On the NP hardness of solving bilinear matrix inequalities and simultaneous stabilization with static output feedback

Proceedings of the
American Control Conference
Seattle, Washington • June 1995

TP4 - 3:30

On the $\mathcal{NP}$-hardness of solving Bilinear Matrix Inequalities and simultaneous stabilization with static output feedback

Onur Toker and *Hitay Özbay*
Department of Electrical Engineering
The Ohio State University
2015 Neil Avenue
Columbus, OH 43210

Abstract

In this paper, it is shown that the problem of checking the solvability of a bilinear matrix inequality (BMI), is $\mathcal{NP}$-hard. A matrix valued function, $F(X,Y)$, is called bilinear if it is linear with respect to each of its arguments, and an inequality of the form, $F(X,Y) > 0$ is called a bilinear matrix inequality. Recently, it was shown that, the static output feedback problem, fixed order controller problem, reduced order $\mathcal{H}^\infty$ controller design problem, and several other control problems can be formulated as BMIs [4, 3]. Based on the results of [4, 3], BMIs seem to be a potentially powerful tool for the formulation of some important control problems. But the main result of this paper shows that the problem of checking the solvability of BMIs is $\mathcal{NP}$-hard, and hence it is rather unlikely to find a polynomial time algorithm for solving general BMI problems. As an independent result, it is also shown that simultaneous stabilization with static output feedback is an $\mathcal{NP}$-hard problem, namely given n plants, the problem of checking the existence of a static gain matrix, K, which stabilizes all of the n plants, is $\mathcal{NP}$-hard.

1. Introduction

In this paper, it is shown that the problem of checking the solvability of a general bilinear matrix inequality is $\mathcal{NP}$-hard (See [2] for the definition of the complexity class $\mathcal{NP}$-hard and related issues). A matrix valued function $F(X,Y)$ is called bilinear if it is linear with respect to each of its arguments, and an inequality of the form $F(X,Y) > 0$, is called a bilinear matrix inequality. The main result of this paper is the $\mathcal{NP}$-hardness of the problem of checking the existence of X, Y such that $F(X,Y)$ is symmetric and positive definite, i.e. $F(X,Y) > 0$. Results of [4, 3] show that some important control problems, including the static output feedback problem, fixed order controller problem, and reduced order $\mathcal{H}^\infty$ controller design problems, can be formulated using BMIs, and hence BMIs seem to be a potentially powerful tool for solving these control problems. But the $\mathcal{NP}$-hardness of checking the solvability of BMIs imply that, it is rather unlikely to find a polynomial time solution procedure for general BMI problems. This result does not eliminate the possibility of conservative approaches or efficient solution procedures for some special type of BMI problems, but shows that the general BMI problem is $\mathcal{NP}$-hard. As an independent result, it is also shown that simultaneous stabilization with static output feedback is $\mathcal{NP}$-hard, namely given n plants, the problem of checking the existence of a static gain matrix, K, which stabilizes all of the n plants, is $\mathcal{NP}$-hard. This result does not give much information about the complexity of the static output feedback problem of a single plant, but shows that the this rationally decidable version of simultaneous stabilization problem [1, 5] is $\mathcal{NP}$-hard.

2. Solvability of BMIs

In this section, first it is shown that the following version of the Knapsack problem [2], is $\mathcal{NP}$-hard.

Lemma 1: Given n integers $a_1, ..., a_n \in \mathbb{Z}$, the problem of checking the existence of n real numbers $x_1, ..., x_n \in \mathbb{R}$, such that

$$1-\alpha < |x_k| < 1+\alpha, \quad k = 1, ..., n, \text{ and}$$

$$|a_1x_1 + ... + a_nx_n| < \beta,$$

where $\alpha = \frac{1}{5n(1+\sum_{k=1}^{n}|a_k|)}$, and $\beta = \frac{1}{5}$, is $\mathcal{NP}$-hard.

