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# On the nonstationary Navier-Stokes system

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# ON THE NONSTATIONARY NAVIER–STOKES SYSTEM

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## INTRODUCTION

Since the appearance of the pioneer work by Kiselev and Ladyženskaia, a large number of works have been published by various authors on the initial value problem for the nonstationary Navier-Stokes equations. As yet the existence of a global solution (in time) has not been proved for the three-dimensional flow for sufficiently general initial conditions; but it now appears that the existence and uniqueness of a genuine solution which is local in time have been established, although a complete proof of such a result does not seem to have been published at the time of writing this report.

The present article is an attempt to deduce an existence and uniqueness theorem by means of Hilbert space theory. Our ultimate objective is the proof of the theorem in its classical form. As a preliminary step to this goal, however, we shall prove

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it here in a somewhat weaker form as an existence and uniqueness theorem on the initial value problem of the Navier-Stokes equation regarded as a *nonlinear operator equation* in an appropriate Hilbert space [1]). The proof that this « strong solution » is actually a genuine solution in the classical sense will be given in a subsequent publication.

The Navier-Stokes system of equations in the $m$-dimensional space, where $m = 2$ or 3, may be written as

$$\text{(I)}\quad \begin{aligned} \partial u/\partial t &= \Delta u - \operatorname{grad} p + f - (u, \operatorname{grad})u && (x \in D,\ t > 0),\\ \operatorname{div} u &= 0 && (x \in D,\ t > 0),\\ u_{|\partial D} &= 0 && (t > 0),\\ u_{|t=0} &= a && (x \in D) \end{aligned}$$

with the well-known notations. $u = u(x, t) = (u_1(x, t), \ldots, u_m(x, t))$ is the velocity field, $p = p(x, t)$ is the pressure, $a = a(x)$ is the initial velocity and $f = f(x, t) = (f_1(x, t), \ldots, f_m(x, t))$ is the external force; the kinematic viscosity is set equal to unity. For simplicity we assume that $D$ is a bounded domain of $R^m$ with a sufficiently smooth boundary $\partial D$.

The system (I) can be transformed into the operator equation

$$\text{(II)}\quad \begin{aligned} du/dt &= - Au + Pf(t) + Fu \qquad (Fu = - P(u, \operatorname{grad})u)\\ u(0) &= a\,. \end{aligned}$$

Here $t \to u(t)$ and $t \to f(t)$ are regarded as functions on real numbers to $\mathcal{H}_\sigma$ and $\mathcal{H}$ respectively. $\mathcal{H}_\sigma$ is the Hilbert space consisting of all *solenoidal* vector function $u$ ($\operatorname{div} u = 0$, $u_{n|\partial D} = 0$ where $n$ denotes the normal component), $P$ is the orthogonal projection of $\mathcal{H} \equiv [L^2(D)]^m$ onto $\mathcal{H}_\sigma$ and $A$ is formally given by $- P\Delta$.

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[1]) In this sense our results are closely related to the results of Sobolevskii [1].

More precisely, $\mathcal{H}_\sigma$ is to be defined as the orthogonal complement in $\mathcal{H}$ of the subspace spanned by the vectors of the form grad $h$ with $h = h(x)$ smooth in $D$. A is a selfadjoint operator in $\mathcal{H}_\sigma$, being the *Friedrichs extension* of the nonnegative symmetric operator $-P\Delta$ in $\mathcal{H}_\sigma$ defined for all $u \in C^2$ with div $u = 0$ and $u_{|\partial D} = 0$ (such functions belong to $\mathcal{H}_\sigma$). Another characterization of $A$ is that it is the selfadjoint operator associated with the nonnegative sesquilinear form

$$(\text{grad } u, \text{grad } v) = \int_D \text{grad } u \cdot \text{grad } \bar{v}\, dx$$

defined for $u, v \in [H_0^1(D)]^m \cap \mathcal{H}_\sigma \equiv \mathcal{H}^1_{0,\sigma}$. In any case $A$ is characterized by the fact that

$$(\text{grad } u, \text{grad } v) = (w, v)\,, \qquad u \in \mathcal{H}^1_{0,\sigma},\, w \in \mathcal{H}_\sigma\,,$$

for all $v \in \mathcal{H}^1_{0,\sigma}$ (or, equivalently, for all solenoidal $v \in \overset{\infty}{C_0}(D)$) implies that $u \in D[A]$ and $Au = w$.

(II) can further be transformed into the nonlinear integral equation

$$\text{(III)} \quad u(t) = e^{-ta} + \int_0^t e^{-(t-s)A} Pf(s)ds + \int_0^t e^{-(t-s)A} Fu(s)ds\,.$$

Here $e^{-tA}$ is a bounded operator on $\mathcal{H}_\sigma$ for each $t \geqslant 0$ and forms a semigroup of bounded operators with the infinitesimal generator $-A$.

The transition (I) $\to$ (II) $\to$ (III) has been quite formal. We shall not try to justify this transition in detail. We shall rather start from (III) and prove the existence and uniqueness of a certain type of solution of (III). Then we shall prove that this solution is actually a unique solution of the operator differential equation (II) under certain additional conditions. In a later publication this « strong » solution will be shown to form, in conjunction with an appropriate function $p = p(x, t)$, a genuine solution of the classical system (I).

