}^{L}D^{\gamma}_{0,t}\left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)\right), \qquad t > 0, \quad n-1 < \gamma < n.$$

Remark 2.2.

If f is an abstract function with values in X, then the integrals in the definition are understood in Bochner's sense.

Suppose $M = \sup\limits_{t\geq 0}\|T(t)\|$ and define

$$\mathcal{T}(t) = \int_0^{\infty} \xi_\alpha(\theta) T(t^{\alpha}\theta)\, d\theta, \qquad \mathcal{S}(t) = \alpha \int_0^{\infty} \theta \xi_\alpha(\theta) T(t^{\alpha}\theta)\, d\theta, \qquad t \geq 0,$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\,\theta^{-1-1/\alpha}\,\varpi_\alpha(\theta^{-1/\alpha}) \geq 0,$$

$$\varpi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\,\frac{\Gamma(n\alpha+1)}{n!}\,\sin(n\pi\alpha), \qquad \theta \in (0, \infty),$$

where ξ_α is a probability density function defined on $(0,\infty)$, that is

$$\xi_\alpha(\theta) \geq 0, \qquad \theta \in (0,\infty), \qquad \int_0^\infty \xi_\alpha(\theta)\,d\theta = 1.$$

In a recent paper, Zhou and Jiao [37] gave some basic properties of $\mathcal{T}$ and $\mathcal{S}$ which will play an important role in the sequel.

Lemma 2.3 ([37, Lemmas 3.2–3.4]).

(i) *For any fixed $t \geq 0$ and any $x \in X$, $\|\mathcal{T}(t)x\| \leq M\|x\|$ and $\|\mathcal{S}(t)x\| \leq M\|x\|/\Gamma(\alpha)$.*

(ii) *$\{\mathcal{T}(t) : t \geq 0\}$ and $\{\mathcal{S}(t) : t \geq 0\}$ are strongly continuous.*

(iii) *For each $t > 0$, $\mathcal{T}(t)$ and $\mathcal{S}(t)$ are compact operators if $T(t)$ is compact.*

Further properties of $\mathcal{T}$ and $\mathcal{S}$ were explored by Wang and Zhou [31, 32].

Suppose that there exists the bounded operator $B\colon X \to X$ given by

$$B = \left[I - \sum_{k=1}^{m} a_k \mathcal{T}(t_k) \right]^{-1}. \tag{2}$$

Applying [33, Theorem 3.3, Remark 3.4] we can give two sufficient conditions for the existence and boundedness of the operator B.

Lemma 2.4.

The operator B defined in (2) exists and is bounded if one of the following two conditions holds:

(C_1) *there are real numbers a_k such that*

$$M \sum_{k=1}^{m} |a_k| < 1; \tag{3}$$

(C_2) *$T(t)$ is compact for each $t > 0$ and the homogeneous linear nonlocal problem*

$$^{C}D_{0,t}^{\alpha} x(t) = Ax(t), \qquad \alpha \in (0,1), \quad t \in J, \qquad x(0) = \sum_{k=1}^{m} a_k x(t_k), \tag{4}$$

has no non-trivial mild solutions.

Proof. Under assumption (C_1), from Lemma 2.3 (i) and (3) we have

$$\left\| \sum_{k=1}^{m} a_k \mathcal{T}(t_k) \right\| \leq M \sum_{k=1}^{m} |a_k| < 1.$$

Thus by the Neumann theorem, B exists and it is bounded. Under assumption (C_2), it is obvious that mild solutions to (4) have the form $x(t) = \mathcal{T}(t)x(0)$, hence

$$x(0) = \sum_{k=1}^{m} a_k x(t_k) = \sum_{k=1}^{m} a_k \mathcal{T}(t_k) x(0).$$

By Lemma 2.3 (iii), $\mathcal{T}(t_k)$ is compact for each $t_k > 0$, $k = 1, 2, \ldots, m$. Thus $\sum_{k=1}^{m} a_k \mathcal{T}(t_k)$ is also compact. Since problem (4) has no non-trivial mild solutions, one obtains the desired result applying the Fredholm alternative theorem. □

Similarly to [36], one can introduce the following definition of mild solutions to (1).

Definition 2.5.

