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ON THE NUMBER OF CERTAIN SUBGRAPHS CONTAINED IN GRAPHS WITH A GIVEN NUMBER OF EDGES

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ABSTRACT

All graphs considered are finite, undirected, with no loops, no multiple edges and no isolated vertices. For two graphs G, H, let $N(G,H)$ denote the number of subgraphs of G isomorphic to H. Define also, for $l \geqq 0$, $N(l,H) = \max N(G,H)$, where the maximum is taken over all graphs G with l edges. We determine $N(l,H)$ precisely for all $l \geqq 0$ when H is a disjoint union of two stars, and also when H is a disjoint union of $r \geqq 3$ stars, each of size s or $s+1$, where $s \geqq r$. We also determine $N(l,H)$ for sufficiently large l when H is a disjoint union of r stars, of sizes $s_1 \geqq s_2 \geqq \cdots \geqq s_r > r$, provided $(s_1 - s_r)^2 < s_1 + s_r - 2r$. We further show that if H is a graph with k edges, then the ratio $N(l,H)/l^k$ tends to a finite limit as $l \to \infty$. This limit is non-zero iff H is a disjoint union of stars.

1. Introduction

All graphs considered are finite, undirected, with no loops, no multiple edges and no isolated vertices. For two graphs G, H, let $N(G,H)$ denote the number of subgraphs of G isomorphic to H. Define also, for $l \geqq 0$, $N(l,H) = \max N(G,H)$, where the maximum is taken over all graphs G with l edges.

Erdös and Hanani [2] determined $N(l,H)$ explicitly when H is a complete graph. We investigated in [1] the asymptotic behaviour of $N(l,H)$ for fixed H as l tends to infinity. Here we determine $N(l,H)$ precisely for all $l \geqq 0$ when H is a disjoint union of two stars (Theorem 5) and also when H is a disjoint union of $r \geqq 3$ stars, each of size s or $s+1$, where $s \geqq r$ (Theorem 3). We also determine $N(l,H)$ for sufficiently large l when H is a disjoint union of r stars of sizes $s_1 \geqq s_2 \geqq \cdots \geqq s_1 > r$, provided $(s_1 - s_r)^2 < s_1 + s_r - 2r$ (Theorem 4). We further

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show that if H is a graph with k edges, then the ratio $N(l, H)/l^k$ tends to a finite limit as $l \to \infty$. This limit is non-zero iff H is a disjoint union of stars (Theorems 1, 2).

2. Notation and definitions

For every set A, $|A|$ is the cardinality of A. G_l is a graph with l edges. For every graph G, $V(G)$ is the set of vertices of G and $E(G)$ is its set of edges. If $e \in E(G)$, the set $N(e)$ of *neighbours* of e is the set of all edges $f \in E(G) \setminus \{e\}$ that are adjacent to e, and the *degree* of e is $d(e) = |N(e)|$.

For $S \subset V(G)$, define $N(S) = \{x \in V(G) : xy \in E(G)$ for some $y \in S\}$. Define also $\delta(G) = \max\{|S| - |N(S)| : S \subset V(G)\}$, $\gamma(G) = \frac{1}{2}(|V(G)| + \delta(G))$. If $x \in V(G)$, $G - x$ is the subgraph of G consisting of the edges of G not incident with x and their vertices.

If G, H, T are graphs and H is a subgraph of T, let $x(G; T, H)$ denote the maximal number r, such that there exist r subgraphs of G isomorphic to T whose intersection includes a subgraph isomorphic to H. ($x(G; T, H) = 0$ if G contains no copy of H.) The operational meaning of this definition is: If H' is a copy of H in G, then H' can be extended to a copy of T in G in at most $x(G; T, H)$ ways.

$I(k)$ is the graph consisting of k independent edges and $K(1, k)$ is the star consisting of k edges incident with one common vertex. Since we do not allow isolated vertices, we agree that $K(1, 0)$ is the *empty* graph.

For nonnegative numbers $j_1, s_1, j_2, s_2, \ldots, j_k, s_k$, $H(j_1 * s_1, j_2 * s_2, \ldots, j_k * s_k)$ is the disjoint union of $j_1 + \cdots + j_k$ stars: j_1 of type $K(1, s_1)$, j_2 of type $K(1, s_2), \ldots, j_k$ of type $K(1, s_k)$. If the multiplicity j_i is 1, we write s_i instead of $1 * s_i$. We also let $HE(r, l)$ denote the graph with l edges which is the disjoint union of r stars, each having $[l/r]$ or $\lceil l/r \rceil$ edges. Note that

$$H(j * (s + 1), (r - j) * s) = HE(r, rs + j)$$

and

$$HE(r, l) = H([l/r], [(l + 1)/r], \ldots, [(l + r - 1)/r]).$$

If H is any disjoint union of r stars and $l \geqq 0$, define

$$g(l, H) = N(HE(r, l), H). \tag{1}$$

In particular, define for $r \geqq j \geqq 1$ and $s \geqq 0$

$$g(l, r, j, s) = g(l, H(j * (s + 1), (r - j) * s)). \tag{2}$$

3. An extremal property of unions of stars

One of the main results obtained in [1] is the following:

THEOREM A (Theorem 5 in [1]). *For every graph H there are positive constants* c_1, c_2 *such that* $c_1 l^{\gamma(H)} \leqq N(l, H) \leqq c_2 l^{\gamma(H)}$ *for all* $l \geqq |E(H)|$.

By the definition of $\gamma(H)$, for every graph H

$$\gamma(H) \geqq \tfrac{1}{2}|V(H)|. \tag{3}$$

The extremal graphs H for which equality holds in (3) were called a.e.c. graphs in [1]. The asymptotic behaviour of $N(l, H)$ for such graphs was determined quite precisely as follows:

THEOREM B (Theorem 4 in [1]). *If H is a.e.c., then*

$$N(l, H) = (1 + O(l^{-1/2})) \cdot \frac{1}{|\mathrm{Aut}\, H|} \cdot (2l)^{|V(H)|/2},$$

where $|\mathrm{Aut}\, H|$ *is the number of automorphisms of H.*

The following simple theorem characterizes the extremal graphs for the opposite inequality for $\gamma(H)$.