In what follows we assume, for simplicity, once for all that $Pf(t)$ is continuous for $t > 0$ (in the strong topology of $\mathcal{H}_\sigma$) so that we are able to work within the realm of Riemann integrals in dealing with the integral equation (III). Actually this is not essential, for otherwise we could use the Bochner instead of the Riemann integrals. In any case, the behavior of $f(t)$ for $t \to 0$ is allowed to be singular to a certain extent.

The main results of the present note are the following. In §1 we construct by a simple successive approximation a solution of (III). The successive approximation converges and yields a local solution $u(t)$ of (III) if the initial velocity $a$ belongs to $D[A^{1/4}]$ and the external force $f(t)$ is not too singular for $t \to 0$ (it suffices that $\| Pf(t) \| = o(t^{-3/4})$), see Theorem 1. This solution is even global if $\| A^{1/4}a \|$ is sufficiently small and $f(t)$ is sufficiently weak in a sense to be specified (Theorem 2). Furthermore, $u(t)$ has the property that $A^\alpha u(t)$ exists for $t > 0$ for any $\alpha < 1$, is Hölder continuous in $t$ (Theorem 3) and satisfies the inequality $\| A^\alpha u(t) \| = o(t^{1/4-\alpha})$, $1/4 < \alpha < 1$. Within the class of such (or somewhat more general) functions the solution of (III) is shown to be unique (Theorem 4 of § 2).

In § 3 we prove that a solution $u(t)$ of (III) satisfies (II) for $t > 0$ if $Pf(t)$ is Hölder continuous for $t > 0$ (Theorem 5).

§ 4 is devoted to the case $m = 2$. It is shown that *(III) has a global solution for any initial velocity* $a \in \mathcal{H}_\sigma$ provided that $Pf(t)$ is not too singular for $t \to 0$, again $\| Pf(t) \| = o(t^{-3/4})$ being sufficient. This solution is again unique within the class of functions similar to those mentioned above, and it is a solution of (II) if $Pf(t)$ is Hölder continuous for $t > 0$ (Theorems 6 and 7).

## 1. Solution of the integral equation for $m = 3$

In this section we consider the three-dimensional flow ($m = 3$). We want to construct the solution of the integral equation (III) by successive approximation, according to the scheme

$$u_0(t) = e^{-ta} + \int_0^t e^{-(t-s)A} Pf(s)ds, \tag{1}$$

$$u_{n+1}(t) = u_n(t) + \int_0^t e^{-(t-s)A} Fu_n(s)ds, \qquad n = 0, 1, 2, \ldots \tag{2}$$

But it is not at all obvious that $u_n(t)$ can be constructed in this way, for the functions in the integrands of (1) and (2) are continuous only in the open interval $(0, t)$. In particular the nonlinear term $Fu_n(s)$ requires a special investigation. To this end we make use of the following lemmas.

LEMMA 1: *$A$ is selfadjoint and strictly positive: $A \geqslant \delta > 0$, so that $A^{-1}$ exists and is bounded with $\| A^{-1} \| \leqslant \delta^{-1}$.*

LEMMA 2: $\| A^{\alpha} e^{-tA} \| \leqslant t^{-\alpha}, \qquad 0 \leqslant \alpha \leqslant e = 2,718 \ldots$

LEMMA 3: *If $u \in D[A^{3/4}]$, then $Fu = - P(u, \text{grad})u \in \mathcal{H}_\sigma$ is defined with*

$$\| Fu \| \leqslant M \| A^{3/4} u \| \; \| A^{1/2} u \| . \tag{3}$$

If $u, v \in D[A^{3/4}]$, then

$$\| Fu - Fv \| \leqslant M(\| A^{3/4} u \| \; \| A^{1/2}(u - v) \| + + \| A^{3/4}(u - v) \| \; \| A^{1/2} v \|) . \tag{4}$$

Here the constant $M$ depends only on $D$.

The proof of Lemmas 1 and 2 are simple and may be omitted. The proof of Lemma 3 will be given in Appendix.

1. Suppose now that the first $n + 1$ approximations $u_0(t), \ldots, u_n(t)$ have been constructed, $A^{\alpha} u_n(t)$ exist and are continuous in $t$ for $0 < t \leqslant T$ for any $\alpha < 1$ and that the following inequalities hold with constants $K_{\alpha n}$:

$$\| A^{\alpha} u_n(t) \| \leqslant K_{\alpha n} t^{1/4 - \alpha} \quad \text{for} \quad 0 < t \leqslant T, \frac{1}{4} \leqslant \alpha < 1 . \tag{5}$$

Then $Fu_n(t)$ is continuous for $0 < t \leqslant T$ by (4) and $\| Fu_n(t) \| \leqslant$

