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ON THE NORMAL BUNDLE OF A MANIFOLD
Mark Mahowald

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In the Michigan lecture notes of 1940 [8] Whitney proved that any manifold in the cobordism class of $P_{2}$ cannot be embedded in $R^{4}$ with a normal field while non-orientable manifolds in the trivial cobordism class may or may not have a normal field. We will give a new proof of this result using some of the recent notions of differential topology. As one would expect, Whitney's theorem is a special case of a more general theorem and for the statement of this theorem we introduce some notation.

Let $M^{n}$ be a compact smooth $n$-manifold. Let $\bar{w}_{i}$ be the dual Stiefel Whitney classes of $M^{n}$.

Definition. Let $\sigma\left(M^{n}\right)=0$ if $\bar{w}_{1} \cdot \bar{w}_{n-1}=0$ and $\sigma\left(M^{n}\right)=1$ if $\bar{w}_{1} \cdot \bar{w}_{n-1} \neq 0$.

Clearly $\sigma\left(M^{n}\right)$ is just a Stiefel Whitney number [6]. Note also that by a result of Massey [5], $\sigma\left(M^{n}\right)=0$ unless $n=2^{j}$.

Theorem 1. For any embedding of $M^{n}$ in $R^{2 n}$ the (twisted) Euler class is congruent to $2 \sigma \bmod 4$.

This result is a slight sharpening of the theorem of Massey [4]; the proof is given in $\S 4$ after some preliminary results in $\S \S 2$ and 3.

Let $\chi$ be the Euler characteristic of $M^{2}$. In Whitney's theorem the role of $\sigma$ in Theorem 1 is played by $\chi$. It is not hard to verify that for 2 -dimension manifolds $\sigma=\chi \bmod 2$. In addition, for 2 -dimensional manifolds we can prove (section 6)

Theorem 2. For each $k$ and each value of $\sigma$ there is a manifold $M^{2}$ and an embedding of $M^{2}$ in $R^{4}$ with twisted Euler class $2 \sigma+4 k$.

We have not been able to show that a single manifold has an embedding for each $k$. Whitney exhibited two embeddings of the Klein bottle, one with a trivial Euler class and one with a non-trivial one.

We also have this weaker result (section 7) for other values of $n$.
THEOREM 3. For every even $n$ there exists a manifold $M^{n}$ and an embedding of $M^{n}$ in $R^{2 n}$ with no normal field.

It is known that if $n \neq 2^{j}$ and $n>3$, then every $n$-manifold embeds

[^0]in $R^{2 n-1}$. Hence this result asserts in addition that some $n$-manifolds have inequivalent embeddings in $R^{2 n}$.

It is interesting to note that the principal lemma yielding Theorem 1 also gives a new proof of the following slightly strengthened version of a result of Levine [2] and Mahowald [3].

Theorem 4. Suppose $M^{n}$ is orientable in addition. If there exists a class $d$ of dimension $(n-k-1) / 2$ such that $d \cup S q^{1} d \cup \bar{w}_{k} \neq 0$, then $M^{n}$ does not embed in $R^{n+k+1}$.

In [3] only the application of this result to give- $P_{n}$ does not embed in $R^{2 n-2}$ if $n=2^{j}+1$-is given.
2. Some lemmas. In this section we will derive some information about a particular secondary cohomology operation. Let $K$ be a semisimplicial complex and let $u \in C^{2 k}(K ; Z)$ such that $\delta u=2 v$. If $w$ is an integer $(\mathrm{a} \bmod j)$ cocycle we write $[w]\left([w]_{j}\right)$ for the cohomology class containing $w$. We have the following results, some of which are well known.
2.1. $S q^{1}[u]_{2}=[v]_{2}$ and $\beta_{2}[u]_{2}=[v]$ where $\beta_{j}$ is the Bockstein coboundary connected with the sequence $0 \rightarrow Z \rightarrow Z \rightarrow Z_{j} \rightarrow 0$.
2.2. If $\mathfrak{p}$ is the Pontriagin square operation $\mathfrak{p}: H^{2 k}\left(K ; Z_{2}\right) \rightarrow$ $H^{4 k}\left(K ; Z_{4}\right)$ then $\mathfrak{p}\left([u]_{2}\right)=\left[u \cup u+u \cup_{1} \delta u\right]_{4}$.
2.3. If $a \in H^{i}(X ; Z)$ then let $\bar{a}$ be its $\bmod 2$ restriction. Then

$$
\beta_{4} \mathfrak{p} \mathcal{p}\left([u]_{2}\right)=\left[v \cup_{1} v+u \cup v\right]
$$

and

$$
\overline{\beta_{4} \mathfrak{p}\left([u]_{2}\right)}=S q^{2 k} S q^{1}[u]_{2}+[u]_{2} \cup[v]_{2} .
$$

