every prime p and let P_1, P_2 be polynomials with $(\deg P_1, \deg P_2) = 1$. If $p_1, p_2, \ldots, p_k$ are the first k primes, and if $n < p_k$, then there is no proper extension of $Q(\zeta_{p_1}, \zeta_{p_2}, \ldots, \zeta_{p_k})$ contained in K of degree $\leqslant n$. So by choosing n bigger than the degrees of P_1 and P_2 and k large enough so that $k > n$ and the coefficients of the P_i and a_1 are in $Q(\zeta_{p_1}, \ldots, \zeta_{p_k})$, we are assured that the sets of the form $\{a_i\}$ are contained in $Q(\zeta_{p_1}, \ldots, \zeta_{p_k})$ and are hence finite. Nevertheless, if $P_1(X) = X^n$ and $P_2(Y) = Y^m$, then taking p to be any prime relatively prime to m and n, we have $P_1(E_p) = E_p = P_2(E)$ where $E_p = \{\zeta_p^k | \ k = 1, 2, \ldots, p\}$. So there are infinitely many sets of the second type. The same phenomenon occurs for P_1, P_2 of the same degree when $K = Q$ ([6]).

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On the number of Abelian groups of a given order

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1. Introduction. Let $A(x)$ denote the number of essentially distinct Abelian groups of order not exceeding x. Then

$$A(x) = A_1 x + A_2 x^{1/2} + A_3 x^{1/3} + \Delta(x)$$

where

$$A_r = \prod_{\substack{\nu=1 \\ \nu \neq r}}^{\infty} \zeta\left(\frac{\nu}{r}\right) \quad (r = 1, 2, 3)$$

and

$$\Delta(x) \ll x^{\theta} \log^{\theta'} x.$$

Results of the above type with the pairs

$$(\theta, \theta') = (\tfrac{1}{2}, 0), (\tfrac{1}{3}, 2), (\tfrac{3}{10}, \tfrac{9}{10}), (\tfrac{20}{69}, \tfrac{21}{23}), (\tfrac{2}{7}, \tfrac{6}{7}), (\tfrac{34}{123}, 0), (\tfrac{7}{27}, 2)$$

were proved by P. Erdös and G. Szekeres [1], D. G. Kendall and R. A. Rankin [2], H. E. Richert [3], W. Schwarz [4], and P. G. Schmidt [5], [6]. As an application of the theory of two dimensional exponent pairs I have developed elsewhere [9], I here show that

$$\Delta(x) \ll x^{105/407} \log^2 x. \tag{1}$$

Here the exponent $\frac{105}{407} = .257\ldots < \frac{7}{27} = .259\ldots$

Actually the method yields exponents smaller than $\frac{105}{407}$, but I shall avoid the computations that will be necessary to obtain the best possible exponent in this way.

2. Lemmas.

LEMMA 1 (Lemma of partial summation). *Let $g(m, n)$ denote any numbers, real or complex, such that, if*

$$G(m, n) = \sum_{\substack{1 \leqslant \mu \leqslant m \\ 1 \leqslant \nu \leqslant n \\ (\mu, \nu) \in D}} g(\mu, \nu)$$

then $|G(m,n)| \leqslant G$ $(1 \leqslant m \leqslant M, 1 \leqslant n \leqslant N)$ *for any arbitrary region* D *contained in the rectangle* $1 \leqslant m \leqslant M$, $1 \leqslant n \leqslant N$. *Let* $h(m,n)$ *denote real numbers* $0 \leqslant h(m,n) \leqslant H$ *such that the three expressions*

$$h(m,n)-h(m+1,n); \quad h(m,n)-h(m,n+1);$$
$$h(m,n)-h(m+1,n)-h(m,n+1)+h(m+1,n+1)$$

keep a fixed sign for all values of m, n *considered. Then*

$$\Big|\sum_{(m,n)\epsilon D} g(m,n)h(m,n)\Big| \leqslant 5GH.$$

LEMMA 2. *Let* M *and* N *be positive integers,* u_m $(\geqslant 0)$ *and* v_n (>0) $(1 \leqslant m \leqslant M, 1 \leqslant n \leqslant N)$ *denote constants. Let* $A_m > 0$, $B_n > 0$. *Then there exists a* q *with the following properties* (Q_1 *and* Q_2 *are given non-negative numbers*):