$\leqslant MK_{3/4,n}K_{1/2,n}t^{-3/4}$ by (3) and (5). Hence $u_{n+1}(t)$ can be constructed by (2). Furthermore we have

$$(6) \qquad A^\alpha u_{n+1}(t) = A^\alpha u_0(t) + \int_0^t A^\alpha e^{-(t-s)A} F u_n(s) ds ,$$

the validity of which is ensured by the absolute convergence of the integral involved. In fact, we have $\| A^\alpha e^{-(t-s)A} \| \leqslant (t-s)^{-\alpha}$ by Lemma 2 and $\| F u_n(s) \|$ is dominated by const. $s^{-3/4}$ as noted above. Thus (6) gives

$$(7) \qquad \| A^\alpha u_{n+1}(t) \| \leqslant \| A^\alpha u_0(t) \| + MK_{3/4,n}K_{1/2,n} \int_0^t (t-s)^{-\alpha} s^{-3/4} ds$$

$$\leqslant (K_{\alpha 0} + B(1-\alpha, 1/4) MK_{3/4,n}K_{1/2,n}) t^{(1/4)-\alpha} ,$$

where $B$ denotes the beta function. (7) shows that (5) is satisfied with $n$ replaced by $n+1$, with

$$(8) \qquad K_{\alpha,n+1} = K_{\alpha 0} + B(1-\alpha, 1/4) MK_{3/4,n}K_{1/2,n} .$$

Furthermore, the continuity of $A^\alpha u_{n+1}(t)$ as function of $t$ can be concluded easily from (6). Therefore, *the successive approximation can be continued indefinitely if* $A^\alpha u_0(t)$ *is continuous in* $t$ *for* $0 < t \leqslant T$ *and* (5) *is satisfied for* $n = 0$. As is seen from (1), this condition is satisfied if $a \in D[A^{1/4}]$ and $\| Pf(t) \| = o(t^{-3/4})$ for $t \to 0$. In fact, we have then by Lemma 2

$$(9) \qquad \| A^\alpha u_0(t) \| \leqslant \| A^{\alpha-(1/4)} e^{-tA} A^{1/4} a \| + \int_0^t \| A^\alpha e^{-(t-s)A} \| N s^{-3/4} ds$$

$$\leqslant (\| A^{1/4} a \| + NB(1-\alpha, 1/4)) t^{(1/4)-\alpha} , \qquad 0 < t \leqslant T , \frac{1}{4} \leqslant \alpha < 1) ,$$

where $N = \sup_{0<t\leqslant T} t^{3/4} \| Pf(t) \|$ . Thus we are able to choose

$$(10) \quad K_{\alpha 0} = \| A^{1/4} a \| + NB(1-\alpha, 1/4) , \qquad N = \sup_{0<t\leqslant T} t^{3/4} \| Pf(t) \| .$$

An examination of (9) shows, however, that a better choice is

$$(11)\quad K_{\alpha 0} = \sup_{0<t\leqslant T} t^{\alpha-(1/4)} \| A^{\alpha-(1/4)} e^{-tA} A^{1/4} a \| + NB(1-\alpha, 1/4)\,.$$

As will be seen, this remark is rather important.

2. The recurrence formulas of (8) form a closed system for $\alpha = 1/2$ and $\alpha = 3/4$. Set $k_n = \max(K_{1/2,n}, K_{3/4,n})$. Then (8) gives a recurrence inequality for $k_n$:

$$(12)\qquad k_{n+1} \leqslant k_0 + \beta M k_n^2\,, \qquad \beta = B(1/4, 1/4)\,.$$

An elementary consideration shows that the sequence $\{k_n\}$ is bounded if

$$(13)\qquad k_0 < 1/4\beta M\,,$$

with

$$(14)\qquad k_n \leqslant K \equiv \frac{1-(1-4\beta M k_0)^{1/2}}{2\beta M} < \frac{1}{2\beta M}\,.$$

Suppose now that (13) is satisfied so that $\{k_n\}$ is bounded as in (14). We shall show that $u_n(t)$ converges uniformly for $0 \leqslant t \leqslant T$. We have

$$(15)\quad w_{n+1}(t) \equiv u_{n+1}(t) - u_n(t) = \int_0^t e^{-(t-s)A}(Fu_n(s) - Fu_{n-1}(s))ds\,,$$

$$n = 0, 1, 2, \ldots$$

(set $u_{-1}(t) = 0$) and by Lemmas 2 and 3.

$$(16)\quad \| A^\alpha w_{n+1}(t) \| \leqslant \int_0^t \| A^\alpha e^{-(t-s)A} \| \; \| Fu_n(s) - Fu_{n-1}(s) \| \, ds$$

$$\leqslant M \int_0^t (t-s)^{-\alpha}(\| A^{3/4} u_n(s) \| \; \| A^{1/2} w_n(s) \| +$$

$$+ \| A^{3/4} w_n(s) \| \; \| A^{1/2} u_{n-1}(s) \| )ds$$

$$\leqslant MK \int_0^t (t-s)^{\alpha}(s^{-1/2} \| A^{1/2} w_n(s) \| + s^{-1/4} \| A^{3/4} w_n(s) \| )ds$$

(the last term in the last integrand should be omitted if $n = 0$). Setting $\alpha = 1/2$ and $\alpha = 3/4$ in (16), we see by induction that the following estimates hold for $0 < t \leqslant T$:

$$(17) \qquad \| A^{1/2} w_n(t) \| \leqslant \frac{(2\beta MK)^n K}{2t^{1/4}}, \qquad \| A^{3/4} w_n(t) \| \leqslant \frac{(2\beta MK)^n K}{2t^{1/2}} .$$