THEOREM 1. *For every graph H*

$$\gamma(H) \leqq |E(H)|, \tag{4}$$

and equality holds if and only if H is a disjoint union of stars.

PROOF. The theorem can be proved quite easily directly from the definition of $\gamma(H)$. However, we prefer to derive it from Theorem A.

Obviously, for every graph H:

$$N(l, H) \leqq \binom{l}{|E(H)|} \leqq \frac{1}{|E(H)|!}\, l^{|E(H)|}.$$

This, together with Theorem A, implies the validity of (4).

Suppose H is a disjoint union of r stars. For every l, put $G_l = HE(r, l)$. One can easily verify that there is a positive constant c such that

$$N(l, H) \geqq N(G_l, H) \geqq c \cdot l^{|E(H)|}$$

for all sufficiently large l. Combining this with Theorem A, we get

$$|E(H)| \leqq \gamma(H)$$

and therefore $\gamma(H) = |E(H)|$.

Now suppose, conversely, that H is not a disjoint union of stars. Then there is an edge $e \in E(H)$ incident with two vertices of degrees $\geqq 2$. Put $H' = H - e$. Obviously $|V(H')| = |V(H)|$, $\delta(H') \geqq \delta(H)$ and thus $\gamma(H') \geqq \gamma(H)$.

Therefore, using inequality (4) for H', we conclude that

$$\gamma(H) \leqq \gamma(H') \leqq |E(H')| < |E(H)|,$$

i.e., inequality (4) is strict for H. □

In view of Theorems A and B, the following conjecture seems quite natural.

CONJECTURE 1. *For every graph H there is a positive constant $b(H)$ such that*

$$\lim_{l\to\infty} N(l,H)/l^{\gamma(H)} = b(H).$$

By Theorem B, Conjecture 1 holds if H is a.e.c. The next theorem shows that it holds also if H is a disjoint union of stars.

THEOREM 2. (i) *Let H be a graph with k edges. For $l \geqq k$ define*

$$h(l) = N(l,H) \Big/ \binom{l}{k}.$$

Then $h(l)$ is a monotone non-increasing function of l for $l \geqq k$.

(ii) *If H is a disjoint union of stars, then the limit*

$$\lim_{l\to\infty} N(l,H)/l^{\gamma(H)}$$

exists and is a positive finite number.

PROOF. (i) Suppose $l > m \geqq k$, and let G_l be a graph such that $N(l,H) = N(G_l,H)$. Let S be the set of all ordered pairs (K,M), where M is a subgraph of G_l with m edges and K is a subgraph of M isomorphic to H. Clearly

$$|S| = N(l,H)\cdot\binom{l-k}{m-k},$$

and

$$|S| \leqq \binom{l}{m}\cdot N(m,H).$$

Therefore,

$$N(m,H) \geqq N(l,H)\cdot\binom{l-k}{m-k}\Big/\binom{l}{m} = N(l,H)\cdot\binom{m}{k}\Big/\binom{l}{k},$$

and $h(m) \geqq h(l)$, as needed.

(ii) By part (i) of the theorem, the limit

$$\lim_{l\to\infty} N(l,H)/l^{|E(H)|}$$

exists for every graph H. By Theorem A and Theorem 1, this limit is positive iff H is a disjoint union of stars (and in this case $\gamma(H)=|E(H)|$), and is zero otherwise. □

By Theorem 1 the disjoint unions of stars form, in a sense, a class dual to the class of a.e.c. graphs. In the next sections we compute $N(l,H)$ precisely for various graphs H in this class.

4. Disjoint unions of stars of nearly equal sizes

In this section we prove the following two theorems:

THEOREM 3. *If $r \geqq 1$ and $k \geqq r^2$ or $k = r^2 - r + 1$, then*

$$N(l, HE(r,k)) = N(HE(r,l), HE(r,k)) \tag{5}$$
$$(= g(l, HE(r,k)) \text{ — see (1)}) \qquad \textit{for all } l \geqq 0.$$

(Recall that if $k = r\cdot s + j$, $1 \leqq j \leqq r$, then $g(l, HE(r,k))$ is denoted by $g(l,r,j,s)$ — see (2).)

THEOREM 4. *If $s_1 \geqq s_2 \geqq \cdots \geqq s_r > r \geqq 2$ and $(s_1 - s_r)^2 < s_1 + s_r - 2r$, then there exists an l_0 such that for all $l > l_0$,*

$$N(l, H(s_1,s_2,\ldots,s_r)) = N(HE(r,l), H(s_1,s_2,\ldots,s_r)) \tag{6}$$
$$(= g(l, H(s_1,s_2,\ldots,s_r)) \text{ — see (1)}).$$

REMARK 1. If $k < r\log r$ and $H = HE(r,k)$, then $N(l,H) \neq N(HE(r,l),H)$, since in this case $N(HE(r+1,l),H) > N(HE(r,l),H)$ for sufficiently large l. (This can be proved by computations similar to those appearing in the next remark.) Thus the condition $k \geqq r^2$ in Theorem 3 is not entirely superfluous (although it is probably not best possible).

REMARK 2. (i) One can easily check that if $H = H(s_1,s_2,\ldots,s_r)$, $(r \geqq 1$, $s_1 \geqq s_2 \geqq \cdots \geqq s_r \geqq 2)$ and $k = |E(H)|$ $(= s_1 + \cdots + s_r)$, then

$$N(HE(r,l),H) = \frac{r!}{|\mathrm{Aut}\, H|}\left(\frac{l}{r}\right)^k \cdot (1 + O(l^{-1})).$$

(Note that $|\mathrm{Aut}\, H| \cdot N(HE(r,l),H)$ is the number of embeddings of H into $HE(r,l)$.)

Therefore, if H falls within the scope of Theorem 4, then the value of the limit

$$\lim_{l\to\infty} N(l,H)/l^k,$$

whose existence was proved in Theorem 2, is $r!/(r^k\,|\mathrm{Aut}\,H|)$.

(ii) Theorem 5 in Section 5 and Lemma 7 of this section show that for $r=2$ the assertion of Theorem 4 holds iff $s_1 \geqq s_2 \geqq 1$ and $(s_1-s_2)^2 < s_1+s_2$, except for $s_1=s_2=1$.

