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ON THE NULLITY OF GRAPHS*

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Abstract. The nullity of a graph G, denoted by $\eta(G)$, is the multiplicity of the eigenvalue zero in its spectrum. It is known that $\eta(G) \leq n-2$ if G is a simple graph on n vertices and G is not isomorphic to nK_1. In this paper, we characterize the extremal graphs attaining the upper bound $n-2$ and the second upper bound $n-3$. The maximum nullity of simple graphs with n vertices and e edges, $M(n,e)$, is also discussed. We obtain an upper bound of $M(n,e)$, and characterize n and e for which the upper bound is achieved.

Key words. Graph eigenvalue, Nullity, Clique, Girth, Diameter.

AMS subject classifications. 05C50.

1. Introduction.

Let G be a simple graph. The vertex set of G is referred to as $V(G)$, the edge set of G as $E(G)$. If W is a nonempty subset of $V(G)$, then the subgraph of G obtained by taking the vertices in W and joining those pairs of vertices in W which are joined in G is called the subgraph of G induced by W and is denoted by $G[W]$. We write $G - \{v_1, \ldots, v_k\}$ for the graph obtained from G by removing the vertices $v_1, \ldots, v_k$ and all edges incident to them.

We define the *union* of G_1 and G_2, denoted by $G_1 \cup G_2$, to be the graph with vertex-set $V(G_1) \cup V(G_2)$ and edge-set $E(G_1) \cup E(G_2)$. If G_1 and G_2 are disjoint we denote their union by $G_1 + G_2$. The disjoint union of k copies of G is often written kG. As usual, the complete graph and cycle of order n are denoted by K_n and C_n, respectively. An isolated vertex is sometimes denoted by K_1.

Let $r \geq 2$ be an integer. A graph G is called *r-partite* if $V(G)$ admits a partition into r classes $X_1, X_2, \ldots, X_r$ such that every edge has its ends in different classes; vertices in the same partition must not be adjacent. Such a partition $(X_1, X_2, \ldots, X_r)$ is called a *r-partition* of the graph. A *complete r-partite graph* is a simple r-partite graph with partition $(X_1, X_2, \ldots, X_r)$ in which each vertex of X_i is joined to each vertex of $G - X_i$; if $|X_i|$=n_i, such a graph is denoted by $K_{n_1,n_2,\ldots,n_r}$. Instead of '2-partite' ('3-partite') one usually says *bipartite* (*tripartite*).

Let G and G' be two graphs. Then G and G' are *isomorphic* if there exists a bijection $\varphi : V(G) \to V(G')$ with $xy \in E(G) \iff \varphi(x)\varphi(y) \in E(G')$ for all $x, y \in V(G)$.

The adjacency matrix $A(G)$ of graph G of order n, having vertex-set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ symmetric matrix $[a_{ij}]$, such that a_{ij}=1 if v_i and v_j are adjacent and 0, otherwise. A graph is said to be singular (non-singular) if its

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adjacency matrix is a singular (non-singular) matrix. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A(G)$ are said to be the eigenvalues of the graph G, and to form the spectrum of this graph. The number of zero eigenvalues in the spectrum of the graph G is called its *nullity* and is denoted by $\eta(G)$. Let $r(A(G))$ be the rank of $A(G)$, clearly, $\eta(G) = n - r(A(G))$. The rank of a graph G is the rank of its adjacency matrix $A(G)$, denoted by $r(G)$. Then $\eta(G) = n - r(G)$. Each of $\eta(G)$ and $r(G)$ determines the other.

It is known that $0 \leq \eta(G) \leq n - 2$ if G is a simple graph on n vertices and G is not isomorphic to nK_1. In [3], L.Collatz and U.Sinogowitz first posed the problem of characterizing all graphs G with $\eta(G) > 0$. This question is of great interest in chemistry, because, as has been shown in [4], for a bipartite graph G (corresponding to an alternant hydrocarbon), if $\eta(G) > 0$, then it indicates the molecule which such a graph represents is unstable. The problem has not yet been solved completely; only for trees and bipartite graph some particular results are known (see [4] and [5]). In recent years, this problem has been investigated by many researchers([5], [7] and [8]).

A natural question is how to characterize the extremal matrices attaining the upper bound $n-2$ and the second upper bound $n-3$. The following theorems answer this question.