Proof. By the coboundary formula [7] which also holds in s.s.c. we have $\delta\left(u \cup u+u \cup_{1} \delta u\right)=4\left(v \cup_{1} v+u \cup v\right)$. This gives the first statement and the second now follows by definition.
2.4. If $u \cup u+\delta p$ is an integer cocycle then $u \cup_{1} v$ is a $\bmod 2$ cocycle and $S q^{1}\left(\left[u \cup_{1} v\right]\right)=S q^{2 k} S q^{1}[u]_{2}+[u]_{2} \cup[v]$.

Proof. By the coboundary formula we have

$$
\begin{aligned}
\delta\left(u \cup_{1} v\right) & =u \cup v-v \cup u+\delta u \cup_{1} v \\
& =2(u \cup v)+2\left(v \cup_{1} v\right)
\end{aligned}
$$

since $\delta(u \cup u)=0$ implies $u \cup v+v \cup u=0$. Now 2.1 completes the proof.
2.5. If $u \cup u=2 b+\delta c$, then $b+u \cup_{1} v$ is a $\bmod 2$ cocycle and

$$
S q^{1}\left[b+u \cup_{1} v\right]_{2}=S q^{2} S q^{1}[u]_{2}+[u]_{2} \cup S q^{1}[u]_{2} .
$$

Proof. Note that $\delta(u \cup u)=2(v \cup u+u \cup v)=2 \delta b$. Hence

$$
v \cup u+u \cup v=\delta b
$$

and the result follows as in 2.4.
In 2.4 we require that $u \cup u+\delta p$ is an integer cocycle, that is, we require that $\beta_{2}[u \cup u]=0$. The universal example for such a class $u$ is obtained by considering a fibering $p: X \rightarrow K\left(A_{2}, 2 k\right)$ with fiber $K(Z, 4 k)$ and $k$-invariant $2 \beta_{4} \mathfrak{p}(\alpha)$ where $\alpha$ is the fundamental class of $K\left(Z_{2}, 2 k\right)$. Let $\alpha^{\prime}=p^{*}(\alpha)$. Then by 2.4, $\alpha^{\prime} \cup_{1} S q^{1} \alpha^{\prime}$ is a cocycle and not a coboundary (since $\alpha^{\prime} \cup S q^{1} \alpha^{\prime} \neq 0$ ). Let $\varepsilon=\alpha^{\prime} \cup_{1} S q^{1} \alpha^{\prime}$.

Let $S A$ be the suspension of $A$ and let $s: H^{j}(A) \rightarrow H^{j+1}(S A)$ be the suspension isomorphism. There is a natural map $f: S K\left(Z_{2}, 2 k-1\right) \rightarrow X$ such that $f^{*}$ is an isomorphism in dimension $2 k$.
2.6. With the above notation there is a class $\beta \in p^{*} H^{*}\left(K\left(Z_{2}, 2 k\right) ; Z_{2}\right)$ (that is a primary operation) such that $f^{*}(\beta+\varepsilon)=s\left(\alpha \cup S q^{1} \alpha\right)$ where $s: H^{j}\left(K\left(Z_{2}, 2 k-1\right)\right) \simeq H^{j+1}\left(S K\left(Z_{2}, 2 k-1\right)\right)$. If $\beta$ satisfies the above equation then $\beta+S q^{2 k}$ will do so too.

Proof. As a vector space $H^{4 k}\left(S K ; Z_{2}\right)$ is generated by

$$
f^{*} p^{*} H^{4 k}\left(K\left(Z_{2}, 2 k\right)\right) \quad \text { and } \quad s\left(\alpha \cup S q^{1} \alpha\right)
$$

Hence $f^{*}(\varepsilon)=\lambda s\left(\alpha \cup S q^{1} \alpha\right)+\beta$ where $\lambda=0$ or 1 and $\beta$ satisfies the theorem. By direct computation we see that

$$
S q^{1} s\left(\alpha \cup S q^{1} \alpha\right)=S q^{2 k} S q^{1} s \alpha \notin f^{*} p^{*} S q^{1} H^{4 k}\left(K(Z, 2 k) ; Z_{2}\right)
$$