A function $x \in C(J, X)$ is called a mild solution to (1) if it satisfies the following equation:

$$x(t) = \mathcal{T}(t) \sum_{k=1}^{m} a_k B(g(t_k)) + g(t), \qquad t \in J, \tag{5}$$

where

$$g(t_k) = \int_0^{t_k} (t_k - s)^{\alpha-1} \mathcal{S}(t_k - s) f(s, x(s))\, ds, \tag{6}$$

$$g(t) = \int_0^{t} (t - s)^{\alpha-1} \mathcal{S}(t - s) f(s, x(s))\, ds, \qquad t \in J. \tag{7}$$

Remark 2.6.

Due to [36] a mild solution to fractional evolution equation (1) with the initial condition is $x(t) = \mathcal{T}(t)x(0) + g(t)$, so taking into account our nonlocal condition, we get

$$x(0) = \sum_{k=1}^{m} a_k \mathcal{T}(t_k) x(0) + \sum_{k=1}^{m} a_k g(t_k).$$

So $x(0) = \sum_{k=1}^{m} a_k B(g(t_k))$ and hence $x(t) = \mathcal{T}(t) \sum_{k=1}^{m} a_k B(g(t_k)) + g(t)$, it is exactly (5).

3. First existence result

Our first existence result is based on the well-known Schaefer fixed point theorem [28].

Theorem 3.1.

Let $F\colon X \to X$ be a continuous mapping of X into X which is compact on each bounded subset of X. Then either

(i) *the equation $x = \lambda F x$ has a solution for $\lambda = 1$, or*

(ii) *the set $\{x \in X : x = \lambda F x \text{ for some } \lambda \in (0, 1)\}$ is unbounded.*

In this section, we will study our problem under the following assumptions:

(H_1) $f\colon J \times X \to X$ satisfies the Carathéodory conditions.

(H_2) There is a function h such that $I_{0,t}^{\alpha} h(t)$ exists for all $t \in J$ and $I_{0,\cdot}^{\alpha} h(\cdot) \in C((0, 1], \mathbb{R}^+)$ with $\lim_{t\to 0^+} I_{0,t}^{\alpha} h(t) = 0$ and a nondecreasing continuous function $\Omega\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t, x)\| \le h(t)\Omega(\|x\|)$$

for all $x \in X$ and for almost every $t \in J$.

Remark 3.2.

In our previous works [34, 36], we assumed that there exists a function $h \in L^{1/\alpha_1}(J, \mathbb{R}^+)$, $\alpha_1 \in [0, \alpha)$, where $L^p(J, \mathbb{R}^+)$ denotes the Banach space of all Lebesgue measurable functions $h\colon J \to \mathbb{R}^+$ with the norm of h given by

$$\|h\|_{L^p(J,\mathbb{R}^+)} = \begin{cases} \left(\int_J |h(t)|^{1/p} dt \right)^p, & 1 < p < \infty, \\ \inf\limits_{\mu(\bar{J})=0} \sup\limits_{t \in J - \bar{J}} |h(t)|, & p = \infty, \end{cases}$$

where $\mu(\bar{J})$ is the Lebesgue measure on $\bar{J}$. However, it is not difficult to verify that the old (strong) condition $h \in L^{1/\alpha_1}(J, \mathbb{R}^+)$, $\alpha_1 \in [0, \alpha)$, implies a new (weak) condition $I^{\alpha}_{0,\cdot} h(\cdot) \in C((0, 1], \mathbb{R}^+)$ with $\lim_{t \to 0^+} I^{\alpha}_{0,t} h(t) = 0$.

(H$_3$) The inequality $\limsup_{\rho \to \infty} \rho \left(M^2 B \Omega(\rho) \sum_{k=1}^{m} |a_k| I^{\alpha}_{0,t_k} h(t_k) + M\Omega(\rho) \sup_{t \in J} I^{\alpha}_{0,t} h(t) \right)^{-1} > 1$ holds.

(H$_4$) $T(t)$ is compact for each $t > 0$.

We consider the following problem:

$$ {}^{C}D^{\alpha}_{0,t} x(t) = Ax(t) + \lambda f(t, x(t)), \qquad \alpha \in (0, 1], \quad \lambda, t \in J, \qquad x(0) = \sum_{k=1}^{m} a_k x(t_k). \tag{8} $$

Define an operator $F\colon C(J, X) \to C(J, X)$ as follows:

$$ (Fx)(t) = (F_1 x)(t) + (F_2 x)(t), \qquad t \in J, $$

where $F_i\colon C(J, X) \to C(J, X)$, $i = 1, 2$, are given by the formulas

$$ (F_1 x)(t) = \mathcal{T}(t) \sum_{k=1}^{m} a_k B(g(t_k)), \qquad (F_2 x)(t) = g(t), $$

where B is the operator defined in (2), $g(t_k)$ is defined in (6) and $g(t)$ is defined in (7). Obviously, a mild solution to equation (8) is a solution to the operator equation

$$ x = \lambda F x \tag{9} $$

and conversely. Thus, we can apply the Schaefer fixed point theorem to derive the existence of solutions to equation (1).