Then it follows from (16) that

$$(18) \qquad \| A^\alpha w_n(t) \| \leqslant B(1-\alpha, 1/4)(2\beta MK)^{n-1} MK^2 t^{(1/4)-\alpha},$$

$$0 < t \leqslant T, \quad \frac{1}{4} \leqslant \alpha < 1 .$$

Since $2\beta MK < 1$ by (14), we conclude that $\sum w_n(t)$ is absolutely and uniformly convergent for $0 \leqslant t \leqslant T$. Thus $u(t) \equiv \lim u_n(t) = \sum w_n(t)$ exists uniformly for $0 \leqslant t \leqslant T$. Similarly it follows from (18) that $\lim_{n\to\infty} A^\alpha u_n(t) = \sum A^\alpha w_n(t)$ exists for $\alpha < 1$ uniformly for $\varepsilon \leqslant t \leqslant T$ for any $\varepsilon > 0$. In view of the closure of the operator $A^\alpha$, the limit must be equal to $A^\alpha u(t)$. Since $A^\alpha u_n(t)$ are continuous for $0 < t \leqslant T$, the same is true with $A^\alpha u(t)$. Since the sequence $\{K_{\alpha n}\}$ is bounded by (8) and the boundedness of $\{k_n\}$, it follows from (5) that

$$(19) \qquad \| A^\alpha u(t) \| \leqslant K_\alpha t^{(1/4)-\alpha}, \quad 0 < t \leqslant T, \quad \frac{1}{4} \leqslant \alpha < 1 .$$

Here the constant $K_\alpha$ may be replaced by $K$ at least for $\alpha = 1/2$ and $3/4$.

The convergence $A^\alpha u_n(t) \to A^\alpha u(t)$ for $\alpha = 1/2$ and $3/4$ implies the convergence $Fu_n(t) \to Fu(t)$ by virtue of (4), where $\| Fu_n(t) \|$ is dominated by const. $t^{-3/4}$ by (3) and (5). Taking the limit $n \to \infty$ in (2), we thus conclude that $u(t)$ is a solution of the integral equation (III) for $0 \leqslant t \leqslant T$. Thus we have proved the existence of a solution of (III) under the condition (13).

3. We shall now show that (13) is satisfied if a $\in D[A^{1/4}]$ and

$$(20) \qquad \| Pf(t) \| = o(t^{-3/4}), \qquad t \to 0 .$$

In fact, $a \in D[A^{1/4}]$ implies that $t^{-\alpha(1/4)}A^{\alpha-(1/4)}e^{-tA}A^{1/4}a \to 0$ for $[1/4] < \alpha < 1$, for $t^{\alpha-(1/4)}A^{\alpha-(1/4)}e^{-tA} \to 0$ *strongly* (this operator is uniformly bounded in $t$ by Lemma 2 and obviously tends to zero when applied to a vector $u$ in $D(A^{\alpha-(1/4)})$ which is dense in $\mathcal{H}_\sigma$). Therefore, the first term on the right of (11) can be made arbitrarily small by choosing $T$ small. The same is true with the second term if (20) is satisfied. Thus $k_0 = \max(K_{1/2,0}K_{3/4,0})$ can be made to satisfy (13) by a sufficiently small choice of $T$. This gives

THEOREM 1: *Let $a \in D[A^{1/4}]$ and let* (20) *be satisfied. Then there is a $T > 0$ such that a solution $u(t)$ of* (III) *exists for $0 \leqslant t \leqslant T$. This $u(t)$ is constructed by the successive approximation described above. $u(t)$ is continuous for $0 \leqslant t \leqslant T$ whereas $A^\alpha u(t)$ is continuous for $0 < t \leqslant T$* and $\| A^\alpha u(t) \| = o(t^{(1/4)-\alpha})$ *for* $t \to 0$ *for* $[1/4] < \alpha < 1$.

REMARK 1: (20) *is satisfied if* $\| f(t) \| = o(t^{-3/4})$ *for* $t \to 0$.

It is interesting to note that the successive approximation leads to a *global* solution if $\| A^{1/4}a \|$ is sufficiently small and if $Pf(t)$ is sufficiently weak. We have namely

THEOREM 2: *We can set $T = \infty$ in Theorem* 1 *if*

$$\| A^{1/4}a \| + \beta \sup_{0<t<\infty} t^{3/4} \| Pf(t) \| < 1/4\beta M, \quad \beta = B(1/4, 1/4). \tag{21}$$

The proof follows immediately by noting that (13) is satisfied by the choice $T = \infty$, as is seen from (10).

REMARK 2: *In the proof of Theorem* 2 *no use has been made of the energy principle, which can be written as*

$$\int_0^t \| A^{1/2}u(t) \|^2 \, dt \leqslant \| a \|^2 - \| u(t) \|^2 + \int_0^t \| A^{-1/2}Pf(t) \|^2 \, dt \,. \tag{22}$$

If the energy principle is taken into account, it is possible to prove the existence of a global solution under conditions weaker than (21), especially with respect to the behavior of $f(t)$ for $t \to \infty$. We shall not go into such a consideration here, however.