$$Q_1 \leqslant q \leqslant Q_2$$

and

$$\sum_{m=1}^{M} A_m q^{u_m} + \sum_{n=1}^{N} B_n q^{-v_n}$$
$$\ll \sum_{m=1}^{M}\sum_{n=1}^{N} (A_m^{v_n} B_n^{u_m})^{1/(u_m+v_n)} + \sum_{m=1}^{M} A_m Q_1^{u_m} + \sum_{n=1}^{N} B_n Q_2^{-v_n}.$$

LEMMA 3. *For arbitrary* $q>0$ *and for any real function* g,

$$\sum_{(m,n)\epsilon D} \psi\big(g(m,n)\big) \ll \frac{|D|}{q} + \sum_{\nu=1}^{\infty}\Big|\sum_{(m,n)\epsilon D} e\big(-\nu g(m,n)\big)\Big| \operatorname{Min}\Big(\frac{1}{\nu}, \frac{q}{\nu^2}\Big)$$

where $e(u) = e^{2\pi i u}$, $\psi(u) = u-[u]-\frac{1}{2}$, $[u]$ *being the integral part of* u *and* $|D|$ *is the area of the region* D.

LEMMA 4. *Let* $f(x)$ *be real with continuous derivatives upto third order in* $(a,b]$. *Let* $0 < \lambda_2 \ll -f''(x) \ll \lambda_2$ *and* $f'''(x) \ll \lambda_3$ *throughout* $(a,b]$. *Let* x_ν *be defined by the equation* $f'(x_\nu) = \nu$ $(\alpha < \nu \leqslant \beta)$ *where* $\alpha = f'(b)$ *and* $\beta = f'(a)$. *Then*

$$\sum_{a<n\leqslant b} e\big(f(n)\big) = e(-\tfrac{1}{8}) \sum_{\alpha<\nu\leqslant\beta} |f''(x_\nu)|^{-1/2} e\big(f(x_\nu)-\nu x_\nu\big) +$$
$$+O\big((b-a)\lambda_3^{1/3}\big)+O(\lambda_2^{-1/2})+O\big(\log\{2+(b-a)\lambda_2\}\big).$$

Lemma 1 above is Lemma 1 of [8] with $p=2$; Lemma 2 above is Lemma 3 of [8]; Lemma 3 above is Lemma 8 of [8] with $s=1$; and Lemma 4 above is Lemma 3 of [7].

3. The theory of two dimensional exponent pairs.

DEFINITION 1. The real function $g(x,y)$ is said to be an *approximation of degree* r to the real function $f(x,y)$ in a region D of the Euclidean plane if f and g possess partial derivatives upto r orders in D and

$$|f_{x^py^q} - g_{x^py^q}| < cg_{x^py^q}$$

for all (x,y) in D and $1 \leqslant p+q \leqslant r$, where c denotes a sufficiently small constant such that $0 < c < \frac{1}{2}$. We write then $f \frac{r}{D} g$.

DEFINITION 2. We shall say that the pair (l_0, l_1) where l_0 and l_1 are absolute constants such that

$$0 \leqslant l_0, l_1 - l_0 \leqslant \tfrac{1}{6}$$

is a *two dimensional exponent pair*, if to every set of real numbers s, t such that $st \neq 0$,

$$(\mu+\mu_1)s+(\mu+\mu_2)t+\mu+\mu'+1 \neq 0$$

where μ, μ' are any non-negative integers and μ_1, μ_2 are either zero or unity, there exists an integer r $(\geqslant 6)$ depending only on s, t such that the inequality

$$\sum_{(m,n)\epsilon D} e\big(f(m,n)\big) \ll (zw)^{l_0}(ab)^{1-l_1}$$

holds with respect to s, t and u whenever the following conditions are satisfied. D is a region contained in the rectangle $a < m \leqslant ua$, $b < n \leqslant ub$;

$$u > 1, \quad z = |vs|a^{-s-1}b^{-t} \gg 1, \quad w = |vt|a^{-s}b^{-t-1} \gg 1,$$
$$a, b \gg 1 \quad \text{and} \quad f \frac{r}{D} vx^{-s}y^{-t}.$$