But by 2.4 $S q^{1} f^{*}(\varepsilon)=S q^{2 k} S q^{1} s \alpha$. Since

$$
S q^{1} \lambda s\left(\alpha \cup S q^{1} \alpha\right)+S q^{1} \beta=S q^{2 k} S q^{1} s \alpha
$$

if and only if $\lambda=1$ and $S q^{1} \beta=0$ we are finished.
In 2.5 we required that $u \cup u \equiv 0 \bmod 2$. The universal example for such a class $u$ is given by a fiber space $p_{1}: Y \rightarrow K\left(Z_{2}, 2 k\right)$ with $K\left(Z_{2}, 4 k-1\right)$ as the fiber and $S q^{2 k}$ as the $k$-invariant. Since there is no homotopy in dimension $4 k$ we have, letting $[u]_{2}=p_{1}^{*} \alpha$ :
2.7. The class $\mu=\left[b+u \cup_{1} v\right] \in H^{4 k}\left(Y ; Z_{2}\right)$ is not spherical and
hence is the universal example of a nontrivial natural cohomology operation which we write as $\mu$ too.

Let $g: S K\left(Z_{2}, 2 k-1\right) \rightarrow Y$ be the natural map inducing an isomorphism $g^{*}$ in dimension $2 k$. By an argument identical to the proof of 2.6 we have 2.8. In the above notation $g^{*}\left(\mu+\beta^{\prime}\right)=s\left(\alpha \cup S q^{1} \alpha\right)$ where $\beta^{\prime} \in p_{1}^{*} H^{*}\left(K\left(Z_{2}, 2 k\right), Z_{2}\right)$. If $\beta^{\prime}$ satisfies the above equation then $\beta^{\prime}+S q^{2 k}$ will do so too.
3. Let $\gamma_{n}$ be the universal $n$-plane bundle and let $I$ be the trivial line bundle. The base space of $I$ will usually be clear from the context. If $\nu$ is any $n$-plane bundle we let $T(\nu)$ be the Thom complex and $U \in H^{n}\left(T ; Z_{2}\right)$ be the Thom class. Recall that in $T, U \cup U$ is equal to $U \cup \bar{w}_{n}$ which is the restriction $\bmod 2$ of an integer class $U \cup \chi$ where $\chi$ is the twisted Euler class (of order 2 if $n$ is odd). Hence $\beta_{2} S q^{n} U=0$. By usual obstruction theory, letting $n=2 k$, we see that there exists a map $g: T\left(\gamma_{2 k}\right) \rightarrow X$ such that $g^{*}$ is an isomorphism in dimension $2 k$.

Lemma 3.1. In the above notation we can find a $\beta$ satisfying 2.6 such that $g^{*}(\beta+\varepsilon)=U \cup \bar{w}_{n-1} \cup \bar{w}_{1}, n=2 k$.

Proof. Consider the diagram:

where $i$ is the map induced by the natural inclusion of $\gamma_{n-1} \oplus I$ in $\gamma_{n}$, and $g^{\prime}$ is defined by requiring $g^{\prime *}(s \alpha)=U^{\prime}$, the Thom class of $T\left(\gamma_{n-1} \oplus I\right)$. Letting $\beta$ be the class of 2.6 , we have $g^{*} f^{*}(\beta+\varepsilon)=$ $s\left(U_{n-1} \cup U_{n-1} \cup \bar{w}_{1}\right)=U^{\prime} \cup \bar{w}_{n-1} \cup \bar{w}_{1}$ where $U_{n-1}$ is the Thom class of $T\left(\gamma_{n-1}\right)$. Hence $g^{*}(\beta+\varepsilon)=U \cup \bar{w}_{n-1} \cup \bar{w}_{1}+\alpha$ where $\alpha \in \operatorname{ker} i^{*}$. But ker $i^{*}$ is generated by $S q^{n} U=U \cup \bar{w}_{n}$. Therefore 2.6 completes the proof.

## 4. Proof of Theorem 1.

Notation. In the remaining sections it will be convenient to use a dot for the cup product.

Let $M^{n}$ be embedded in $R^{2 n}$ and let $T(\eta)$ be the Thom complex of the normal bundle. By [6] $M^{n}$ has a normal field if $n=1 \bmod 2$ (it even embeds in $R^{2 n-1}$ ) so we suppose $n$ is even. The group $H^{2 n}(T(n) ; Z)=Z$ and is generated by a class $b$ such that $2 j b=U \cdot \lambda$
( $\bar{w}_{n}$ is zero, hence $\lambda$ is zero $\bmod 2$ ). The cohomology operation $\mu$ is defined on $U$ and by 2.7 and 3.1 we have $\mu(U)=\left[U \cdot \bar{w}_{1} \cdot \bar{w}_{n-1}+j b\right]_{2}$. Since the top cohomology class of the Thom complex of a normal bundle to an embedding is spherical [6], $\mu(U)=0$. Therefore $j b=U \cdot \bar{w}_{1} \cdot \bar{w}_{n-1}$ $(\bmod 2)$.
5. Proof of Theorem 4. Suppose we have an embedding of the kind described. Let $E$ and $E_{0}$ be the normal disk and sphere bundle respectively. Consider the sequence