Theorem 1.1. *Suppose that G is a simple graph on n vertices and $n \geq 2$. Then $\eta(G) = n - 2$ if and only if G is isomorphic to $K_{n_1,n_2} + kK_1$, where $n_1 + n_2 + k = n$, $n_1, n_2 > 0$, and $k \geq 0$.*

Theorem 1.2. *Suppose that G is a simple graph on n vertices and $n \geq 3$. Then $\eta(G) = n-3$ if and only if G is isomorphic to $K_{n_1,n_2,n_3}+kK_1$, where $n_1+n_2+n_3+k = n$, $n_1, n_2, n_3 > 0$, and $k \geq 0$.*

We now introduce the definition of maximum nullity number, which is closely related to the upper bound of $\eta(G)$. Let $\Gamma(n, e)$ be the set of all simple graphs with n vertices and e edges. The maximum nullity number of simple graphs with n vertices and e edges, $M(n, e)$, is $\max\{\eta(A) : A \in \Gamma(n, e)\}$, where $n \geq 1$ and $0 \leq e \leq \binom{n}{2}$.

This paper is organized as follows. Theorem 1.1 and Theorem 1.2 are proved in section 3. In order to prove them, we obtain some inequalities concerning $\eta(G)$ in section 2. In section 4, we obtain an upper bound of $M(n, e)$, and characterize n and e for which the upper bound is achieved.

2. Some inequalities concerning $\eta(G)$. A *path* is a graph P of the form $V(P) = \{v_1, v_2, \ldots, v_k\}$ and $E(P) = \{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\}$, where the vertices $v_1, v_2, \ldots, v_k$ are all distinct. We say that P is a path from v_1 to v_k, or a (v_1,v_k)-path. It can be denoted by P_k. The number of edges of the path is its *length*. The *distance* $d(x, y)$ in G of two vertices x, y is the length of a shortest (x, y)-path in G; if no such path exists, we define $d(x, y)$ to be infinite. The greatest distance between any two vertices in G is the *diameter* of G, denoted by $diam(G)$.

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LEMMA 2.1. (see [6]) (i) *The adjacency matrix of the complete graph* K_n, $A(K_n)$, *has 2 distinct eigenvalues* $n-1$, -1 *with multiplicities* 1, $n-1$ *where* $n>1$.
(ii) *The eigenvalues of* C_n *are* $\lambda_r = 2cos\frac{2\pi r}{n}$, *where* $r = 0, \ldots, n-1$.
(iii) *The eigenvalues of* P_n *are* $\lambda_r = 2cos\frac{\pi r}{n+1}$, *where* $r = 1, 2, \ldots, n$.

LEMMA 2.2. (i) $r(K_n) = \begin{cases} 0 & \text{if } n = 1; \\ n & \text{if } n > 1. \end{cases}$
(ii) $r(C_n) = \begin{cases} n-2, & \text{if } n \equiv 0(mod4); \\ n, & \text{otherwise.} \end{cases}$
(iii) $r(P_n) = \begin{cases} n-1, & \text{if } n \text{ is odd}; \\ n, & \text{otherwise.} \end{cases}$

Proof. (i) and (iii) are direct consequences from Lemma 2.1.

(ii) We have $\lambda_r = 0$ if and only if $2cos\frac{2\pi r}{n} = 0$ if and only if $\frac{2\pi r}{n} = \pi/2$ or $3\pi/2$. Therefore $\lambda_r = 0$ if and only if $r = n/4$ or $r = 3n/4$. Hence (ii) holds. ∎

The following result is straightforward.

LEMMA 2.3. (i) *Let* H *be an induced subgraph of* G. *Then* $r(H) \leq r(G)$.
(ii) *Let* $G = G_1 + G_2$, *then* $r(G) = r(G_1) + r(G_2)$, *i.e.,* $\eta(G) = \eta(G_1) + \eta(G_2)$.

In the remainder of this section, we give some inequalities concerning $\eta(G)$.

PROPOSITION 2.4. *Let* G *be a simple graph on* n *vertices and* K_p *be a subgraph of* G, *where* $2 \leq p \leq n$. *Then* $\eta(G) \leq n-p$.