THEOREM 3: *In Theorem* 1, *$A^\alpha u(t)$ is uniformly Hölder continuous in any closed interval $[\varepsilon, T]$ with $\varepsilon > 0$ if $\alpha < 1$.*

Proof. It is easily seen that $u_0(t)$ is Hölder continuous in $[\varepsilon, T]$ with any exponent $\gamma < 1 - \alpha$ (this is a special case of what is proved below). Thus it suffices to show that the last term on the right of (III) is Hölder continuous when multiplied with $A^{\alpha}$. Let us denote by $v(t)$ this term. Then

$$A^{\alpha}v(t+h) - A^{\alpha}v(t) = \int_0^t (e^{-hA} - 1)A^{\alpha}e^{-(t-s)A}Fu(s)ds + \int_t^{t+h} A^{\alpha}e^{-(t+h-s)A}Fu(s)ds$$

and therefore

$$\| A^{\alpha}v(t+h) - A^{\alpha}v(t) \| \leqslant \| (e^{-hA} - 1)A^{-\gamma} \| \int_0^t \| A^{\gamma+\alpha}e^{-(t-s)A} \| \, \| Fu(s) \| ds + \int_t^{t+h} \| A^{\alpha}e^{-(t+h-s)A} \| \, \| Fu(s) \| ds$$

$$\leqslant h^{\gamma} \int_0^t \frac{1}{(t-s)^{\alpha+\gamma}} \cdot \frac{MK^2}{s^{3/4}} ds + \int_t^{t+h} \frac{1}{(t+h-s)^{\alpha}} \frac{MK^2}{s^{3/4}} ds$$

$$\leqslant MK^2 B(1-\alpha-\gamma, 1/4)h^{\gamma} + \frac{MK^2}{1-\alpha} t^{-3/4}h^{1-\alpha}$$

for $\alpha + \gamma < 1$, $\gamma > 0$. (Here we have made use of the estimate $\| (e^{-hA} - 1)A^{-\gamma} \| \leqslant h^{\gamma}$, which is easily proved). This shows that $A^{\alpha}v(t)$ is Hölder continuous in $[\varepsilon, T]$ with any exponent $\gamma < 1 - \alpha$.

## 2. Uniqueness of the solution

We shall now show that the solutions $u(t)$ of (III) obtained in Theorems 1 and 2 are unique within a certain class of functions.

DEFINITION 1: *We denote by* $\mathcal{S} = \mathcal{S}[0, T]$ *the set of all functions* $[0, T] \ni t \to u(t) \in \mathcal{H}_\sigma$ *with the following properties*: *i)* $u(t)$ *is continuous on* $[0, T]$. *ii)* $A^{1/2}u(t)$ *exists and is continuous for* $t \in (0, T]$ *and* $\| A^{1/2}u(t) \| = o(t^{-1/4})$, $t \to 0$, *and iii)* $A^{3/4}u(t)$ *is continuous on* $(0, T]$ *and* $\| A^{3/4}u \| = o(t^{-1/2})$ *for* $t \to 0$.

Note that *iii)* implies the existence and the continuity in $t$ of $A^\alpha u(t)$ for any $\alpha \leqslant 3/4$. If $u \in \mathcal{S}$, $Fu(t)$ exists and is continuous by Lemma 3, and $\| Fu(t) \| = o(t^{-3/4})$.

For a $u(t) \in \mathcal{S}$ the last integral on the right of (III) is defined, so that $u(t)$ can be a solution of (III). In fact, the solutions $u(t)$ of Theorems 1 and 2 belong to $\mathcal{S}$ (with $T = \infty$ in Theorem 2).

We now have the uniqueness theorem

THEOREM 4: *The solutions of Theorems* 1 *and* 2 *are unique* [2]) *within the class* $\mathcal{S}$ (*with* $T = \infty$ *in Theorem* 2).

Proof. Let $v(t) \in \mathcal{S}[0, T_1]$ be another solution of (III) with the same initial value $a$. First we show that $v(t) = u(t)$ for sufficiently small $t$.

Since both $u(t)$ and $v(t)$ satisfy (III) with the same $a$, we have

$$(23) \qquad w(t) \equiv v(t) - u(t) = \\ = \int_0^t e^{-(t-s)A}(Fv(s) - Fu(s))ds\,, \qquad 0 < t \leqslant T_0\,.$$

We can apply to (23) the same estimation as applied to (15) and (16). It follows without difficulty that

$$(24) \qquad \| A^{1/2}w(t) \| \leqslant \frac{(2\beta MK)^n K}{2t^{1/4}}\,, \qquad n = 1, 2, 3, \ldots$$

and similarly for $\| A^{3/4}w(t) \|$; here $K$ may be made arbitrarily

[2]) Actually the uniqueness can be proved within the class wider than $\mathcal{S}$ obtained by dropping the condition *iii)* in Definition 1.

small by choosing $T_0$ sufficiently small. Hence we have $A^{1/2}w(t) = 0$, $w(t) = 0$ on letting $n \to \infty$.

To show that $v(t) = u(t)$ for $0 \leqslant t \leqslant T' \equiv \min(T, T_1)$, it suffices to note that any solution $v(t) \in S$ of (III) is also a solution of (III) with the initial time $t_0 > 0$, that is,

$$v(t) = e^{-(t-t_0)A}v(t_0) + \int_{t_0}^{t} e^{-(t-s)A}Pf(s)ds + \int_{t_0}^{t} e^{-(t-s)A}Fv(s)ds \tag{25}$$

for $t_0 < t \leqslant T_1$. This would be obvious if (III) were equivalent to the differential equation (II). But this can be verified easily by direct calculation without reference to (II). Note that $Pf(t)$ and $Fv(t)$ are continuous for $t_0 \leqslant t \leqslant T$ by hypotheses.