$$
T(\eta)=E / E_{0} \xrightarrow{\tau} S E_{0} \xrightarrow{S f} S K\left(Z_{2}, j\right) \xrightarrow{g} Y
$$

where $g$ is defined in the paragraph just before 2.8 and $S f$ is the suspension of the map $f: E_{0} \rightarrow K\left(Z_{2}, j\right)$ satisfying $f^{*}(\alpha)=a \cdot d$ where $a$ is any class such that $\tau^{*}(s a)=U$. The map $\tau$ is the natural map. ${ }^{1}$ Let $\lambda=f S f \tau$. Clearly $\lambda$ is a defining map for $\mu$. We have $g^{*} \mu=$ $s\left(\alpha \cdot S q^{1} \alpha\right)$ by 2.8. By direct computation $f^{*}\left(\alpha \cdot S q^{1} \alpha\right)=a \cdot \bar{w}_{k} \cdot d \cdot S q^{1} d+b$ where $b$ is in $\operatorname{ker} \tau^{*}$. Finally $\lambda^{*}(\mu)=U \cdot \bar{w}_{k} \cdot d \cdot S q^{1} d$ which is in the top cohomology class of $T(\eta)$ and hence must be zero. This contradiction proves the theorem.
6. Proof of Theorem 2. Let $f^{\prime}: S^{4} \rightarrow T\left(\gamma^{2}\right)$ be any map. By Theorem 36 [6] the $\operatorname{map} f^{\prime}$ is homotopic to a map $f: S^{4} \rightarrow T\left(\gamma^{2}\right)$ which is transverse regular on $G_{2, k}$ (the grassmann manifold of 2 planes in $R^{2+k}$ which, if $k>3$, is universal for classifying 2 plane bundles over 2-manifolds. Then $f^{-1}\left(G_{2, k}\right)=M^{2}$ is a sub-manifold of $S^{4}$ and $f / M^{2}: M^{2} \rightarrow G_{2, k}$ is the classifying map of the normal bundle to an embedding of $M^{2}$ in $R^{4} \subset S^{4}$. All that remains is to investigate the structure of $\pi_{4}\left(T\left(\gamma^{2}\right)\right.$ ).

Lemma 6.1. The first few homotopy groups of $T\left(\gamma^{2}\right)$ are

$$
\begin{array}{ccccc}
i & 1 & 2 & 3 & 4 \\
\pi_{i}\left(T\left(\gamma^{2}\right)\right) & 0 & Z_{2} & 0 & Z .
\end{array}
$$

The k-invariant with which the $Z$ group is added is $2 \beta_{4} \mathfrak{p}(\alpha)$ where $\alpha$ is the fundamental class of $K\left(Z_{2}, 2\right)$.

Remark. It is interesting to note that this portion of the Postnikov tower for $T\left(\gamma^{2}\right)$ is the same as the corresponding portion for $\widetilde{G}_{n}, n>4$ where $\widetilde{G}_{n}$ is the classifying space for oriented $n$-plane bundles. Indeed the $k$-invariants computed in [1] agree with these

[^1]given here. The class $w_{4} \in H^{4}\left(\widetilde{G}_{n} ; Z_{2}\right)$ is associated with $U \cdot w_{1}^{2}$ in $H^{4}\left(T\left(\gamma^{2}\right) ; Z_{2}\right)$ while $w_{2}^{2}$ and $U \cdot w_{2}$ are similarly associated.

Proof of the lemma. Since the Thom class of $T\left(\gamma^{2}\right)$ is also the fundamental class and since $S q^{1} U \neq 0$, the Hurewicz isomorphism theorem proves that $\pi_{2}\left(T\left(\gamma^{2}\right)\right)=Z_{2}$. Now $H^{3}\left(T\left(\gamma^{2}\right) ; J\right)=Z_{2}$ if $J=Z$ or $Z_{2 k}$ for any $k$ and zero for other $Z_{p}$. Hence any homotopy group in dimension 3 . must be attached with a nontrivial $k$-invtriant. But $H^{4}\left(K\left(Z_{2}, 2\right) ; Z_{2}\right)$. is generated by $S q^{2} \alpha$ and $S q^{2} U=U \cdot w_{2}$ in $H^{*}\left(T\left(\gamma^{2}\right)\right)$ and so $\pi^{2}\left(T\left(\gamma^{2}\right)\right)=0$.