THEOREM 1. $(0,0)$ *is a two dimensional exponent pair.*

THEOREM 2. *If* (λ_0, λ_1) *is a two dimensional exponent pair, so is*

$$(l_0, l_1) \quad \textit{where} \quad l_0 = \frac{\frac{1}{2}-\lambda_1}{K+4(K-1)(\frac{1}{2}-\lambda_1)}, \quad l_1 = \frac{\frac{1}{2}-\lambda_0+k(\frac{1}{2}-\lambda_1)}{K+4(K-1)(\frac{1}{2}-\lambda_1)};$$

k being any integer greater than or equal to unity and $K = 2^k$.

Theorem 1 above is trivial, while Theorem 2 above is Theorem 8′ of [9] with $p = 2$.

4. General inequalities for two dimensional exponential sums.

LEMMA 5. *Let* $f(x,y)$ *be real in a region* D *contained in the rectangle* $M < x \leqslant M'$, $N < y \leqslant N'$; *where* $M' \leqslant 2M$ *and* $N' \leqslant 2N$. *Then*

$$\sum_{(m,n)\epsilon D} e\big(f(m,n)\big) \ll MNq^{-1/2} + \Big\{\frac{MN}{q}\sum_{1\leqslant u\leqslant q-1}\Big|\sum_{m,n} e\big(f(m+u,n)-f(m,n)\big)\Big|\Big\}^{1/2}$$

where the summation on the right side is taken over the lattice points (m, n) for which both $(m+u, n)$ and (m, n) are in D, u being an integer and the only restriction on q being $0 < q \leqslant M$.

THEOREM 3. *Let $f(x, y)$ possess continuous second order partial derivatives in the rectangle $M < x \leqslant M'$, $N < y \leqslant N'$ containing the region D, where $M' \leqslant 2M$, $N' \leqslant 2N$. Let*

$$f_{x^2} \gg \ll \frac{\lambda}{M^2}, \quad f_{xy} \ll \frac{\lambda}{MN}, \quad f_{y^2} \gg \ll \frac{\lambda}{N^2},$$

$$f_{x^2}f_{y^2} - f_{xy}^2 \gg \frac{\lambda^2}{M^2N^2}$$

for all values of x and y considered; where $\lambda > 0$; $M, N \geqslant 1$. Then

$$\sum_{(m,n)\epsilon D} e\big(f(m, n)\big) \ll (\lambda^{1/2} + \lambda^{-1/2}M)\,(\lambda^{c/2}N^{1-c} + \lambda^{-1/2}N)$$

for any real number c such that $0 \leqslant c \leqslant 1$.

Lemma 5 above is Lemma 2 of [8] with $p = 2$. When $c = 1$, Theorem 3 above is an immediate consequence of Theorem 1 of [8] with $p = 2$. When $c = 0$, Theorem 3 above is got by applying Theorem 1 of [8] with $p = 1$ to the sum $\sum_m e\big(f(m, n)\big)$ for each fixed n. The general case for any c in $0 \leqslant c \leqslant 1$ is now obvious.

THEOREM 4. *Let $f(x, y)$ possess continuous third order partial derivatives in the rectangle $M < x \leqslant M'$, $N < y \leqslant N'$ containing the region D, where $M' \leqslant 2M$, $N' \leqslant 2N$. Let*

$$f_{x^{2+j}} \gg \ll \frac{\lambda}{M^{2+j}}, \quad f_{x^{1+j}y} \ll \frac{\lambda}{M^{1+j}N}, \quad f_{x^jy^2} \gg \ll \frac{\lambda}{M^jN^2},$$

$$H(f_{x^j}) = f_{x^{2+j}}f_{x^jy^2} - f_{x^{1+j}y}^2 \gg \frac{\lambda}{M^{2+2j}N^2} \qquad (j = 0, 1)$$

for all the values of x and y considered, where $\lambda \gg M \gg 1$ and $\lambda \gg N \gg 1$. Then

$$\sum_{(m,n)\epsilon D} e\big(f(m, n)\big) \ll \lambda^{(1+c)/2(3+c)}M^{1/2}N^{3/(3+c)} \qquad (0 \leqslant c \leqslant 1).$$