Proof : For a given vector, $a = [a_1 \dots a_n]^T \in \mathbb{Z}^n$, the above problem has a solution if there exists a vector, $x \in \{-1,+1\}^n$ satisfying $a^Tx = 0$.
Conversely, if the above problem has a solution, then

$$x = \overline{x} + \Delta x, \text{ where } \overline{x}_k \in \{-1,1\}, \|\Delta x\|_\infty < \alpha.$$

Note that, $a^T\overline{x}$ is an integer, $|a^T\overline{x}| \le |a^Tx| + |a^T\Delta x|$, and

$$|a^T\Delta x| \le \|a\|_2\|\Delta x\|_2 \le n\|a\|_\infty\|\Delta x\|_\infty \le \frac{1}{5},$$

$$|a^Tx| \le \frac{1}{5},$$

therefore $a^T\overline{x} = 0$.
This shows that the above problem has a solution iff the integer Knapsack problem [2] has a solution. By the $\mathcal{NP}$-hardness of the integer Knapsack problem, $\mathcal{NP}$-hardness of the above problem follows. □
The following lemma shows that, given an affine matrix subspace, the problem of checking the existence of a stable matrix in that affine subspace, is $\mathcal{NP}$-hard.

Lemma 2: Given $(n+1)$ real matrices, $M_0, ..., M_n$, the problem of checking the existence of n real numbers $x_1, ..., x_n$, such that $M_0 + x_1M_1 + ... + x_nM_n$ is stable, is $\mathcal{NP}$-hard.

Proof: For a given vector $a = [a_1 \dots a_n]^T \in \mathbb{Z}^n$, define $\alpha = \frac{1}{5n(1+\sum_{k=1}^{n}|a_k|)}$, and $\beta = \frac{1}{5}$,

$$A_{1,k}(x) = -(1+\alpha) - x_k, A_{2,k}(x) = -(1+\alpha) + x_k,$$

$$A_{3,k} = \begin{bmatrix} \frac{1}{2}(1-\alpha)^2 & -x_k \\ x_k & -2 \end{bmatrix}, \quad k = 1, ..., n,$$

and $A_r(x) = \mathrm{diag}(A_{r,1}(x), ..., A_{r,n}(x))$, $r = 1,2,3$.
Note that, the existence of a real vector, x, such that

$$M(x) = \mathrm{diag}(A_1(x), A_2(x), A_3(x), -\beta - a^Tx, -\beta + a^Tx),$$

is stable, is equivalent to the solvability of the problem of Lemma 1. Since $M(x)$ characterizes an affine matrix subspace, by the $\mathcal{NP}$-hardness result of

Lemma 1, it follows that the problem of checking the existence of a stable matrix in a given affine matrix subspace, is an $\mathcal{NP}$-hard problem. □

Remark 1: Given n real matrices, $M_1, ..., M_n$, the problem of checking the existence of n real numbers $x_1, ..., x_n$, such that $x_1 M_1 + ... + x_n M_n$ is stable, is also $\mathcal{NP}$-hard. Because, given $(n+1)$ matrices $M_0, ..., M_n$, define $N_k = \text{diag}(M_k, -\delta_{k,0})$, for $k = 0, ..., n$. Then, there exists n real numbers $x_1, ..., x_n$, such that $M_0 + x_1 M_1 + ... x_n M_n$ is stable, iff there exists $(n+1)$ real numbers $x_0, ..., x_n$ such that $x_0 N_0 + ... + x_n N_n$ is stable. By the $\mathcal{NP}$-hardness result of Lemma 2, it follows that, given a matrix subspace, the problem of checking the existence of a stable matrix in that subspace, is $\mathcal{NP}$-hard too.

Theorem 1: Given a matrix valued bilinear form $F(\cdot,\cdot)$, the problem of checking the existence of (real) matrices X, Y such that $F(X,Y) > 0$, is $\mathcal{NP}$-hard.

Proof: Given $(n+1)$ matrices $M_0, ..., M_n$, define $M(x) = M_0 + x_1 M_1 + ... + x_n M_n$, $N(x) = x_0 M_0 + ... + x_n M_n$, $S(P) = (P + P^T)/2$, and

$$F(x,P) = \begin{bmatrix} -N(x)^T S(P) - S(P) N(x) & 0 \\ 0 & x_0 S(P) \end{bmatrix}.$$

Then if there exists a real x and P such that $F(x,P) > 0$, then $x_0 \neq 0$, and $\frac{1}{x_0} N(x)$ is stable, therefore there exist n real numbers $x_1, ..., x_n$ such that $M_0 + x_1 M_1 + ... + x_n M_n$ is stable. Conversely, if there exist such n real numbers, then the BMI, $F(x,P) > 0$, is solvable. This shows that the problem of Lemma 2 polynomially reduces to the problem of solvability of BMIs, therefore checking the solvability of a BMI is an $\mathcal{NP}$-hard problem.