We begin with some lemmas. After Lemma 2 we shall briefly outline the strategy of the proof of Theorems 3 and 4.

LEMMA 1. *If G, T, H are graphs and H is a subgraph of T, then*

$$N(G,T) \leqq N(G,H)\cdot\frac{x(G;T,H)}{N(T,H)}.$$

PROOF. Let S be the set of all ordered pairs (A,B), where B is a subgraph of G isomorphic to T, and A is a subgraph of B isomorphic to H. Obviously

$$|S| = N(G,T)\cdot N(T,H),$$

and

$$|S| \leqq N(G,H)\cdot x(G;T,H).$$

This clearly implies the desired result. □

LEMMA 2. *If H is any disjoint union of stars, then*

$$N(l,H) \geqq g(l,H)$$

for all $l \geqq 0$.

PROOF. Obvious. □

We shall prove Theorem 3 according to the following scheme: First we prove (Lemma 5) that for $H=H(r*s)$ and all G_l, $N(G_l,H) \leqq N(HE(r,l),H)$. This proves Theorem 3 for disjoint unions of equal stars. (In order to perform the induction, we are forced to consider at the same time also the graphs $H(s+1,(r-1)*s)$.)

In order to prove Theorem 3 for $T=H(j*(s+1),(r-j)*s)$, we show that for all G_l, $x(G_l,T,H) \leqq x(HE(r,l);T,H)$, and use Lemma 1. Lemma 1 holds as equality for $G_l = HE(r,l)$.

The structure of the proof of Theorem 4 is similar.

LEMMA 3. *Let H be a graph. For every $e \in E(H)$, let $S(e)$ denote the subgraph of H spanned by $N(e)$ and let $T(e)$ denote the subgraph of H spanned by $E(H)\setminus\{N(e)\cup\{e\}\}$. Define an equivalence relation $\sim$ on $E(H)$ as follows: $e \sim e'$ iff $S(e)$ and $T(e)$ are isomorphic to $S(e')$ and $T(e')$, respectively. Let $e_1, e_2, \ldots, e_q$ be a system of representatives of the equivalence classes of $E(H)$. Define*

$$L(H) = \{(S_1, T_1), (S_2, T_2), \ldots, (S_q, T_q)\}$$

where $S_i = S(e_i)$, $T_i = T(e_i)$ for $1 \leqq i \leqq q$. Denote by γ_i the number of edges of H equivalent to e_i. Let $c_1, c_2, \ldots, c_q$ be non-negative real numbers whose sum is 1.

(i) *If $G = G_l$ is a graph with l edges $f_1, \ldots, f_l$ and $d_j = d(f_j)$ for $1 \leqq j \leqq l$, then*

$$N(G_l, H) \leqq \sum_{j=1}^{l} \sum_{i=1}^{q} \frac{c_i}{\gamma_i} N(d_j, S_i) \cdot N(l-1-d_j, T_i). \tag{7}$$

(ii)

$$N(l, H) \leqq l \cdot \max\left\{ \sum_{i=1}^{q} \frac{c_i}{\gamma_i} N(k, S_i) \cdot N(l-1-k, T_i) : 0 \leqq k \leqq l-1 \right\}.$$

PROOF. Part (ii) follows immediately from part (i). To prove (i) fix i, $1 \leqq i \leqq q$ and denote by F the set of all ordered pairs (f, A), where A is a subgraph of G, $f \in E(A)$, and A is isomorphic to H by an isomorphism that carries f to one of the γ_i edges of H equivalent to e_i. Clearly

$$|F| = N(G, H) \cdot \gamma_i. \tag{8}$$

Let f_j be a fixed edge of G. If $(f_j, A) \in F$, then clearly $E(A) \cap N(f_j)$ is a copy of S_i and $E(A) \cap (E(G)\setminus(N(f_j)\cup\{f_j\}))$ is a copy of T_i. (Here $N(f_j)$ denotes, of course, the set of edges of G adjacent to f_j.) Thus, the number of pairs $(f_j, A) \in F$ does not exceed

$$N(d_j, S_i) \cdot N(l-1-d_j, T_i).$$

This shows that

$$|F| \leqq \sum_{j=1}^{l} N(d_j, S_i) \cdot N(l-1-d_j, T_i).$$

From this and (8) we obtain

$$N(G_l, H) \leqq \sum_{j=1}^{l} \frac{1}{\gamma_i} N(d_j, S_i) N(l-1-d_j, T_i).$$

Since the last inequality holds for each i, $1 \leqq i \leqq q$, it implies (7). □

The following technical lemma is used in the proof of Theorem 3. We omit its (easy) proof.

LEMMA 4. *Let l, r, s, x be integers, $r>0$, $s>0$, $l \geqq (r+1)s$, $0 \leqq x < l-1$.*
(i) *Define*

$$h(x)=\binom{x}{s-1}\prod_{i=0}^{r-1}\binom{\left\lceil \frac{l-1-x+i}{r}\right\rceil}{s}.$$

If $x \geqq l/(r+1)-1$, then $h(x+1)\leqq h(x)$.
(ii) *Put $x=\lceil l/(r+1)\rceil-1$; then*

$$g(l,r+1,r+1,s-1)=\binom{x}{s-1}g(l-1-x,r,r,s-1)$$
$$+\,g(l-1,r+1,r+1,r+1,s-1).$$

(*See* (2).)
(iii)

$$g(l,r+1,r+1,s-1)\geqq g(l,r+1,1,s-1)\cdot\frac{(\lceil l/(r+1)\rceil-(s-1))^r}{(r+1)\cdot s^r}.$$

The next lemma proves Theorem 3 if $k\equiv 0$ or $1 \pmod r$.

LEMMA 5. (i) *If $s\geqq r\geqq 0$, then*

$$N(l,H(s+1,r*s))=g(l,r+1,1,s)\left(=\frac{l-(r+1)\cdot s}{s+1}\prod_{i=0}^{r}\binom{\left\lfloor\frac{l+i}{r+1}\right\rfloor}{s}\right)$$

for all $l>0$.
(ii) *If $s\geqq r+1\geqq 1$, then*

$$N(l,H((r+1)*s))=g(l,r+1,r+1,s-1)\left(=\prod_{i=0}^{r}\binom{\left\lfloor\frac{l+i}{r+1}\right\rfloor}{s}\right)$$

for all $l\geqq 0$.