Proof. We assume without loss of generality that $M \gg N$. Let

$$F(x, y) = f(x+u, y) - f(x, y) = u\int_0^1 f_x(x+ut, y)\,dt.$$

Then $F(x, y)$ satisfies the conditions of Theorem 3 with $u\lambda/M$ in the place of λ and so we have by Theorem 3 and Lemma 5,

$$(2) \qquad \frac{1}{MN}S = \frac{1}{MN}\sum_{(m,n)\epsilon D} e\big(f(m, n)\big) \ll q^{-1/2} + \Big\{\frac{1}{MNq}\sum_{1\leqslant u\leqslant q-1} |S'|\Big\}^{1/2}$$

where $0 < q \leqslant M$ and

$$\frac{1}{MNq}\sum_{1\leqslant u\leqslant q-1} |S'| \ll \frac{1}{MNq}\sum_{\substack{1\leqslant u\leqslant q-1\\ u\lambda\geqslant M^2}} \Big(\frac{u\lambda}{M}\Big)^{(1+c)/2} N^{1-c} +$$
$$+\frac{1}{MNq}\sum_{MN\leqslant u\lambda< M^2} \Big(\frac{u\lambda}{M}\Big)^{-(1-c)/2} MN^{1-c} + \frac{1}{MNq}\sum_{u\lambda<MN}\Big(\frac{u\lambda}{M}\Big)^{-1} MN$$
$$\ll (q\lambda)^{(1+c)/2}M^{-(3+c)/2}N^{-c} + \frac{1}{q}\Big\{M^{(3+c)/2}N^{-c}\lambda^{-1} + \frac{M}{\lambda}\Big(\frac{MN}{\lambda}\Big)^{\varepsilon}\Big\}$$

where $\varepsilon > 0$ and is arbitrarily small.

Hence

$$(3) \qquad \frac{1}{MN}S \ll q^{-1/2} + (q\lambda)^{(1+c)/4}M^{-(3+c)/4}N^{-c/2}$$

provided

$$(4) \qquad M^{(3+c)/2}N^{-c}\lambda^{-1} \ll 1 \quad \text{and} \quad M^{1+\varepsilon}N^{\varepsilon}\lambda^{-1-\varepsilon} \ll 1.$$

Applying now Lemma 2 with $Q_1 = 0$ and $Q_2 = M$, we have

$$(5) \qquad S \ll \lambda^{(1+c)/2(3+c)}M^{1/2}N^{3/(3+c)} + M^{1/2}N.$$

Since $\lambda \gg N$ and $c \leqslant 1$, the second term is smaller than the first and hence Theorem 4 follows subject to the conditions (4).

Let now $\lambda^{1+\varepsilon}M^{-1-\varepsilon}N^{-\varepsilon} \ll 1$ so that $N \gg \Big(\frac{\lambda}{M}\Big)^{1+1/\varepsilon}$. Then

$$\lambda^{(1+c)/2(3+c)}M^{1/2}N^{3/(3+c)} = \lambda^{(2+c)/(3+c)}N^{2/(3+c)}\Big(\frac{\lambda}{M}\Big)^{-1/2}N^{1/(3+c)}$$
$$\gg \lambda^{(2+c)/(3+c)}N^{2/(3+c)}\Big(\frac{\lambda}{M}\Big)^{\left(1+\frac{1}{\varepsilon}\right)\frac{1}{3+c}-\frac{1}{2}}$$
$$\gg \lambda^{(2+c)/(3+c)}N^{2/(3+c)} = \lambda^{\frac{1}{2}\left(1+\frac{1+c}{3+c}\right)}N^{1-\frac{1+c}{3+c}}$$

since $\lambda \gg M$ and $1/\varepsilon$ is arbitrarily large. Theorem 4 now follows from Theorem 3.