Lemma 3.3.
There exists a constant $R^* > 0$ independent of the parameter $\lambda \in J$ such that $\|x\|_C \le R^*$ for every solution x to equation (9).

Proof. Denote $R_0 = \|x\|_C$. Taking into account our conditions and Lemma 2.4 (C$_1$), (C$_2$), it follows from (5) that

$$ \|x(t)\| \le \|(F_1 x)(t)\| + \|(F_2 x)(t)\| \le M \sum_{k=1}^{m} |a_k| \|B\| \|g(t_k)\| + \|g(t)\|, \qquad t \in J. \tag{10} $$

Note that

$$ \begin{aligned} \|g(t_k)\| &\le \int_0^{t_k} (t_k - s)^{\alpha - 1} \|\mathcal{S}(t_k - s)\| \|f(s, x(s))\| \, ds \le \frac{M}{\Gamma(\alpha)} \int_0^{t_k} (t_k - s)^{\alpha-1} h(s) \Omega(\|x\|_C) \, ds \\ &\le \frac{M\Omega(R_0)}{\Gamma(\alpha)} \int_0^{t_k} (t_k - s)^{\alpha - 1} h(s) \, ds = M\Omega(R_0) I^{\alpha}_{0,t_k} h(t_k), \qquad k = 1, 2, \dots, m, \end{aligned} $$

and

$$ \|g(t)\| \le \frac{M\Omega(R_0)}{\Gamma(\alpha)} \int_0^{t} (t - s)^{\alpha-1} h(s) \, ds = M\Omega(R_0) \sup_{t \in J} I^{\alpha}_{0,t} h(t), \qquad t \in J. \tag{11} $$

From (10)–(11), one has

$$ R_0 = \|x\|_C \le M^2 \|B\| \Omega(R_0) \sum_{k=1}^{m} |a_k| I^{\alpha}_{0,t_k} h(t_k) + M\Omega(R_0) \sup_{t \in J} I^{\alpha}_{0,t} h(t), \qquad t \in J, $$

which implies

$$R_0\left(M^2\|B\|\Omega(R_0)\sum_{k=1}^{m}|a_k|I_{0,t_k}^{\alpha}h(t_k)+M\Omega(R_0)\sup_{t\in J}I_{0,t}^{\alpha}h(t)\right)^{-1}\leq 1. \tag{12}$$

However, according to (H_3), there exists $R^*>0$ such that for all $R>R^*$ we have

$$R\left(M^2\|B\|\Omega(R)\sum_{k=1}^{m}|a_k|I_{0,t_k}^{\alpha}h(t_k)+M\Omega(R)\sup_{t\in J}I_{0,t}^{\alpha}h(t)\right)^{-1}>1. \tag{13}$$

Now, comparing (12) and (13), we deduce that $R_0\leq R^*$. As a result, we find that $\|x\|_C\leq R^*$. This completes the proof. □

Let $\mathfrak{B}_{R^*}=\{x\in C(J,X):\|x\|_C\leq R^*\}$. Then $\mathfrak{B}_{R^*}$ is a bounded closed and convex subset in $C(J,X)$. By Lemma 3.3, we can derive the following result.

Lemma 3.4.

The operator F maps $\mathfrak{B}_{R^*}$ into itself.

Lemma 3.5.

The operator $F\colon\mathfrak{B}_{R^*}\to\mathfrak{B}_{R^*}$ is completely continuous.

Proof. For our purpose, we only need to check that $F_i\colon\mathfrak{B}_{R^*}\to\mathfrak{B}_{R^*}$, $i=1,2$, is completely continuous. Firstly, by repeating the procedure of our previous work (see Step III in the proof of [36, Theorem 3.1]), one can obtain that $F_2\colon\mathfrak{B}_{R^*}\to\mathfrak{B}_{R^*}$ is completely continuous. We only emphasize that the main difference is that the condition $h\in L^{1/\alpha_1}(J,\mathbb{R}^+)$, $\alpha_1\in[0,\alpha)$, is replaced by the new condition $I_{0,\cdot}^{\alpha}h(\cdot)\in C((0,1],\mathbb{R}^+)$ with $\lim_{t\to 0^+}I_{0,t}^{\alpha}h(t)=0$.