(Note that the graphs in Lemma 5 are unions of $r+1$ stars, not r.)

PROOF. By Lemma 2

$$N(l,H(s+1,r*s))\geqq g(l,r+1,1,s),$$

and

$$N(l,H((r+1)*s))\geqq g(l,r+1,r+1,s-1)$$

for all $s\geqq r\geqq 0$ and $l\geqq 0$.

To complete the proof we show, by induction on r, that

$$(9)\quad N(l, H(s+1, r*s)) \leqq g(l, r+1, 1, s) \left(= \frac{l-(r+1)s}{s+1} \cdot \prod_{i=0}^{r} \binom{\left[\frac{l+i}{r+1}\right]}{s} \right)$$

and

$$(10)\quad N(l, H((r+1)*s)) \leqq g(l, r+1, r+1, s-1) \left(= \prod_{i=0}^{r} \binom{\left[\frac{l+i}{r+1}\right]}{s} \right)$$

For $r=0$, (9) and (10) are trivial. Assuming they hold for $r-1$, we shall prove them for r ($r \geqq 1$) according to the following scheme:

(i) $(9)_{r-1}$ & $(10)_{r-1} \Rightarrow (9)_r$.

(ii) $(10)_{r-1}$ & $(9)_r \Rightarrow (10)_r$.

(i) Suppose $s \geqq r$. If $l \leqq (r+1)\cdot s$, (9) is trivial. Thus we may assume that $l > (r+1)\cdot s$. Put $H = H(s+1, r*s)$. Using the notation of Lemma 3

$$L(H) = \{(K_{1,s}, H(r*s)), (K_{1,s-1}, H(s+1, (r-1)*s))\}$$

and $\gamma_1 = s+1$, $\gamma_2 = r\cdot s$. Applying part (ii) of Lemma 3 with $c_1 = (l - r\cdot s)/l$, $c_2 = r\cdot s/l$, we obtain

$$N(l,H) \leqq l \max \left\{ \frac{c_1}{s+1} \cdot N(k, K_{1,s}) \cdot N(l-1-k, H(r*s)) + \frac{c_2}{rs} N(k, K_{1,s-1}) \cdot N(l-1-k, H(s+1, (r-1)*s)) : 0 \leqq k \leqq l-1 \right\}.$$

Put $y = l-1-k$. By the induction hypothesis, the last inequality implies

$$N(l,H) \leqq l \max \left\{ \frac{c_1}{s+1} \cdot \binom{k}{s} \cdot \prod_{i=0}^{r-1} \binom{\left[\frac{y+i}{r}\right]}{s} + \frac{c_2}{r\cdot s} \binom{k}{s-1} \cdot \frac{y-rs}{s+1} \prod_{i=0}^{r-1} \binom{\left[\frac{y+i}{r}\right]}{s} : 0 \leqq k \leqq l-1 \right\}$$

$$= \max \left\{ \frac{l-(r+1)\cdot s}{s+1} \cdot \binom{k+1}{s} \cdot \prod_{i=0}^{r-1} \binom{\left[\frac{y+i}{r}\right]}{s} : 0 \leqq k \leqq l-1 \right\}$$

$$= \frac{l-(r+1)s}{s+1} \cdot \prod_{i=0}^{r} \binom{\left[\frac{l+i}{r+1}\right]}{s}.$$

(The last equality holds since the maximum of $\prod_{i=0}^{r}\binom{x_i}{s}$, where $x_0,\ldots,x_r$ are nonnegative integers whose sum is preassigned, is attained when the difference between any two $x_i - s$ does not exceed 1.) The last inequality is just (9).

(ii) Suppose $s \geqq r+1$. We prove (10) by induction on l. If $l < (r+1)\cdot s$, (10) is trivial. Assume (10) holds for $l-1$, and let G_l be a graph ($l \geqq (r+1)\cdot s$). To complete the proof we must show that

$$N(G_l, H) \leqq g(l, r+1, r+1, s-1), \tag{11}$$

where

$$H = H((r+1)*s).$$

Let e be an edge of maximal degree in G_l and put $d = d(e)$. We consider two possible cases.

Case I. $d \geqq \lceil l/(r+1)\rceil - 1$

In this case the number N_1 of copies of H in G_l that contain e does not exceed

$$\binom{d}{s-1}\cdot N(l-1-d, H(r*s)).$$

By the induction hypothesis

$$N_1 \leqq \binom{d}{s-1}\prod_{i=0}^{r-1}\binom{\left[\frac{l-1-d+i}{r}\right]}{s},$$

and by part (i) of Lemma 4

$$N_1 \leqq \binom{x}{s-1}\cdot\prod_{i=0}^{r-1}\binom{\left[\frac{l-1-x+i}{r}\right]}{s} = \binom{x}{s-1}\cdot g(l-1-x, r, r, s-1),$$

where

$$x = \lceil l/(r+1)\rceil - 1.$$

Let N_2 be the number of copies of H that do not contain e. By the induction hypothesis

$$N_2 \leqq g(l-1, r+1, r+1, s-1).$$

Combining the last three formulas with part (ii) of Lemma 4, we obtain

$$N(G_l, H) = N_1 + N_2 \leqq \binom{x}{s-1} g(l-1-x, r, r, s-1) + g(l-1, r+1, r+1, s-1)$$

$$= g(l, r+1, r+1, s-1),$$

which is the required inequality (11).

Case II. $d \leqq \lceil l/(r+1) \rceil - 2 \leqq [l/(r+1)] - 1$

In this case the degree of every edge of G_l does not exceed $[l/(r+1)] - 1$. It follows that

$$x(G_l; K(1,s), K(1,s-1)) \leqq [l/(r+1)] - 1 - (s-2) = [l/(r+1)] - (s-1)$$

(see Section 2 for the definition of $x(G; T, H)$), and thus

$$x(G_l; H, H(s, r*(s-1))) \leqq ([l/(r+1)] - (s-1))^r.$$

This, together with Lemma 1, relation (9) (with s replaced by $s-1$), and part (iii) of Lemma 4, implies

$$\begin{aligned} N(G_l, H) &\leqq N(G_l, H(s, r*(s-1))) \cdot ([l/(r+1)] - (s-1))^r/(r+1) \cdot s^r \\ &\leqq g(l, r+1, 1, s-1) \cdot ([l/(r+1)] - (s-1))^r/(r+1) \cdot s^r \\ &\leqq g(l, r+1, r+1, s-1), \end{aligned}$$

as needed.