Lastly, let $\lambda M^{-(3+c)/2}N^c \ll 1$. Then

$$\lambda^{(1+c)/2(3+c)}M^{1/2}N^{3/(3+c)} = \lambda^{1/2}N\lambda^{-1/(3+c)}M^{1/2}N^{-c/(3+c)} \gg \lambda^{1/2}N.$$

Theorem 4 again follows from Theorem 3. The proof of Theorem 4 is now complete.

5. The main theorems.

THEOREM 5. *If $\varrho, \sigma > 0$ and if (λ_0, λ_1) is any two dimensional exponent pair, then*

$$\sum_{(m,n)\in D} \psi(zm^{-\varrho}n^{-\sigma})$$
$$\ll \{F^{1/2+\lambda_0-\lambda_1}M^{1/2+2\lambda_0}N^{3/2-2\lambda_1}\}^{\frac{1}{3/2+\lambda_0-\lambda_1}} + F^{1/4}M^{1/4}N + F^{-1/2}MN$$

where D is any region contained in the rectangle $M < m \leqslant 2M$, $N < n \leqslant 2N$, $F = zM^{-\varrho}N^{-\sigma}$, and $F \gg M \gg 1$, $F \gg N \gg 1$.

Proof. We consider the sum

$$S_0 = \sum_{(m,n)\in D} e(-\nu zm^{-\varrho}n^{-\sigma}) \qquad (\nu \geqslant 1;\ \varrho, \sigma > 0). \tag{6}$$

Fixing n, we apply Lemma 4 to S_0. Here $f(x) = -\nu zx^{-\varrho}n^{-\sigma}$, $a < x \leqslant b$ where $a = a(n)$ and $b = b(n)$ are such that $M \leqslant a < b \leqslant 2M$; $\lambda_2 = \nu FM^{-2}$, and $\lambda_3 = \nu FM^{-3}$. Let Δ be the transform of the region D by the transformation $u = f'(x)$, $v = y$; i.e. $(u, v)\in\Delta \Leftrightarrow (x, y)\in D$. Then by Lemma 4, we have

$$S_0 = e(-\tfrac{1}{8}) \sum_{(\mu,n)\in\Delta} |f''(x_\mu)|^{-1/2} e\big(f(x_\mu) - \mu x_\mu\big) + O\big((\nu F)^{1/3}N\big) + O\big((\nu F)^{-1/2}MN\big) \tag{7}$$

since

$$N\log\left(2 + \frac{\nu F}{M}\right) \ll N(\nu F)^{1/3}.$$

Now

$$f'(x_\mu) = \nu z\varrho x_\mu^{-\varrho-1}n^{-\sigma} = \mu.$$

Hence

$$x_\mu = (\nu z\varrho\mu^{-1}n^{-\sigma})^{1/(1+\varrho)}$$

and

$$f(x_\mu) - \mu x_\mu = -\frac{1+\varrho}{\varrho}(\nu z\varrho\mu^\varrho n^{-\sigma})^{1/(1+\varrho)},$$
$$|f''(x_\mu)|^{-1/2} = (\nu F)^{-1/2}M.$$

Applying now Lemma 1 to (7) we get

$$S_0 \ll (\nu F)^{-1/2}M|S_1| + (\nu F)^{1/3}N + (\nu F)^{-1/2}MN \tag{8}$$

where

$$S_1 = \sum_{(\mu,n)\in\Delta'} e\left(-\frac{1+\varrho}{\varrho}(\nu z\varrho\mu^\varrho n^{-\sigma})^{1/(1+\varrho)}\right); \qquad \Delta' \subseteq \Delta. \tag{9}$$