Now $H^{4}\left(T\left(\gamma^{2}\right) ; Z\right)=Z$, generated by $U \cdot \chi$ where $\chi$ is the twisted Euler class. Hence the rank of $\pi_{4}\left(T\left(\gamma^{2}\right)\right)$ is 1 . Since the restriction $\bmod 2$ of $U \cdot \chi$ is $S q^{2} U$, the $Z$ component is attached with a nontrivial $k$-invariant. Finally $H^{5}\left(K\left(Z_{2}, 2\right) ; Z\right)=Z_{4}$ generated by $\beta_{4} \mathfrak{p}(\alpha)$ and $\overline{\left(\beta_{4} \mathfrak{p}(\alpha)\right)}=S q^{2} S q^{1} \alpha+\alpha S q^{1} \alpha$ (see 2.3) and since $S q^{2} S q^{1} U+U \cdot U w_{1}=$ $U \cdot w_{2} \cdot w_{1} \neq 0$ the $k$-invariant for the $Z$ component can not be $\beta_{4} \mathfrak{p}(\alpha)$. Therefore it must be $2 \beta_{4} \mathfrak{p}(\alpha)$.

Let $p: X \rightarrow K\left(Z_{2}, 2\right)$ be the fiber map having $2 \beta_{4} p(\alpha)$ as $k$-invariant and $K(Z, 4)$ as fiber. By 2.4 we see that $H^{4}\left(X ; Z_{2}\right)=Z_{2}+Z_{2}$ generated by a new class $\alpha^{\prime} \cup_{1} S q^{1} \alpha^{\prime}$ and by $S q^{2} \alpha^{\prime}$ where $\alpha^{\prime}=p^{*} \alpha$. Hence the natural map $f: T\left(\gamma^{2}\right) \rightarrow X$ induces an isomorphism $f^{*}: H^{i}(X) \rightarrow H^{i}\left(T\left(\gamma^{2}\right)\right)$ for all coefficient groups if $i \leqq 4$. To complete the proof of the lemma. we note that $f^{*}$ is also an isomorphism in dimension 5.

Now we can complete the proof of Theorem 2. Since the order of the $k$-invariant is $2, f^{\prime *}(U \cdot \chi)=2 j \gg$ where $K$ is a generator of $H^{4}\left(S^{4} ; Z\right)$ and $j=\left[f^{\prime}\right]$, the homotopy class of $f^{\prime}$ in $\pi_{4}$ under some identification with the integers. Let $\eta$ be the normal bundle for the embedding of $M^{2}$ in $R^{4}$ constructed above. Then the composite

$$
S^{4} \xrightarrow{\lambda_{1}} T(\eta) \xrightarrow{\lambda_{2}} T\left(\gamma^{2}\right)
$$

(where $\lambda_{2}$ is the natural map and $\lambda_{1}$ is obtained by collapsing the complement of a normal neighborhood of $M^{2}$ to a point) is just $f^{\prime}$. Since $\lambda_{1}^{*}$ is an isomorphism in dimension 4, the twisted Euler class of the embedding is $2 j$ times the twisted fundamental cohomology class.
7. Proof of Theorem 3. Let $T\left(\gamma^{n}\right)$ be the Thom complex of the universal $n$-plane bundle, $n$ even. Then $H_{n}\left(T\left(\gamma^{n}\right) ; Z\right)=Z_{2}$ generated by the cycle dual to the Thom class $U$. Since $T\left(\gamma_{n}\right)$ is $(n-1)$-connected, we have $\pi_{n}\left(T\left(\gamma^{n}\right)\right)=Z_{2}$. Therefore by Serre's theorem, ([6], page 109) rank $H^{2 n}\left(T\left(\gamma^{n}\right) ; Z\right)=\operatorname{rank} \pi_{2 n}\left(T\left(\gamma^{n}\right)\right)$. In particular there is a map $f: S^{2 n} \rightarrow T\left(\gamma^{n}\right)$ such that $f^{*}(U \cdot \chi) \neq 0$ where $\chi$ is the twisted Euler class. Now following the argument of $\S 6$ we construct the desired manifold.

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[^0]:    Received October 2, 1963.

[^1]:    ${ }^{1}$ If we realize $E / E_{0}$ by adding a cone over $E_{0}$ to $E$, then $E$ is naturally embedded in $E \cup_{c} E_{0}$ and $\tau: E \cup_{c} E_{0} \rightarrow E \cup_{c} E_{0} / E$.