This settles Case II and thus completes part (ii) of the induction on r. □

LEMMA 6. *For $s \geqq r \geqq j \geqq 1$ and $l \geqq r \cdot s$, let*

$$x(l, r, j, s) = x(HE(r, l); H(j*(s+1), (r-j)*s), H(r*s)).$$

(i) *If G_l is a graph, then*

$$x(G_l; H(j*(s+1), (r-j)*s), H(r*s)) \leqq x(l, r, j, s).$$

(ii) $g(l, r, j, s) = g(l, r, r, s-1) \cdot x(l, r, j, s)/(s+1)^j$.

PROOF. Put $H = H(r*s)$ and $T = H(j*(s+1), (r-j)*s)$.

(i) Let $\bar{H}$ be a copy of H in G_l. Let $e_1, \ldots, e_r$ be r independent edges in $\bar{H}$. For every $1 \leqq i \leqq r$, let y_i be the number of edges in $E(G_l) \setminus E(\bar{H})$ that are adjacent to e_i and are not adjacent to any e_j $(j \neq i)$. Clearly

$$\sum_{i=1}^{r} y_i \leqq l - r \cdot s. \tag{12}$$

It is easily checked that the number of copies of T in G_l that contain $\bar{H}$ does not exceed

$$\sum \{ y_{i_1} y_{i_2} \cdots y_{i_j} : 1 \leqq i_1 < i_2 < \cdots < i_j \leqq r \}.$$

An easy computation shows that the last sum, in which the $y_i - s$ are nonnegative integers that satisfy (12), attains its maximum when the difference between any two $y_i - s$ does not exceed 1, and their sum is $l - rs$. Since this maximum is precisely $x(l, r, j, s)$, we conclude that

$$x(G_l; T, H) \leqq x(l, r, j, s),$$

as needed.

(ii) Put $G = HE(r, l)$. By definition

$$N(G, H) = g(l, r, r, s-1),$$

and

$$N(G, T) = g(l, r, j, s).$$

Clearly

$$N(T, H) = (s+1)^j,$$

and every copy of H in G is included in precisely $x(l, r, j, s)$ copies of T in G. Thus

$$N(G, H) \cdot x(l, r, j, s) = N(G, T) \cdot N(T, H),$$

which, together with the previous three equalities, implies the validity of (ii). □

Proof of Theorem 3. Suppose $s \geqq r \geqq j \geqq 1$. Put $H = H(r * s)$ and $T = H(j * (s+1), (r-j) * s)$. By Lemma 2

$$N(l, T) \geqq g(l, r, j, s).$$

Let G_l be a graph. In order to complete the proof, we must show that

$$N(G_l, T) \leqq g(l, r, j, s).$$

If $l < r \cdot s$, this is trivial. Thus we may assume that $l \geqq r \cdot s$. By part (ii) of Lemma 5

$$N(G_l, H) \leqq g(l, r, r, s-1).$$

By part (i) of Lemma 6

$$x(G_l; T, H) \leqq x(l, r, j, s).$$

Clearly

$$N(T, H) = (s+1)^j.$$

Combining the last three formulas with Lemma 1 and part (ii) of Lemma 6, we obtain

$$N(G_l, T) \leqq \frac{N(G_l, H) \cdot x(G_l; T, H)}{N(T, H)} \leqq \frac{g(l, r, r, s-1) \cdot x(l, r, j, s)}{(s+1)^j} = g(l, r, g, s). \qquad \square$$

In order to prove Theorem 4, we need another definition and three lemmas. If H is a graph, $r>0$ and $l \geqq 0$, define $NS_r(l,H)=\max N(G,H)$, where the maximum is taken over all graphs G with l edges that are disjoint unions of r stars.

LEMMA 7. *Suppose* $s \geqq t \geqq 1$.

(i) *If*

$$(s-t)^2 < s+t, \tag{13}$$

then for $l > l_0(s,t)$,

$$NS_2(l, H(s,t)) = g(l, H(s,t)). \tag{14}$$

(ii) *If* $(s-t)^2 \geqq s+t$, *then for all* $l \geqq s+t$,

$$NS_2(l, H(s,t)) > g(l, H(s,t)).$$

PROOF. (i) An easy computation shows that if $s=t$, then (14) holds for all $l \geqq 0$. Thus we assume that $s>t$. Clearly, if $l \geqq s+t$, then

$$NS_2(l,H(s,t)) = \max\left\{\binom{l/2-\varepsilon}{s}\binom{l/2+\varepsilon}{t}+\binom{l/2-\varepsilon}{t}\binom{l/2+\varepsilon}{s}:\right.$$
$$\left. 0 \leqq \varepsilon \leqq l/2-t, 2 \mid l-2\varepsilon\right\}.$$

However,

$$\binom{l/2-\varepsilon}{s}\binom{l/2+\varepsilon}{t}+\binom{l/2-\varepsilon}{t}\binom{l/2+\varepsilon}{s}=\frac{1}{s!\,t!}h(\varepsilon)$$

where

$$h(\varepsilon)=\prod_{i=0}^{t-1}\left(\left(\frac{l}{2}-i\right)^2-\varepsilon^2\right)\cdot\left(\prod_{k=t}^{s-1}\left(\frac{l}{2}-\varepsilon-k\right)+\prod_{k=t}^{s-1}\left(\frac{l}{2}+\varepsilon-k\right)\right)$$
$$=\prod_{i=0}^{t-1}\left(\left(\frac{l}{2}-i\right)^2-\varepsilon^2\right)2(s_0+s_2\varepsilon^2+\cdots+s_{2r}\varepsilon^{2r}),$$

$r=[(s-t)/2]$, and

$$s_{2i}=\sum_{t\leqq j_1<j_2<\cdots<j_{2i}<s}\left(\prod_{\substack{t\leqq k<s\\ k\neq j_1,\dots,j_{2i}}}\left(\frac{l}{2}-k\right)\right)$$
$$=\binom{s-t}{2i}\left(\frac{l}{2}\right)^{s-t-2i}\cdot(1+O(l^{-1})) \qquad (0\leqq i\leqq r).$$