Since $\varrho, \sigma > 0$, it is easily seen that $-\varrho/(1+\varrho)$ and $\sigma/(1+\varrho)$ satisfy the conditions for s and t in the Definition 2 and hence we get

$$S_1 \ll (\nu F)^{2\lambda_0}\left(\frac{\nu F}{M}N\right)^{1-\lambda_0-\lambda_1} \tag{10}$$

where (λ_0, λ_1) is any two dimensional exponent pair. Substituting (10) into (8) we get

$$S_0 \ll \left(\frac{\nu F}{MN}\right)^{1/2+\lambda_0-\lambda_1} M^{1/2+2\lambda_0}N^{3/2-2\lambda_1} + (\nu F)^{1/3}N + (\nu F)^{-1/2}MN. \tag{11}$$

Applying now Lemma 3 to the sum

$$S = \sum_{(m,n)\in D} \psi(zm^{-\varrho}n^{-\sigma}) \tag{12}$$

we have

$$S \ll \frac{MN}{q} + \sum_{\nu=1}^{\infty} |S_0| \min\left(\frac{1}{\nu}, \frac{q}{\nu^2}\right). \tag{13}$$

If $0 < \alpha < 1$,

$$\sum_{\nu=1}^{\infty} \nu^{-\alpha}\mathrm{Min}\left(\frac{1}{\nu}, \frac{q}{\nu^2}\right) \leqslant \sum_{\nu=1}^{\infty} \nu^{-\alpha-1} \ll 1$$

and

$$\sum_{\nu=1}^{\infty} \nu^{\alpha}\mathrm{Min}\left(\frac{1}{\nu}, \frac{q}{\nu^2}\right) = \sum_{\nu\leqslant q} \nu^{\alpha-1} + q\sum_{\nu>q} \nu^{\alpha-2} \ll q^\alpha.$$

Hence, substituting (11) into (13) we get

$$S \ll \frac{MN}{q} + \left(\frac{qF}{MN}\right)^{1/2+\lambda_0-\lambda_1} M^{1/2+2\lambda_0}N^{3/2-2\lambda_1} + (qF)^{1/3}N + F^{-1/2}MN. \tag{14}$$

Theorem 5 now follows from Lemma 2 with $Q_1 = 0$, $Q_2 = \infty$.

THEOREM 6. *If D is any region contained in the rectangle $M < m \leqslant 2M$, $N < n \leqslant 2N$, and $\varrho, \sigma > 0$, then*

$$\sum_{(m,n)\in D} \psi(zm^{-\varrho}n^{-\sigma}) \ll F^{1/5}M^{3/5}N^{4/5}$$

where $F = zM^{-\varrho}N^{-\sigma}$ and $F \gg M \gg 1$, $F \gg N \gg 1$.

Proof. We apply Theorem 4 to the sum

$$S_0 = \sum_{(m,n)\in D} e(-\nu z m^{-\varrho} n^{-\sigma}) \qquad (\nu \geqslant 1).$$

Here $f(x, y) = -\nu z x^{-\varrho} y^{-\sigma}$. We take $c = 1$.

$$\lambda = \nu z M^{-\varrho} N^{-\sigma} = \nu F.$$

We get

$$(15) \qquad S_0 \ll (\nu F)^{1/4} M^{1/2} N^{3/4}.$$

Substituting (15) into (13) we get

$$(16) \qquad S \ll \frac{MN}{q} + (qF)^{1/4} M^{1/2} N^{3/4}.$$

Applying now Lemma 2 with $Q_1 = 0$ and $Q_2 = M$, we have

$$(17) \qquad S \ll F^{1/5} M^{3/5} N^{4/5} + M^{1/2} N.$$

Since the second term in the above is smaller than the first, we have Theorem 6.

Lemma 6. *Let $\Delta_3(x)$ be the remainder term in the asymptotic equation*

$$\sum_{n_1 n_2^2 n_3^3 \leqslant x} 1 = A_1^* x + A_2^* x^{1/2} + A_3^* x^{1/3} + \Delta_3(x)$$

where

$$A_r^* = \prod_{\substack{\nu=1 \\ \nu \neq r}}^{3} \zeta\left(\frac{\nu}{r}\right) \qquad (r = 1, 2, 3).$$

If $\Delta_3(x) \ll x^{\theta} \log^{\theta'} x$ with $\theta > 1/4$, then $\Delta(x) \ll x^{\theta} \log^{\theta'} x$.