Secondly, one can check that $F_1\colon\mathfrak{B}_{R^*}\to\mathfrak{B}_{R^*}$ is continuous (by (H_1), (H_2) and Lemma 2.3 (i)) and $F_1\colon\mathfrak{B}_{R^*}\to\mathfrak{B}_{R^*}$ is compact since $\mathcal{T}(t)$ is compact for each $t>0$ (by (H_4) and Lemma 2.3 (iii)). □

Now, we can state the main result of this section.

Theorem 3.6.

Assume that (H_1)–(H_4) hold and condition (C_1) (or (C_2)) is satisfied. Then equation (1) has at least one solution $u\in C(J,X)$ and the set of solutions to equation (1) is bounded in $C(J,X)$.

Proof. Obviously, the set $\{x\in C(J,X):x=\lambda Fx,\ 0<\lambda<1\}$ is bounded due to Lemma 3.4. Now we can apply Theorem 3.1 to derive that F has a fixed point in $\mathfrak{B}_{R^*}$ which is just the mild solution to equation (1). □

4. Second existence result

Our second existence result is based on the O'Regan fixed point theorem [26].

Theorem 4.1.

Let U be an open set in a closed, convex set $\mathcal{C}$ of X. Assume $0\in U$, $T(\overline{U})$ is bounded and $T\colon\overline{U}\to\mathcal{C}$ is given by $T=T_1+T_2$ where $T_1\colon\overline{U}\to X$ is completely continuous, and $T_2\colon\overline{U}\to X$ is a nonlinear contraction. Then either

(i) *T has a fixed point in $\overline{U}$, or*

(ii) *there is a point $x\in\partial U$ and $\lambda\in(0,1)$ with $x=\lambda T(x)$.*

In addition to (H_1), (H_4) and (C_1) (or (C_2)), motivated by Boucherif and Precup [2, 3], we introduce the following two assumptions:

(H_5) There exists a function h such that $I^{\alpha}_{0,t}h(t)$ exists for every $t \in [0, t_m]$ and $I^{\alpha}_{0,\cdot}h(\cdot) \in C((0, t_m], \mathbb{R}^+)$ with $\lim_{t\to 0^+} I^{\alpha}_{0,t}h(t) = 0$ and a nondecreasing continuous function $\Omega\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|f(t, x)\| \le h(t)\Omega(\|x\|)$ for all $x \in X$ and for a.e. $t \in [0, t_m]$, and for every $t \in [t_m, 1]$ there exists a function l such that $I^{\alpha}_{t_m,t}l(t)$ exists and $I^{\alpha}_{t_m,\cdot}l(\cdot) \in C([t_m, 1], \mathbb{R}^+)$ such that

$$\|f(t, x)\| \le l(t), \tag{14}$$

for all $x \in X$ and for a.e. $t \in [t_m, 1]$. Moreover, Ω has the property

$$r > M\Omega(r)\left(\sum_{k=1}^{m} |a_k|\|B\| + 1\right) \sup_{t\in[0,t_m]} I^{\alpha}_{0,t}h(t) \tag{15}$$

for all $r > R_1^* > 0$.

(H_6) There exists a function q such that $I^{\alpha}_{t_m,t}q(t)$ exists for every $t \in [t_m, 1]$ and $I^{\alpha}_{t_m,\cdot}q(\cdot) \in C([t_m, 1], \mathbb{R}^+)$ with $M \sup_{t\in[t_m,1]} I^{\alpha}_{0,t}q(t) \le 1$ and a nondecreasing continuous function $\Psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\Psi(r) < r$ for $r > 0$ such that

$$\|f(t, x) - f(t, y)\| \le q(t)\Psi(\|x - y\|)$$

for a.e. $t \in [t_m, 1]$ and for all $x, y \in X$.

Consider equation (8) again and the equivalent equation

$$x = \lambda T x, \tag{16}$$

where $T\colon C(J, X) \to C(J, X)$ is defined by $(Tx)(t) = (T_1x)(t) + (T_2x)(t)$, $t \in J$, $T_i\colon C(J, X) \to C(J, X)$, $i = 1, 2$, are given by

$$(T_1x)(t) = \begin{cases} \mathcal{T}(t)\sum_{k=1}^{m} a_k B(g(t_k)) + g(t), & t \in [0, t_m),\\ \mathcal{T}(t)\sum_{k=1}^{m} a_k B(g(t_k)) + \int_0^{t_m} (t-s)^{\alpha-1}\mathcal{S}(t-s) f(s, x(s))\, ds, & t \in [t_m, 1], \end{cases}$$

$$(T_2x)(t) = \begin{cases} 0, & t \in [0, t_m),\\ \int_{t_m}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s) f(s, x(s))\, ds, & t \in [t_m, 1]. \end{cases}$$

We first prove that solutions to equation (16) are a priori bounded.