We prove part (i) by showing that if $(s-t)^2 < s+t$ and l is sufficiently large, then h is a decreasing function of ε for $0 \leqq \varepsilon \leqq l/2 - t$. Define $q(z) = h(\sqrt{z})$. Clearly

$$q'(z) = q(z) \cdot (-A(z) + B(z))$$

where

$$A(z) = \sum_{i=0}^{t-1} \frac{1}{((l/2)-i)^2 - z},$$

$$B(z) = \frac{s_2 + 2s_4 z + \cdots + r \cdot s_{2r} \cdot z^{r-1}}{s_0 + s_2 z + \cdots + s_{2r-2} z^{r-1} + s_{2r} z^r}.$$

By the definitions of $h(\varepsilon)$ and the coefficients s_{2i}, $q(z) > 0$ for $0 \leqq z \leqq (l/2-t)^2$, if $l \geqq 2s$. Clearly $A(z) \geqq 4t/l^2$ for $0 \leqq z \leqq (l/2-t)^2$. We claim that if $(s-t)^2 < s+t$ and l is sufficiently large, then

$$\frac{i s_{2i} z^{i-1}}{s_{2i-2} z^{i-1}} < \frac{4t}{l^2} \qquad \text{for all } i, \quad 1 \leqq i \leqq r,$$

and thus $B(z) < 4t/l^2$. Indeed

$$\frac{i s_{2i}}{s_{2i-2}} = \frac{i(s-t-2i+2)(s-t-2i+1)}{2i(2i-1)\cdot(l/2)^2} \cdot (1 + O(l^{-1}))$$
$$\leqq \frac{2(s-t)(s-t-1)}{l^2} \cdot (1 + O(l^{-1})) < 4t/l^2.$$

We conclude that if $(s-t)^2 < s+t$ and l is sufficiently large, then $q'(z) < 0$ for $0 \leqq z \leqq (l/2-t)^2$, and thus $h(\varepsilon)$ is a decreasing function for $0 \leqq \varepsilon \leqq l/2 - t$ and (14) follows.

(ii) Suppose $(s-t)^2 \geqq s+t$ and $l \geqq s+t$. Clearly $s \geqq 3$, $s - t \geqq 2$. We consider two possible cases.

Case 1. $l = 2m$ *is even*

If $m < s$, then

$$NS_2(l, H(s,t)) \geqq 1 > 0 = g(l, H(s,t)).$$

If $m \geqq s$ one can easily check that

$$NS_2(l, H(s,t)) - g(l, H(s,t))$$
$$\geqq \binom{m+1}{s}\binom{m-1}{t} + \binom{m-1}{s}\binom{m+1}{t} - 2\binom{m}{s}\binom{m}{t}$$
$$= \frac{m!(m-1)!}{s!t!(m-s+1)!(m-t+1)!}(((s-t)^2 - (s+t))m + s(s-1) + t(t-1))$$
$$> 0.$$

Case 2. $l = 2m+1$ *is odd*

If $m+1 < s$, then

$$NS_2(l, H(s,t)) \geqq 1 > 0 = g(l, H(s,t)).$$

If $m+1 \geqq s$, one can easily check that

$$\begin{aligned} &NS_2(l, H(s,t)) - g(l, H(s,t)) \\ &\geqq \binom{m+2}{s}\binom{m-1}{t} + \binom{m-1}{s}\binom{m+2}{t} - \binom{m+1}{s}\binom{m}{t} - \binom{m}{s}\binom{m+1}{t} \\ &= \frac{(m+1)!\,(m-1)!}{s!\,t!\,(m-s+2)!\,(m-t+2)!} \cdot (am^2 - bm - c), \end{aligned}$$

where

$$a = 2((s-t)^2 - (s+t)),$$

$$b = (s-t)^2(s+t-3) - 4s^2 - 4t^2 + 6s + 6t,$$

and

$$c = 2t(t-1)(t-2) + 2s(s-1)(s-2).$$

Thus $a \geqq 0$, and by substituting $(s+t) + a/2$ for $(s-t)^2$ in b, we obtain

$$\begin{aligned} am^2 - bm - c &= \frac{a}{2} m(2m - s - t + 4) + 2s(s-1)(m-s+2) \\ &\quad + 2t(t-1)(m-t+2) \\ &\geqq 2s(s-1)(m-s+2) > 0. \end{aligned}$$

This completes the proof of part (ii). (It is worth noting that if $(s-t)^2 = s+t$, then $NS_2(l, H(s,t))/g(l, H(s,t)) \to 1$ as $l \to \infty$, whereas if $(s-t)^2 > s+t$, this limit is larger; this will be a consequence of Lemmas 12 and 13.) □

LEMMA 8. *If* $s_1 \geqq s_2 \geqq \cdots \geqq s_r \geqq 1$ *and* $(s_1 - s_r)^2 < s_1 + s_r$, *then for all sufficiently large* l,

$$NS_r(l, H(s_1, \ldots, s_r)) = g(l, H(s_1, \ldots, s_r)).$$

PROOF. One can easily check that $(s_i - s_j)^2 < s_i + s_j$ for all $1 \leqq i < j \leqq r$. By Lemma 7 there exists an l_0 such that

$$NS_2(l, H(s_i, s_j)) = g(l, H(s_i, s_j))$$

holds for all $1 \leqq i < j \leqq r$ and $l > l_0$.