Proof. Immediate from Lemma 2.2(i) and Lemma 2.3(i). ∎

A *clique* of a simple graph G is a subset S of $V(G)$ such that $G[S]$ is complete. A clique S is maximum if G has no clique S' with $|S'| > |S|$. The number of vertices in a maximum clique of G is called the clique number of G and is denoted by $\omega(G)$. The following inequality is clear from the above result.

COROLLARY 2.5. *Let* G *be a simple graph on* n *vertices and* G *is not isomorphic to* nK_1. *Then* $\eta(G) + \omega(G) \leq n$.

PROPOSITION 2.6. *Let* G *be a simple graph on* n *vertices and let* C_p *be an induced subgraph of* G, *where* $3 \leq p \leq n$. *Then*
$\eta(G) \leq \begin{cases} n-p+2, & \text{if } p \equiv 0(mod4); \\ n-p, & \text{otherwise.} \end{cases}$

Proof. This follows from Lemma 2.2(ii) and Lemma 2.3(i). ∎

The length of the shortest cycle in a graph G is the *girth* of G, denoted by $gir(G)$. A relation between $gir(G)$ and $\eta(G)$ is given here.

COROLLARY 2.7. *If* G *is simple graph on* n *vertices and* G *has at least one cycle, then*
$\eta(G) \leq \begin{cases} n-gir(G)+2, & \text{if } gir(G) \equiv 0(mod4); \\ n-gir(G), & \text{otherwise.} \end{cases}$

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PROPOSITION 2.8. *Let G be a simple graph on n vertices and let P_k be an induced subgraph of G, where $2 \le k \le n$. Then*
$\eta(G) \le \begin{cases} n-k+1, & \textit{if } k \textit{ is odd;} \\ n-k, & \textit{otherwise.} \end{cases}$

Proof. This is a direct consequence of Lemma 2.2(iii) and Lemma 2.3(i). □

COROLLARY 2.9. *Suppose x and y are two vertices in G and there exists an (x,y)-path in G. Then*
$\eta(G) \le \begin{cases} n-d(x,y), & \textit{if } d(x,y) \textit{ is even;} \\ n-d(x,y)-1, & \textit{otherwise.} \end{cases}$

Proof. Let P_k be the shortest path between x and y. Suppose $v_1, v_2, \ldots, v_k$ are the vertices of P_k. Then $G[v_1, v_2, \ldots, v_k]$ is P_k. From Proposition 2.8, we have
$\eta(G) \le \begin{cases} n-d(x,y), & \text{if } d(x,y) \text{ is even;} \\ n-d(x,y)-1, & \text{otherwise.} \end{cases}$ □

COROLLARY 2.10. *Suppose G is simple connected graph on n vertices. Then*
$\eta(G) \le \begin{cases} n-diam(G), & \textit{if } diam(G) \textit{ is even;} \\ n-diam(G)-1, & \textit{otherwise.} \end{cases}$

3. Extremal matrices and graphs. For any vertex $x \in V(G)$, define $\Gamma(x) = \{v : v \in V(G) \text{ and } v \text{ is adjacent to } x\}$. We first give the following lemma.

LEMMA 3.1. *Suppose that G is a simple graph on n vertices and G has no isolated vertex. Let x be an arbitrary vertex in G. Let $Y = \Gamma(x)$ and $X = V(G) - Y$. If $r(G) \le 3$, then*
(i) No two vertices in X are adjacent.
(ii) Each vertex from X and each vertex from Y are adjacent.

Proof. (i) Suppose $x_1 \in X$, $x_2 \in X$, and x_1 and x_2 are adjacent. Since $x_1 \in X$, x_1 and x are not adjacent. Similarly we have x_2 and x are not adjacent. Since G has no isolated vertex, x is not an isolated vertex. Then Y is not an empty set. Select any vertex y in Y. Then $G[x_1, x_2, y]$ is isomorphic to $K_2 + K_1$, $K_{1,2}$ or K_3.

If $G[x_1, x_2, y]$ is isomorphic to K_2+K_1, then $G[x, x_1, x_2, y]$ is isomorphic to P_2+P_2. Since $r(P_2 + P_2) = r(P_2) + r(P_2) = 2 + 2 = 4$ by Lemma 2.3, we have $r(G) \ge 4$, a contradiction.