Lemma 7. *Let (α, β, γ) be any permutation of the integers $(1, 2, 3)$; and let*

$$S_{\alpha,\beta,\gamma}(x) = \sum_{\substack{m^{\alpha+\beta} n^{\gamma} \leqslant x \\ m > n}} \psi\left(\left(\frac{x}{m^{\beta} n^{\gamma}}\right)^{1/\alpha}\right).$$

Then

$$\Delta_3(x) = -\sum_{(\alpha,\beta,\gamma)} S_{\alpha,\beta,\gamma}(x) + O(x^{1/6}).$$

Lemmas 6 and 7 above are Hilfssatz 1 and 2 of [6] respectively. We are now in a position to prove our main Theorem.

Theorem 7.

$$\Delta(x) \ll x^{(3-10\theta)/(11-34\theta)} \log^2 x$$

where $\theta = \mathrm{Sup}(\lambda_1 - \lambda_0)$; (λ_0, λ_1) being any two dimensional exponent pair such that $\lambda_1 + 3\lambda_0 = \frac{1}{2}$.

Proof. In the following proof, we shall write $\theta = \lambda_1 - \lambda_0$ where (λ_0, λ_1) is any two dimensional exponent pair such that $\lambda_1 + 3\lambda_0 = \frac{1}{2}$; so that $\lambda_1 = \frac{1}{8} + \frac{3}{4}\theta$, $\lambda_0 = \frac{1}{8} - \frac{1}{4}\theta$. We also assume $\frac{1}{14} < \theta < \frac{1}{6}$. Let

$$(18) \qquad S_{\alpha,\beta,\gamma}(x) = \sum_{\substack{m^{\alpha+\beta} n^{\gamma} \leqslant x \\ m > n}} \psi\left(\left(\frac{x}{m^{\beta} n^{\gamma}}\right)^{1/\alpha}\right)$$

where (α, β, γ) is any permutation of the integers $(1, 2, 3)$. By Theorem 5, we have

$$(19) \qquad S_{\alpha,\beta,\gamma}(x, M, N) = \sum_{\substack{m^{\alpha+\beta} n^{\gamma} \leqslant x \\ m > n \\ M < m \leqslant 2M \\ N < n \leqslant 2N}} \psi\left(\left(\frac{x}{m^{\beta} n^{\gamma}}\right)^{1/\alpha}\right)$$

$$\ll \{F^{1/2-\theta} M^{3/4-\theta/2} N^{5/4-3\theta/2}\}^{\frac{1}{3/2-\theta}} + F^{1/4} M^{1/4} N + F^{-1/2} MN$$

where $F = (xM^{-\beta} N^{-\gamma})^{1/\alpha}$.

Now

$$(20) \qquad F^{1/4} M^{1/4} N = \left\{x (M^{\alpha+\beta} N^{\gamma})^{\alpha-1} \left(\frac{N}{M}\right)^{\alpha(4-\gamma)}\right\}^{1/4\alpha} \ll x^{1/4}$$

and

$$(21) \qquad F^{-1/2} MN = \left\{x^{-1} (M^{\alpha+\beta} N^{\gamma})^{\alpha/2+1} \left(\frac{N}{M}\right)^{\alpha(4-\gamma)/2}\right\}^{1/2\alpha} \ll x^{1/4}$$

since $M^{\alpha+\beta} N^{\gamma} \ll x$ and $N \ll M$.