Assume that $l > r \cdot l_0$ and suppose that

$$NS_r(l, H(s_1, \dots, s_r)) = N(H(l_1, \dots, l_r), H(s_1, \dots, s_r))$$

where

$$l_1 \geqq \cdots \geqq l_r, \qquad l_1 + \cdots + l_r = l.$$

If $l_1 - l_r \leqq 1$, we have nothing to prove. Otherwise $l_1 + l_r > l_0$. Define $l'_1 = \lceil (l_1 + l_r)/2 \rceil$, $l'_2 = [(l_1 + l_r)/2]$ and $l'_i = l_i$ for $3 \leqq i \leqq r-1$. By Lemma 7 one can easily show that

$$N(H(l'_1, \dots, l'_r), H(s_1, \dots, s_r)) \geqq N(H(l_1, \dots, l_r), H(s_1, \dots, s_r)).$$

Therefore

$$NS_r(l, H(s_1, \dots, s_r)) = N(H(l'_1, \dots, l'_r), H(s_1, \dots, s_r)).$$

By repeatedly applying this argument to pairs of l'_i's that differ by more than one, we finally obtain that

$$NS_r(l, H(s_1, \dots, s_r)) = N(HE(r, l), H(s_1, \dots, s_r)) = g(l, H(s_1, \dots, s_r)). \qquad \square$$

LEMMA 9. *Suppose* $s_1 \geqq s_2 \geqq \cdots \geqq s_r > r \geqq 2$, $(s_1 - s_r)^2 < s_1 + s_r - 2r$ *and define*

$$x(l, r, s_1, \dots, s_r) = x(HE(r, l); H(s_1, \dots, s_r), H(r * r)).$$

Clearly

$$x(l, r, s_1, \dots, s_r) = g(l - r^2, H(s_1 - r, \dots, s_r - r))$$

provided $l \geqq r^2$.

(i) *For all sufficiently large* l

$$x(l, r, s_1, \dots, s_r) = NS_r(l - r^2, H(s_1 - r, \dots, s_r - r)).$$

(ii) *For all sufficiently large* l *and for every graph* G_l *with* l *edges,*

$$x(G_l; H(s_1, \dots, s_r), H(r * r)) \leqq x(l, r, s_1, \dots, s_r).$$

(iii)

$$g(l, H(s_1, \dots, s_r)) = g(l, H(r * r)) \cdot \frac{x(l, r, s_1, \dots, s_r)}{N(H(s_1, \dots, s_r), H(r * r))}.$$

PROOF. Part (i) is just a restatement of Lemma 8, and the proof of part (iii) is the same as that of part (ii) of Lemma 6. To prove part (ii) put $H = H(r * r)$, $T = H(s_1, \dots, s_r)$. Let $\bar{H}$ be a copy of H in G_l. Let $v_1, \dots, v_r$ be the centers of the stars of $\bar{H}$. For every $1 \leqq i \leqq r$, let y_i be the number of edges in $E(G_l)\setminus E(\bar{H})$ that are incident with v_i and are not incident with any v_j $(j \neq i)$. Clearly

$$\sum_{i=1}^{r} y_i \leqq l - r^2.$$

It is easily checked that the number of copies of T in G_l that contain $\bar{H}$ does not exceed

$$N(H(y_1, \ldots, y_r), H(s_1 - r, \ldots, s_r - r)) \leqq NS_r(l - r^2, H(s_1 - r, \ldots, s_r - r)).$$

Combining this with part (i) of the lemma, we obtain part (ii). □

PROOF OF THEOREM 4. Suppose $s_1 \geqq \cdots \geqq s_r > r \geqq 2$, $(s_1 - s_r)^2 < s_1 + s_r - 2r$. Put $H = H(r * r)$, $T = H(s_1, \ldots, s_r)$. By Lemma 2

$$N(l, T) \geqq g(l, t).$$

Let G_l be a graph. In order to complete the proof, we must show that

$$N(G_l, T) \leqq g(l, T).$$

By Theorem 3

$$N(G_l, H) \leqq g(l, H).$$

By part (ii) of Lemma 9, for all sufficiently large l,

$$x(g_l\,; T, H) \leqq x(l, r, s_1, \ldots, s_r).$$

Combining these two inequalities with Lemma 1 and part (iii) of Lemma 9, we find that for all sufficiently large l

$$N(G_l, T) \leqq \frac{N(G_l, H) \cdot x(G_l\,; T, H)}{N(T, H)} \leqq g(l, H) \cdot \frac{x(l, r, s_1, \ldots, s_r)}{N(T, H)} = g(l, T). \quad \square$$

5. Disjoint unions of two stars

Our aim in this section is to determine $N(l, H(s, t))$ for all $l, s, t \geqq 1$. Clearly $N(l, H(1,1)) = \binom{l}{2}$. In the sequel we shall exclude this trivial case.

Define, for $s \geqq t \geqq 1$, $s \geqq 2$ and $l \geqq 0$

$$\begin{aligned} f(l, s, t) &= NS_2(l, H(s, t)) \\ &= \max\{N(H(v, l - v), H(s, t)) : \lceil l/2 \rceil \leqq v \leqq l\} \\ &= \begin{cases} \binom{[l/2]}{s} \cdot \binom{\lceil l/2 \rceil}{s} & \text{if } s = t, \\ \max\left\{\left[\binom{v}{s} \cdot \binom{l - v}{t} + \binom{v}{t} \cdot \binom{l - v}{s}\right] : \lceil l/2 \rceil \leqq v \leqq l\right\} & \text{if } s > t. \end{cases} \end{aligned}$$

THEOREM 5. *If $s \geqq t \geqq 1$, $s \geqq 2$ and $l \geqq 0$, then*

$$N(l, H(s, T)) = f(l, s, t).$$

We first need a few more notations and lemmas. We call two vertices x_1, x_2 of a graph G *equivalent* if there is an automorphism of G that maps x_1 onto x_2. Obviously, this is an equivalence relation on $V(G)$. A system of representatives of the equivalence classes is called an SRV of G.

If G, T are graphs, $y \in V(G)$ and $z \in V(T)$, let $N(G, y; T, z)$ denote the number of subgraphs of G that contain y and are isomorphic to T with an isomorphism that carries y to z.

In this section we denote the vertices of $H(s,t)$ by $a_1, a_2, b_1, \ldots, b_s, c_1, \ldots, c_t$. a_1 is joined by edges to $b_1, \ldots, b_s$, and a_2 is joined to $c_1, \ldots, c_t$.

We begin with two simple lemmas.