If $G[x_1, x_2, y]$ is isomorphic to $K_{1,2}$, then $G[x, x_1, x_2, y]$ is isomorphic to P_4. Therefore $r(G) \ge r(P_4) = 4$, a contradiction.

If $G[x_1, x_2, y]$ is isomorphic to K_3, then using the fact that neither x_1 nor x_2 is adjacent to x, we can verify that $r(G[x, x_1, x_2, y]) = 4$, a contradiction.

Therefore no two vertices in X are adjacent.

(ii) Suppose not, then there exist $x_1 \in X$ and $y_1 \in Y$ such that x_1 and y_1 are not adjacent. Since x and y_1 are adjacent, we have x and x_1 are distinct. Due to the fact that G has no isolated vertex, we can choose a vertex z in G which is adjacent to x_1. By (i) we see $z \in Y$. Then x and z are adjacent.

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If y_1 and z are not adjacent, then $G[x, x_1, y_1, z]$ is isomorphic to P_4. Hence $r(G[x, x_1, y_1, z]) > 3$, a contradiction.

If y_1 and z are adjacent, then using the fact that neither y_1 nor x is adjacent to x_1, we can verify that $r(G[x, x_1, y_1, z]) = 4$, a contradiction. Thus each vertex from X and each vertex from Y are adjacent. □

In order to prove Theorem 1.1, we prove the following lemma.

Lemma 3.2. *Suppose that G is a simple graph on n vertices ($n \geq 2$) and G has no isolated vertex. Then $\eta(G) = n - 2$ if and only if G is isomorphic to a complete bipartite graph K_{n_1,n_2}, where $n_1 + n_2 = n$, $n_1, n_2 > 0$.*

Proof. The sufficiency is clear.

To prove the necessity, choose an arbitrary vertex x in G. Let $Y = \Gamma(x)$ and $X = V(G) - Y$. Since G has no isolated vertex, x is not an isolated vertex. Then Y is not an empty set. Since $x \in X$, X is not empty.

We now prove any two vertices in Y are not adjacent. Suppose that there exist $y_1 \in Y$ and $y_2 \in Y$ such that y_1 and y_2 are adjacent. Then $G[x, y_1, y_2]$ is a triangle. By Proposition 2.4, we have $\eta(G) \leq n - 3$, a contradiction.

From Lemma 3.1, we know that
(i) any two vertices in X are not adjacent, and
(ii) any vertex from X and any vertex from Y are adjacent. Hence G is isomorphic to a complete bipartite graph. □

Theorem 1.1 is immediate from the above lemma.

Two matrices A_1 and A_2 that are related by $B = P^{-1}AP$ where P is a permutation matrix, are said to be *permutation similar*. Graphs G_1 and G_2 are isomorphic if and only if $A(G_1)$ and $A(G_2)$ are permutation similar.

We denote by $J_{p,q}$ the $p \times q$ matrix of all 1's. Sometimes we simply use J to denote an all 1's matrix of appropriate or undetermined size. Similar conventions are used for zeros matrices with O replacing J. Let A_1 and A_2 be two matrices. Define $A_1 \oplus A_2 = \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix}$ and $A_1 \underline{\oplus} A_2 = \begin{bmatrix} A_1 & J \\ J & A_2 \end{bmatrix}$.

Then Theorem 1.1 can be written in the following equivalent form.

Theorem 3.3. *Suppose that G is a simple graph on n vertices and $n \geq 2$. Then $\eta(G) = n - 2$ if and only if $A(G)$ is permutation similar to matrix $O_{n_1,n_1} \underline{\oplus} O_{n_2,n_2} \oplus O_{k,k}$, where $n_1 + n_2 + k = n$, $n_1, n_2 > 0$, and $k \geq 0$.*

Some lemmas are given before we prove Theorem 1.2.

Lemma 3.4. *Let A be a symmetric $n \times n$ matrix and let the rank of A be k. Then there exists a nonsingular principal minor of order k.*

Lemma 3.5. *Suppose that G is a simple graph on n vertices ($n \geq 3$) and G has no isolated vertex. Then $\eta(G) = n - 3$ if and only if G is isomorphic to a complete tripartite graph K_{n_1,n_2,n_3}, where $n_1, n_2, n_3 > 0$.*

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Proof. If G is isomorphic to a complete tripartite graph, then $A(G)$ is permutation similar to $O\underline{\oplus}O\underline{\oplus}O$. Thus we can verify that $r(G) = 3$, i.e., $\eta(G) = n - 3$. The sufficiency follows.