Thus both $u(t)$ and $v(t)$ satisfy (25). If $u(t_0) = v(t_0)$, it follows as above that $u(t) = v(t)$ in an interval $t_0 \leqslant t \leqslant t_0 + h$ with an $h > 0$ (note that both $u(t)$ and $v(t)$ are of class $S\,[t_0, T']$). Furthermore, it is easily seen that $h$ has a positive lower bound when $t_0$ is changed.

## 3. Solution of the differential equation

The solution $u(t)$ of the integral equation (III) constructed in Theorem 1 or 2 may be regarded as a solution of the operator differential equation (II) in a generalized sense. But it need not be so in the strict sense.

DEFINITION 2: *By a strict solution of* (II) *we mean a function* $u(t)$ *such that* $u(t)$ *is continuous in* $[0, T]$ *and continuously differentiable in* $(0, T]$, $Au(t)$ *exists and is continuous in* $(0, T]$ *and* (II) *is satisfied in* $(0, T]$.

As is seen from Lemma 3, $Fu(t)$ exists and is continuous in $(0, T]$ for a strict solution $u(t)$.

It follows from a well known a fact in semigroup theory that a strict solution of (II) is a solution of the integral equation (III) if $Pf(t)$ and $Fu(t)$ are integrable at $t \to 0$. The converse is in general not true, but we have

THEOREM 5: *Let $Pf(t)$ be uniformly Hölder continuous in any closed subinterval $[\varepsilon, T]$, $\varepsilon > 0$. Then the solntion $u(t)$ of Theorem 1 is a strict solution of* (II). *It is unique as a strict solution belonging to the ctass* $\mathcal{S}\,[0, T]$.

Proof. (III) may be written

$$(27)\quad u(t) = e^{-tA}a + \int_0^t e^{-(t-s)A}g(s)ds\,, \qquad g(t) = Pf(t) + Fu(t)\,.$$

Here $g(t)$ is uniformly Hölder continuous in $[\varepsilon, T]$. For $Pf(t)$ this is an assumption and for $Fu(t)$ this follows from Lemma 3, since $A^\alpha u(t)$ is Hölder continuous for $\alpha = 1/2$ and $3/4$ by Theorem 3. Hence follows by a standard argument (see e. g. Kato [1]) that $u(t)$ is a strict solution of (II). The uniqueness follows from Theorem 4 since a strict solution of (II) of class $\mathcal{S}$ is also a solution of (III) by the remark above.

REMARK 3: *In this paper we shall not go into the question whether $u(t) = u(x, t)$ of Theorem 1 is a genuine solution of* (I) *in the classical sense. It will be shown elsewhere that this is actually the case if we make some additional assumptions on $a = a(x)$ and $f(t) = f(x, t)$. Here we note only that the existence of $Au(t)$ for the strict solution $u(t)$ of* (II) *implies that $u(t) \in H^2(D)$, that is, $u(x, t)$ is twice strongly differentiable in $x$, according to a result of Cattabriga* [1].

## 4. Two dimensional case

In this section we shall indicate briefly how the foregoing results can be improved in the case $m = 2$.

We note once for all that all the results proved above for $m = 3$ remain valid in the present case. This is due to that lemmas 1, 2, 3 are true for $m = 2$ as well as for $m = 3$.

Actually, however, the assumptions in Theorems 1 and 2 can be weakened considerably for $m = 2$. To show this we need a sharper estimate on $Fu$ than that given by Lemma 3, which is indeed optimal in a certain sense for $m = 3$ but not so for $m = 2$. In what follows we shall use the following lemma valid for $m = 2$.

LEMMA 3′: *Let* $u \in D[A^{1/2}]$. *Then* $Fu \in \mathcal{H}_\sigma$ *is defined, with*

$$\| A^{-1/4}Fu \| \leqslant M \| A^{1/2}u \| \; \| A^{1/4}u \| . \tag{3′}$$

If $u, v \in D[A^{1/2}]$, we have

$$\| A^{-1/4}(Fu - Fv) \| \leqslant M( \| A^{1/2}u \| \; \| A^{1/4}(u - v) \| + \\ + \| A^{1/2}(u - v) \| \; \| A^{1/4}v \| ) . \tag{4′}$$

The proof of Lemma 3′ will be given in Appendix.