Also

$$(22) \qquad F^{\frac{1}{2}-\theta} M^{\frac{3}{4}-\frac{\theta}{2}} N^{\frac{5}{4}-\frac{3\theta}{2}} = x^{\frac{1}{\alpha}\left(\frac{1}{2}-\theta\right)} (M^{\alpha+\beta} N^{\gamma})^{\frac{1}{12}(5-6\theta)-\frac{1}{\alpha}\left(\frac{1}{2}-\theta\right)} \left(\frac{N}{M}\right)^{\left(\frac{5-6\theta}{12}\right)(3-\gamma)}$$

$$\ll x^{\frac{1}{12}(5-6\theta)} \quad \text{if } \alpha \geqslant 2,$$

$$\ll x^{\left(\frac{3-10\theta}{11-34\theta}\right)\left(\frac{3}{2}-\theta\right)}.$$

Again, if $\alpha = 1$, $(MN)^3 \ll M^{1+\beta} N^{\gamma}$ and hence,

$$(23) \qquad F^{\frac{1}{2}-\theta} M^{\frac{3}{4}-\frac{\theta}{2}} N^{\frac{5}{4}-\frac{3\theta}{2}} = x^{\frac{1}{2}-\theta} (M^{-1-\beta} N^{-\gamma})^{\frac{1}{2}-\theta} (MN)^{\frac{5-6\theta}{4}}$$

$$\ll x^{\frac{1}{2}-\theta} (M^{1+\beta} N^{\gamma})^{\frac{1}{2}\left(\theta-\frac{1}{6}\right)} \ll x^{\left(\frac{3-10\theta}{11-34\theta}\right)\left(\frac{3}{2}-\theta\right)}$$

if $M^{1+\beta} N^{\gamma} \geqslant x^{1-\frac{14\theta-1}{11-34\theta}}$.

Next, we consider the case $\alpha = 1$ and

$$M^{1+\beta} N^{\gamma} \ll x^{1-\frac{14\theta-1}{11-34\theta}}.$$

In this case, we have, by Theorem 6,

$$S_{\alpha,\beta,\gamma}(x, M, N) \ll (FM^3N^4)^{1/5} \tag{24}$$

where $F = xM^{-\beta}N^{-\gamma}$.

Now

$$FM^3N^4 = x(M^{1+\beta}N^{\gamma})^{1/3}\left(\frac{N}{M}\right)^{\frac{4}{3}(3-\gamma)} \ll x^{\frac{4}{3}-\frac{1}{3}\frac{14\theta-1}{11-34\theta}} = x^{\frac{5(3-10\theta)}{11-34\theta}}. \tag{25}$$

It now follows from inequalities (19) to (25) that

$$S_{\alpha,\beta,\gamma}(x) \ll x^{\frac{3-10\theta}{11-34\theta}} \log^2 x. \tag{26}$$

Theorem 7 is now immediate from Lemmas 6 and 7.

THEOREM 8. $\Delta(x) \ll x^{105/407}\log^2 x$.

Proof. By Theorem 1, $(0, 0)$ is a two dimensional exponent pair. Applying Theorem 2 to the pair $(0, 0)$ with $k = 2$, we get the pair $(\frac{1}{20}, \frac{3}{20})$. Again applying Theorem 2 with $k = 1$ to the pair $(\frac{1}{20}, \frac{3}{20})$ we get the pair $(\frac{7}{68}, \frac{16}{68})$. Theorem 2 with $k = 1$, when applied to the pair $(\frac{7}{68}, \frac{16}{68})$ yields the pair $(\frac{18}{208}, \frac{45}{208})$. Theorem 2 with $k = 2$, when applied to the pair $(\frac{7}{68}, \frac{16}{68})$ yields the pair $(\frac{18}{488}, \frac{63}{488})$. Since the set of two dimensional exponent pairs is obviously a convex set, we multiply the pair $(\frac{18}{208}, \frac{45}{208})$ by $\frac{650}{711}$ and the pair $(\frac{18}{488}, \frac{63}{488})$ by $\frac{61}{711}$ and add. Then we get the pair $(\frac{13}{158}, \frac{33}{158})$. We apply finally Theorem 2 with $k = 1$ to this pair. Then we get the pair $(\lambda_0, \lambda_1) = (\frac{23}{250}, \frac{56}{250})$. Since this pair satisfies the condition $\lambda_1 + 3\lambda_0 = \frac{1}{2}$ we have $\theta \geqslant \frac{33}{250}$ where θ is defined in the statement of Theorem 7. Theorem 8 is now an immediate consequence of Theorem 7.

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