To prove the necessity, choose an arbitrary vertex x in G. Let $Y = \Gamma(x)$ and $X = V(G) - Y$. Since G has no isolated vertex, x is not an isolated vertex. Then Y is not an empty set. Since $x \in X$, X is not empty.

By Lemma 3.1, we have the following results.

Claim 3.6. *Any two vertices in X are not adjacent.*

Claim 3.7. *Any vertex from X and any vertex from Y are adjacent.*

We now consider $G - X$, and prove

Claim 3.8. $r(G - X) \leq 2$.

Proof. Suppose $r(G - X) > 2$. Due to the fact that $r(G - X) \leq r(G) = 3$, we see $r(G - X) = 3$. By Lemma 3.4, there exists an induced subgraph H of $G - X$ such that H is order 3 and $r(H) = 3$. Then H is a triangle. Since x is adjacent to each vertex of H, K_4 is a subgraph of G. Therefore $\eta(G) \leq n - 4$, a contradiction. □

Furthermore, we can show

Claim 3.9. $r(G - X) = 2$.

Proof. Suppose $r(G - X) < 2$, then $r(G - X) = 0$. Hence $G - X = O$. Therefore $r(G) = 2$, which contradicts $\eta(G) = n - 3$. □

By Theorem 1.1, $G - X$ is isomorphic to $K_{n_1,n_2} + kK_1$, where $n_1, n_2 > 0$, and $k \geq 0$.

If $k > 0$, then $A(G)$ is permutation similar to

$$\begin{bmatrix} O & J & J & J \\ J & O & J & O \\ J & J & O & O \\ J & O & O & O \end{bmatrix}.$$

Then $r(G) = 4$, a contradiction. Thus $k = 0$. So $G - X$ is isomorphic to K_{n_1,n_2}.

By Claim 3.6 and 3.7, we see G is isomorphic to a complete tripartite graph K_{n_1,n_2,n_3}, where $n_1, n_2, n_3 > 0$. □

Theorem 1.2 is immediate from the above lemma. Theorem 1.2 also has the following equivalent form.

Theorem 3.10. *Suppose that G is a simple graph on n vertices and $n \geq 3$. Then $\eta(G) = n - 3$ if and only if $A(G)$ is permutation similar to matrix*

$$O_{n_1,n_1}\underline{\oplus}O_{n_2,n_2}\underline{\oplus}O_{n_3,n_3} \oplus O_{k,k},$$

where $n_1 + n_2 + n_3 + k = n$, $n_1, n_2, n_3 > 0$, and $k \geq 0$.

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4. Maximum nullity number of graphs. In the first section, we define

$$M(n,e) = \max\{\eta(A) : A \in \Gamma(n,e)\}$$

where $\Gamma(n,e)$ is the set of all simple graphs with n vertices and e edges. In this section an upper bound of $M(n,e)$ is given. Let $g(m) = max\{k : k \mid m \text{ and } k \leq \sqrt{m}\}$, where m is a positive integer, e.g., $g(1) = 1, g(2) = 1, g(4) = 2$.

THEOREM 4.1. *The following results hold:*
(i) $M(n,0) = n$. $M(n, \binom{n}{2}) = 0$.
(ii) $M(n,1) = n-2$ *for* $n \geq 2$.
(iii) $M(n, \binom{n}{2} - 1) = 1$ *for* $n > 2$.
(iv) $M(n,e) \leq n-2$ *for* $e > 0$.
(v) $M(n,e) = n-2$ *if* $e > 0$ *and* $g(e) + e/g(e) \leq n$.
(vi) $M(n,e) \leq n-3$ *if* $e > 0$ *and* $g(e) + e/g(e) > n$.

Proof. (i) and (ii) are immediate from the definition.

(iii) Suppose $G \in \Gamma(n, \binom{n}{2} - 1)$. Then G is isomorphic to K_n with one edge deleted. Thus there exist two identical rows (columns) in $A(G)$. Therefore $A(G)$ is singular and $\eta(G) \geq 1$.