With the use of Lemma 3′ in place of Lemma 3, we repeat the construction of the solution by successive approximation as in § 1. The only changes required are the following. (5) is to be replaced by

$$\| A^\alpha u_n(t) \| \leqslant K_{\alpha n} t^{-\alpha} , \quad 0 < t \leqslant T , \quad 0 \leqslant \alpha < 3/4 . \tag{5′}$$

The $(t - s)^{-\alpha}$ in the integrand of (7) should be replaced by $(t - s)^{-(1/4)-\alpha}$ ; this is due to the estimate

$$\| A^\alpha e^{-(t-s)A} Fu(s) \| = \| A^{\alpha+(1/4)} e^{-(t-s)A} \| \; \| A^{-1/4}Fu_n(s) \| \leqslant \\ \leqslant \frac{M \| A^{1/2}u_n(s) \| \; \| A^{1/4}u_n(s) \|}{(t - s)^{\alpha+(1/4)}}$$

resulting from Lemma 2 and (3′). $K_{3/4,n}$ and $K_{1/2,n}$ in (7) and (8) should be replaced by $K_{1/2,n}$ and $K_{1/4,n}$, respectively, and $B(1 - \alpha, 1/4)$ by $B(3/4 - \alpha, 1/4)$. (9) should be replaced by

$$\| A^\alpha u_0(t) \| \leqslant ( \| a \| + N'B(3/4 - \alpha, 1/4))t^{-\alpha} , \\ 0 < t \leqslant T , \quad 0 \leqslant \alpha < 3/4 , \tag{9′}$$

where $N' = \sup\limits_{0<t\leqslant T} t^{3/4} \| A^{-1/4}Pf(t) \|$. (11) should be replaced by

$$K_{\alpha 0} = \sup_{0<t\leqslant T} t^\alpha \| A^\alpha e^{-tA} a \| + N'B(3/4 - \alpha, 1/4) . \tag{10′}$$

Setting $k_n = \max(K_{14/,n}, K_{1/2,n})$ this time, we are again led to the recurrence inequality (12). If follows that (13) is sufficient for the convergence of the successive approximation.

In this way we arrive at the following theorem.

THEOREM 6: *Let $a \in \mathcal{H}_\sigma$ be arbitrary and let*

$$(20') \qquad \| A^{-1/4} Pf(t) \| = o(t^{-3/4}), \quad t \to 0 .$$

*Then there is a $T > 0$ such that a solution $u(t)$ of* (III) *exists for $0 \leqslant t \leqslant T$. This $u(t)$ is constructed by the new successive approximation. $u(t)$ is continuous for $0 \leqslant t \leqslant T$, $A^\alpha u(t)$ is continuous for $0 < t \leqslant T$ and*

$$(19') \quad \| A^\alpha u(t) \| \leqslant K_\alpha t^{-\alpha}, \quad 0 < t \leqslant T, \quad 0 \leqslant \alpha < 3/4 .$$

REMARK 4: (20′) *is satisfied if $\| Pf(t) \| = o(t^{-3/4})$ for $t \to 0$.*

REMARK 5: *Theorem 6 is incomplete in several respects. In the first place, the existence and continuity of $A^\alpha u(t)$ is directly proved only for $0 \leqslant \alpha < 3/4$. Actually this is true for any $\alpha < 1$, for the new solution $u(t)$ must coincide with that given by Theorem* 1 *because a uniqueness theorem analogous to Theorem* 4 *can be proved within a class $\mathcal{S}'$ which is defined by replacing $A^{3/4}\, u(t)$ in iii) of Definition* 1 *by $A^{1/4}\, u(t)$. Since $\mathcal{S} \subset \mathcal{S}'$, the $u(t)$ of Theorem* 1 *must coincide with the $u(t)$ of Theorem* 6 *in the common interval of existence. Nevertheless, the inequality* (19′) *cannot be extended to all $\alpha < 1$ by this argument. Of course* (19′) *is essential only in the immediate neighborhood of $t = 0$, for $A^\alpha u(t)$ is continuous for $t > 0$ for any $\alpha < 1$ as noted above.*

*In the second place, Theorem* 6 *is of a local nature. In the twodimensional case, however, a local solution is known to be a global one (see Sobolevskii* [1]). *Thus Theorem* 6 *is only of an intermediate nature, and we have the following stronger theorem.*

THEOREM 7: *Under the assumptions of Theorem* 6, *there is a global solution $u(t)$ of* (III) *such that $u(t)$ is continuous for $t \geqslant 0$, $A^\alpha u(t)$ exists and is continuous for $t > 0$ for any $\alpha < 1$ and* (19′) *holds near $t = 0$. This solution is unique within the class $\mathcal{S}'\,[0, \infty)$. If $Pf(t)$ is uniformly Hölder continuous in any finite closed subinterval of $(0, \infty)$, then $u(t)$ is a strict global solution of* (II).

## APPENDIX

Proof of Lemma 3

We have

$$\| Fu \| \leqslant \| (u, \operatorname{grad} u) \| \leqslant$$

$$\leqslant \left( \int_D | u |^2 \, | \operatorname{grad} u |^2 \, dx \right)^{1/2} \leqslant \| u \|_{L^6} \, \| \operatorname{grad} u \|_{L^3}$$

by the Hölder inequality. But $\| u \|_{L^6} \leqslant \text{const.} \| \operatorname{grad} u \| = \| A^{1/2} u \|$ by the Sobolev inequality and by the definition of $A$ as the selfadjoint operator associated with the sesquilinear form $(\operatorname{grad} u, \operatorname{grad} v)$. Therefore, (3) is proved if we show that

$$\text{(A1)} \qquad \| \operatorname{grad} u \|_{L^3} \leqslant \text{const.} \| A^{3/4} u \| \, .$$

(4) follows in a similar way taking into account that $Fu$ is of the second order in $u$.

We shall prove (A1) as an application of a theorem on interpolation spaces (see Lions [1]).