Lemma 4.2.

There exist $R_i^* > 0$, $i = 1, 2$, independent of the parameter λ, such that $\|x\|_{C([0,t_m],X)} \le R_1^*$ and $\|x\|_{C([t_m,1],X)} \le R_2^*$, that is $\|x\|_C \le R^* = \max\{R_1^*, R_2^*\}$ for every solution x of the equation (16).

Proof. **Case 1.** We prove that there exists $R_1^* > 0$ such that $\|x\|_{C([0,t_m],X)} \le R_1^*$. For $t \in [0, t_m]$ and $\lambda \in J$, denote $R_{[0,t_m]} = \|x\|_{C([0,t_m],X)}$, we have

$$\begin{aligned} \|x(t)\| &\le \lambda\|(T_1x)(t)\| + \|(T_2x)(t)\| \le M\sum_{k=1}^{m} |a_k|\|B\|\|g(t_k)\| + \|g(t)\| \\ &\le M\sum_{k=1}^{m} |a_k|\|B\|\frac{M}{\Gamma(\alpha)}\int_0^{t_k} (t_k-s)^{\alpha-1}h(s)\Omega\big(R_{[0,t_m]}\big)\, ds + \frac{M}{\Gamma(\alpha)}\int_0^{t} (t-s)^{\alpha-1}h(s)\Omega\big(R_{[0,t_m]}\big)\, ds \\ &\le M\Omega\big(R_{[0,t_m]}\big)\left(\sum_{k=1}^{m} |a_k|\|B\| + 1\right) \sup_{t\in[0,t_m]} I^{\alpha}_{0,t}h(t), \end{aligned}$$

which implies

$$R_{[0,t_m]} \leq M\Omega\big(R_{[0,t_m]}\big)\left(\sum_{k=1}^{m}|a_k|\|B\|+1\right)\sup_{t\in[0,t_m]} I^{\alpha}_{0,t}h(t).$$

From (15) we find that there exists $R_1^* \geq R_{[0,t_m]} > 0$ such that $\|x\|_{C([0,t_m],X)} \leq R_1^*$.

Case 2. We prove that there exists $R_2^* > 0$ such that $\|x\|_{C([t_m,1],X)} \leq R_2^*$. For $t \in [t_m, 1]$ and $\lambda \in J$, keeping in mind our assumptions, we find that

$$\begin{aligned}
\|x(t)\| \leq{}& M\sum_{k=1}^{m}|a_k|\|B\|\frac{M}{\Gamma(\alpha)}\int_0^{t_k}(t_k-s)^{\alpha-1}h(s)\Omega(R_1^*)\,ds\\
&+\frac{M}{\Gamma(\alpha)}\int_0^{t_m}(t-s)^{\alpha-1}h(s)\Omega(R_1^*)\,ds+\frac{M}{\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}h(s)\,ds\\
\leq{}& M\Omega(R_1^*)\left(\sum_{k=1}^{m}|a_k|\|B\|+1\right)\sup_{t\in[0,t_m]} I^{\alpha}_{0,t}h(t)+M\sup_{t\in[t_m,1]} I^{\alpha}_{t_m,t}l(t),
\end{aligned}$$

which implies $\|x\|_{C([t_m,1],X)} \leq R_2^*$, where

$$R_2^* = M\left[\Omega(R_1^*)\left(\sum_{k=1}^{m}|a_k|\|B\|+1\right)\sup_{t\in[0,t_m]} I^{\alpha}_{0,t}h(t)+\sup_{t\in[t_m,1]} I^{\alpha}_{t_m,t}l(t)\right].$$

Let $R^* = \max\{R_1^*, R_2^*\}$. Then all solutions of the equation (16) satisfy $\|x\|_C \leq R^*$, where R^* is independent of the parameter λ. □

Denote $\mathcal{D} = \{x \in C(J, X) : \|x\|_C < R^* + 1\}$. We can proceed as in the proof of Lemma 4.2 to derive the following result.

Lemma 4.3.
$T(\overline{\mathcal{D}})$ is bounded.

One can proceed as in the proof of Lemma 3.5 to obtain the following result.