Since G contains K_{n-1}, by Proposition 2.4, we have $\eta(G) \leq 1$. Hence $\eta(G) = 1$. Therefore $M(n, \binom{n}{2} - 1) = 1$.

(iv) From the fact that $\eta(G) \leq n-2$ if G is a simple graph on n vertices and G is not isomorphic to nK_1, we see that $M(n,e) \leq n-2$ for $e > 0$.

(v) Let $n_1 = g(e)$, $n_2 = e/g(e)$ and $k = n - n_1 - n_2$. Then $G = K_{n_1,n_2} + kK_1 \in \Gamma(n,e)$ and $\eta(G) = n-2$. Hence $M(n,e) = n-2$.

(vi) Suppose $M(n,e) > n-3$. Since $M(n,e) \leq n-2$, we have $M(n,e) = n-2$. Then there exists $G \in \Gamma(n,e)$ such that $\eta(G) = n-2$. Hence $G = K_{n_1,n_2} + kK_1$. Therefore $n_1 \times n_2 = e$ and $n_1 + n_2 + k = n$. Without loss of generality, we may assume $n_1 \leq n_2$. Then $n_1 \leq \sqrt{e}$. Since $n_1 \mid n$, $n_1 \leq g(e)$.

Since $n_1 \leq \sqrt{e}$ and $g(e) \leq \sqrt{e}$, $g(e)n_1 \leq e$. Then $1 - \frac{e}{g(e)n_1} \leq 0$.

Since

$$g(e) + e/g(e) - n_1 - n_2 = g(e) - n_1 + e/g(e) - n_2 = g(e) - n_1 + e/g(e) - e/n_1$$

$$= g(e) - n_1 + e\frac{n_1 - g(e)}{g(e)n_1} = (g(e) - n_1)(1 - \frac{e}{g(e)n_1}) \leq 0,$$

$g(e) + e/g(e) \leq n_1 + n_2 \leq n$, which contradicts to $g(e) + e/g(e) > n$. $\square$

The following immediate corollary gives an upper bound for $M(n,e)$ and characterizes when the upper bound is achieved.

COROLLARY 4.2. *Suppose* $e > 0$. *Then* $M(n,e) \leq n-2$ *and the equality holds if and only if* $g(e) + e/g(e) \leq n$.

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Here we give a necessary condition for $M(n,e) = n-3$.

THEOREM 4.3. *If* $M(n,e) = n-3$, *then* $e \le n^2/3$.

Proof. Due to the fact that $M(n,e) = n-3$, there exists $G \in \Gamma(n,e)$ such that $\eta(G) = n-3$. Hence $G = K_{n_1,n_2,n_3} + kK_1$. Therefore $n_1+n_2+n_3 \le n$ and $n_1n_2 + n_2n_3 + n_1n_3 = e$.

Since

$$(n_1+n_2+n_3)^2 = n_1^2+n_2^2+n_3^2+2(n_1n_2+n_2n_3+n_1n_3)$$

$$\ge n_1n_2+n_2n_3+n_1n_3+2(n_1n_2+n_2n_3+n_1n_3) = 3e,$$

then $n^2 \ge 3e$, i.e., $e \le n^2/3$. □

The following corollary is immediate.

COROLLARY 4.4. *If* $n^2/3 < e \le \binom{n}{2}$, *then* $M(n,e) \le n-4$.

Finally we give a table for the exact values of $M(n,e)$, where $1 \le n \le 5$.

n	$e=0$	1	2	3	4	5	6	7	8	9	10
1	1	-	-	-	-	-	-	-	-	-	-
2	2	0	-	-	-	-	-	-	-	-	-
3	3	1	1	0	-	-	-	-	-	-	-
4	4	2	2	2	2	1	0	-	-	-	-
5	5	3	3	3	3	2	3	2	2	1	0

$M(5,5) = 2$ is obtained by Theorem 4.1(vi) and the fact that $\eta(K_{1,1,2}+K_1) = 2$. $M(5,7) = 2$ is from Theorem 4.1(vi) and the fact that $\eta(K_{1,1,3}) = 2$, and $M(5,8) = 2$ is from Theorem 4.1(vi) and the fact that $\eta(K_{1,2,2}) = 2$.

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