3. Simultaneous stabilization with static output feedback

In this section, it is shown that given n plants, the problem of checking the existence of a (real) static gain matrix, K, which stabilizes all of the n plants, is $\mathcal{NP}$-hard. Simultaneous stabilizability of n plants with static output feedback, is rationally decidable [1, 5], but Theorem 2 shows that this problem is $\mathcal{NP}$-hard, namely although the problem can be solved (possibly in doubly exponential amount of time) using only rational operations, it is rather unlikely to find a polynomial time solution procedure for that problem. This result does not give information about the computational complexity of the static output feedback problem of a single plant, but shows that the simultaneous version of the problem is $\mathcal{NP}$-hard.

Theorem 2: Given n plants with state space realizations, (A_k, B_k, C_k), $k = 1, ..., n$, the problem of checking the existence of a (real) static gain matrix, K, such that $A_k + B_k K C_k$ is stable for $k = 1, ..., n$, is $\mathcal{NP}$-hard.

Proof: For a given vector, $a = [a_1 \ ... \ a_n]^T \in \mathbb{Z}^n$, define $\alpha = \frac{1}{5n(1+\sum_{k=1}^n |a_k|)}$, and $\beta = \frac{1}{5}$,

$$A_{1,k}(x) = -(1+\alpha) - x_k, A_{2,k}(x) = -(1+\alpha) + x_k,$$

$$A_{3,k}(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{9}{2}x_k & -1-x_k & -1-x_k \end{bmatrix},$$

for $k = 1, ..., n$, and $A_4(x) = -\beta - a^T x$, $A_5(x) = -\beta + a^T x$. Note that $A_{3,k}$ is stable iff $0 < x_k < \frac{1}{2}$ or $x_k > 2$. Therefore, there exists a real vector, x, such that

$$A_{1,k}(x), A_{2,k}(x), A_{3,k}(\frac{3}{4(1-\alpha)}x + \frac{5}{4}), A_{4,k}(x), A_{5,k}(x),$$

are stable for $k = 1, ..., n$, iff the problem of Lemma 1 has a solution. Furthermore, the above $A(x)$ functions are of the form $A + BKC$ for some real A, B, C, and $K = x$. Therefore, given n plants, the problem of checking the existence of a static gain matrix which stabilizes all of the n plants, is $\mathcal{NP}$-hard.

4. Concluding remarks

In this paper, it was shown that the problem of checking the solvability of a general bilinear matrix inequality, is $\mathcal{NP}$-hard. Although BMIs can be used to formulate some important control problems [4, 3], and hence seems to be a potentially powerful tool, the main result of this paper shows that it is rather unlikely to find both efficient and nonconservative solution procedures for general BMI problems. As an independent result, it was also shown that, simultaneous stabilization with static output feedback problem is also $\mathcal{NP}$-hard. This shows that simultaneous version of the static output feedback problem is $\mathcal{NP}$-hard.

Acknowldegements

Authors would like to thank E. Yaz for pointing out important references on the subject. This work was supported in part by NSF under grant No. MSS-9203418, and by AFOSR under grant No. F49620-93-1-0288

References

[1] V. Blondel and M. Gevers, "Simultaneous Stabilizability of Three Linear Systems Is Rationally Undecidable,", *Mathematics of Control, Signals and Systems* (1993), pp. 135–145.

[2] M. Garey and D. Johnson, *Computers and Intractability : A Guide to the Theory of $\mathcal{NP}$-completeness*, W. H. Freeman, San Francisco, 1979.

[3] M. G. Safanov, K. C. Goh, and J. H. Ly, "Controller synthesis via bilinear matrix inequalities", *Proc. of American Control Conference*, 1994.

[4] T. Iwasaki, and R. E. Skelton, "A complete solution to the general $\mathcal{H}^\infty$ control problem: LMI existence conditions and state space formulas", *Proc. of American Control Conference*, 1993.

[5] A. Tarski, "A Decision method for elementary algebra and geometry," Berkeley note, 1951.