Lemma 4.4.
The operator $T_1\colon \overline{\mathcal{D}} \to C(J, X)$ is completely continuous.

Lemma 4.5.
The operator $T_2\colon \overline{\mathcal{D}} \to C(J, X)$ is a nonlinear contraction.

Proof. From the definition of T_2 we only need to show that $T_2\colon \overline{\mathcal{D}} \to C([t_m, 1], X)$ is a nonlinear contraction. In fact, for any $x, y \in \overline{\mathcal{D}}$ and $t \in [t_m, 1]$, we have

$$\begin{aligned}
\|(T_2x)(t)-(T_2y)(t)\| \leq{}& \int_{t_m}^{t}(t-s)^{\alpha-1}\big\|\mathcal{S}(t-s)[f(s,x(s))-f(s,y(s))]\big\|\,ds\\
\leq{}& \frac{M}{\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}q(s)\Psi(\|x(s)-y(s)\|)\,ds\\
\leq{}& \frac{M\Psi(\|x-y\|_C)}{\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}q(s)\,ds \leq \left(M\sup_{t\in[t_m,1]} I^{\alpha}_{t_m,t}q(t)\right)\Psi(\|x-y\|_C),
\end{aligned}$$

which implies $\|T_2x - T_2y\|_C \leq \Psi(\|x - y\|_C)$. □

Now, we are ready to present the main result of this section.

Theorem 4.6.

Assume that (H_1), (H_4), (H_5) *and* (H_6) *hold and condition* (C_1) (*or* (C_1)) *is satisfied. Then equation* (1) *has at least one solution* $u \in C(J, X)$.

Proof. By Lemma 4.2 we see that (ii) in Theorem 4.1 does not hold for $U = \mathcal{D}$. Therefore, from Theorem 4.1, T has a fixed point in $\mathcal{D}$ which is just the mild solution to the equation (1). This completes the proof. □

Finally, we try to change the conditions (H_5) and (H_6) to the following parallel conditions:

(H_5') Condition (H_5) is assumed without (14).

(H_6') Denoting $\delta = \lim_{r\to+\infty} \inf \Psi(r)/r \le 1$, condition ($H_6$) is assumed in addition with

$$M\delta \sup_{t\in[t_m,1]} I_{0,t}^{\alpha} q(t) < 1.$$

Corollary 4.7.

The existence result in Theorem 4.6 also holds even if (H_5) *and* (H_6) *are replaced by the conditions* (H_5') *and* (H_6') *respectively.*

Proof. Indeed, we can modify Case 2 in the proof of Lemma 4.2 as follows:

$$\begin{aligned}\|x(t)\| \le{}& M\sum_{k=1}^{m}|a_k|\|B\|\frac{M}{\Gamma(\alpha)}\int_0^{t_k}(t_k-s)^{\alpha-1}h(s)\Omega(R_1^*)\,ds + \frac{M}{\Gamma(\alpha)}\int_0^{t_m}(t-s)^{\alpha-1}h(s)\Omega(R_1^*)\,ds \\ &+ \frac{M}{\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}\|f(s,0)\|\,ds + \frac{M}{\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}q(s)\Psi(\|x(s)\|)\,ds \\ \le{}& M\Omega(R_1^*)\left(\sum_{k=1}^{m}|a_k|\|B\|+1\right)\sup_{t\in[0,t_m]} I_{0,t}^{\alpha}h(t) + \frac{M\sup_{t\in[t_m,t]}\|f(t,0)\|(1-t_m)^{\alpha}}{\Gamma(\alpha+1)} \\ &+ \frac{M}{\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}q(s)(\delta\|x(s)\|+\delta_1)\,ds,\end{aligned}$$

for some $\delta_1 \ge 0$. Then we have

$$\begin{aligned}R_2^* = \frac{1}{1-M\delta\sup_{t\in[t_m,1]} I_{0,t}^{\alpha}q(t)}\Bigg\{& M\Omega(R_1^*)\left(\sum_{k=1}^{m}|a_k|\|B\|+1\right)\sup_{t\in[0,t_m]} I_{0,t}^{\alpha}h(t) \\ &+ \frac{M}{\Gamma(\alpha+1)}\sup_{t\in[t_m,t]}\|f(t,0)\|(1-t_m)^{\alpha} + M\delta_1\sup_{t\in[t_m,1]} I_{0,t}^{\alpha}q(t)\Bigg\}.\end{aligned}$$

The rest proof is standard. So we omit it here. □

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