It has been shown by Cattabriga [1] that $D[A] \subset [H^2(D)]^3$ (the inclusion being algebraic and topological, as in the following inclusions). Hence the operator grad sends $D[A]$ into grad $[H^2(D)]^3 = [H^1(D)]^9 \subset [L^6(D)]^9$ (the exponent 9 refers to the tensor function with 9 components). On the other hand grad sends $D[A^{1/2}] = [\dot{H}^1(D)]^3 \subset [H^1(D)]^3$ into grad $[H^1(D)]^3 = [L^2(D)]^9$. By interpolation it follows that grad sends $D[A^{3/4}]$ into $[L^3(D)]^9$, for $1/6 + 1/2 = 2/3$. This is equivalent to (A1).

Proof of Lemma 3′.

For any $u \in \mathcal{H}_\sigma$ we have

$$\text{(A2)} \qquad \| A^{1/2} u \| = \| \operatorname{grad} u \| = \| B^{1/2} X u \| \, ,$$

where $X$ is the injection of $\mathcal{H}_\sigma$ into $\mathcal{H}$ and $B$ is the selfadjoint operator $-\Delta$ acting in $\mathcal{H} = [L^2(D)]^3$ with the boundary condition

$u_{|\partial D} = 0$. It follows from the Heinz inequality (see Kato [2]; this is again an interpolation theorem) that

$$\text{(A3)} \qquad \| B^{1/4}Xu \| \leqslant \| A^{1/4}u \| , \quad u \in D[A^{1/4}] \subset \mathcal{H}_\sigma .$$

The relation dual to (A3) has the form

$$\text{(A4)} \qquad \| A^{-1/4}Pv \| \leqslant \| B^{-1/4}v \| .$$

Now we have $\| A^{-1/4}Fu \| = \| A^{-1/4}P(u, \operatorname{grad})u \| \leqslant \| B^{-1/4} (Xu, \operatorname{grad})Xu \|$ by (A4). In view of (A2) and (A3), (3′) will be proved if we show that

$$\text{(A5)} \qquad \| B^{-1/4}(v, \operatorname{grad})v \| \leqslant \text{const.} \| B^{1/2}v \| \; \| B^{1/4}v \| .$$

Since $B$ is the direct sum of the operator $-\Delta$ (with the zero boundary condition) acting in the space $L^2(D)$ of scalar functions, it suffices to prove (A5) for such a scalar function $v$.

Now $B^{-1/4}$ is an integral operator with a kernel having a singularity of the form $| x - y |^{-3/2}$. Hence (note that $m = 2$)

$$\text{(A6)} \qquad \| B^{-1/4}(v, \operatorname{grad})v \|^2 \leqslant$$

$$\leqslant \text{const.} \iiint_{D^3} \frac{| v(y) | \; | \operatorname{grad} v(y) | \; | v(z) | \; | \operatorname{grad} v(z) |}{| x - y |^{3/2} \, | x - z |^{3/2}} \, dxdydz$$

$$\leqslant \text{const.} \iint_{D^2} \frac{| v(y) | \; | \operatorname{grad} v(y) | \; | v(z) | \; | \operatorname{grad} v(z) |}{| y - z |} \, dydz$$

$$\leqslant \text{const.} \left( \iint_{D^2} \frac{| v(y) |^2 \, | \operatorname{grad} v(z) |^2}{| y - z |} \, dydz \right)^{1/2} \cdot$$

$$\cdot \left( \iint_{D^2} \frac{| v(z) |^2 \, | \operatorname{grad} v(y) |^2}{| y - z |} \, dydz \right)^{1/2} .$$

We shall now show that

$$\text{(A7)} \qquad \int_D | v(x) |^2 \, | x - a |^{-1} \, dx \leqslant C \, \| B^{1/4}v \|^2$$

for any $a \in D$, with the constant $C$ independent of $a$. Then the first integral on the last member of (A6) is majorized by $C \| B^{1/4} v \|^2 \| B^{1/2} v \|^2$ because $\| \operatorname{grad} v \| = \| B^{1/2} v \|$, and the same is true with the second integral, leading to the desired result (A5).

To prove (A7), set $w = B^{1/4} v$ so that $v = B^{-1/4} w$ and

$$| v(x) | \leqslant \text{const.} \int_D \frac{| w(y) |}{| x - y |^{3/2}} dy ,$$

$$\text{(A8)} \qquad | v(x) |^2 \leqslant \text{const.} \left( \int_D \frac{| x - a |^{1/2}}{| y - a | \, | x - y |^{3/2}} dy \right) \cdot$$

$$\cdot \left( \int_D \frac{| y - a | \, | w(y) |^2}{| x - a |^{1/2} | x - y |^{3/2}} dy \right) .$$

Since the first integral on the right of (A8) is majorized by a constant independent of $a$, we have

$$\int_D \frac{| v(x) |^2}{| x - a |} dx \leqslant \text{const.} \int_D | w(y) |^2 dy \int_D \frac{| y - a |}{| x - a |^{3/2} | x - y |^{3/2}} dx$$

$$\leqslant \text{const.} \int_D | w(y) |^2 dy \leqslant \text{const.} \| w \|^2 = \text{const.} \| B^{1/4} v \|^2 .$$

This completes the proof of (3′). The proof of (4′) is similar and is based on the fact that $Fu$ is of the second order in $u$.

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