LEMMA 10. *Let $H_1, H_2, \ldots, H_n$ be n pairwise nonisomorphic graphs, each having k edges. Then, for every graph G_l with l edges*:

$$\sum_{i=1}^{n} N(G_l, H_i) \leqq \binom{l}{k}.$$

PROOF. Obvious. □

LEMMA 11. *Let G, H be graphs, $y \in V(G)$, $G' = G - y$, and let $\{x_1, x_2, \ldots, x_k\} \subset V(H)$ be an* SRV *of H. Then*

$$N(G, H) = N(G'; H) + \sum_{i=1}^{k} N(G, y; H, x_i).$$

PROOF. This is a direct consequence of the definitions. □

PROOF OF THEOREM 5. Clearly

$$N(l, H(s,t)) \geqq f(l, s, t)$$

for all $l \geqq 0$. Thus we only have to show that

$$N(l, H(s,t)) \leqq f(l, s, t) \tag{15}$$

for all s, t such that $s \geqq t \geqq 1$, $s \geqq 2$ and all $l \geqq 0$.

We prove (15) for every fixed t by induction on s. By Theorem 3, (15) holds for

$$\max(t, 2) \leqq s \leqq t + 1.$$

Assuming it holds for $s - 1$, let us prove it for s ($s \geqq t + 2$). Put $H = H(s, t)$.

Suppose $l > 0$ and let G_l be a graph satisfying $N(G_l, H) = N(l, H)$. By the induction hypothesis

$$N(l, H(s-1, t)) = f(l, s-1, t).$$

Let u be the maximal degree of a vertex of G_l. We first show that $u \geqq l/2$. Let $v \geqq \lceil l/2 \rceil$ be a number that satisfies

$$f(l, s-1, t) = \binom{v}{s-1} \cdot \binom{l-v}{t} + \binom{l-v}{s-1}\binom{v}{t}.$$

Clearly we may assume that $f(l, s, t) > 0$ (i.e., $l \geqq s + t$), since otherwise there is nothing to prove. Thus $u \geqq s$. By Lemma 1

$$\begin{aligned}
f(l, s-1, t) &\geqq N(G_l, H(s-1, t)) \\
&\geqq N(G_l, H(s,t)) \cdot \frac{N(H(s,t), H(s-1,t))}{x(G_l\,; H(s,t), H(s-1,t))} \\
&\geqq f(l, s, t) \cdot \frac{s}{u-s+1} \\
&\geqq \frac{s}{u-s+1}\left[\binom{v}{s}\binom{l-v}{t} + \binom{l-v}{s}\binom{v}{t}\right] \\
&= \frac{v-s+1}{u-s+1}\binom{v}{s-1} \cdot \binom{l-v}{t} + \frac{l-v-s+1}{u-s+1}\binom{l-v}{s-1}\binom{v}{t} \\
&\geqq \frac{l/2-s+1}{u-s+1} f(l, s-1, t).
\end{aligned}$$

(The last inequality is true since $v \geqq l - v$ and $s - 1 \geqq t$ imply

$$\binom{v}{s-1} \cdot \binom{l-v}{t} \geqq \binom{l-v}{s-1} \cdot \binom{v}{t}.\,)$$

By our assumption $f(l, s-1, t) > 0$, and thus the preceding inequality implies that $u \geqq l/2$.

Let x be a vertex of degree u in G_l. Define $G' = G'_{l-u} = G_l - x$.

The rest of the proof is divided into two cases.

Case 1. $t = 1$

In this case $\{a_1, a_2, b_1\}$ is an SRV for H. By Lemma 11:

$$N(G_l, H) = N(G', H) + N(G_l, x\,; H, a_1) + N(G_l, x\,; H, a_2) + N(G_l, x\,; H, b_1).$$

By Lemma 1

$$N(G', H) \leqq N(G', H(s-1, 1)) \cdot \frac{l-u-s}{s}.$$

Obviously

$$N(G_l, x; H, a_1) \leqq \binom{u}{s} \cdot (l-u),$$

$$N(G_l, x; H, a_1) \leqq u \cdot N(G', K(1,s)),$$

and

$$N(G_l, x; H, b_1) \leqq N(G', H(s-1,1)).$$

Substituting these four inequalities into the preceding equality, we obtain

$$N(G_l, H) \leqq \binom{u}{s} \cdot (l-u) + u \cdot N(G', K(1,s)) + N(G', H(s-1,1)) \cdot \frac{l-u}{s}.$$

By Lemma 10

$$N(G', K(1,s)) + N(G', H(s-1,1)) \leqq \binom{l-u}{s}.$$

As $u \geqq l/2$, the last two inequalities imply

$$\begin{aligned} N(G_l, H) &\leqq \binom{u}{s} \cdot (l-u) + u \cdot [N(G', K(1,s)) + N(G', H(s-1,1))] \\ &\leqq \binom{u}{s} \cdot (l-u) + u \cdot \binom{l-u}{s} \\ &\leqq f(l,s,1). \end{aligned}$$

This completes the proof of Case 1.

Case 2. $t \geqq 2$

In this case $\{a_1, a_2, b_1, c_1\}$ is an SRV for H. By Lemma 11

$$\begin{aligned} N(G_l, H) = N(G', H) &+ N(G_l, x : H, a_1) + N(G_l, x; H, a_2) \\ &+ N(G_l, x; H, b_1) + N(G_l, x; H, c_1). \end{aligned}$$

By Lemma 1

$$N(G', H) \leqq N(G', H(s-1,1)) \cdot \binom{l-u-s}{t} \cdot \frac{1}{st}.$$

Obviously

$$N(G_l, x; H, a_1) \leqq \binom{u}{s}\binom{l-u}{t},$$

and

$$N(G_l, x; H, a_2) \leqq \binom{u}{t} \cdot N(G', K(1, s)).$$

By Lemma 1

$$N(G_l, x; H, b_1) \leqq N(G', H(s-1, t)) \leqq N(G', H(s-1, 1)) \cdot \binom{l-u-s}{t-1} \cdot \frac{1}{t},$$

and

$$N(G_l, x; H, c_1) \leqq 2 \cdot N(G', H(s, t-1))$$

$$\leqq 2N(G', H(s-1, 1)) \cdot \binom{l-u-s}{t-1} \frac{1}{s \cdot (t-1)}.$$

(The factor 2 is needed only if $t = 2$.)

These six inequalities imply

$$N(G_l, H) \leqq \binom{u}{s} \cdot \binom{l-u}{t} + \binom{u}{t} N(G', K(1, s))$$

$$+ N(G', H(s-1, 1)) \left(\frac{1}{st} \binom{l-u-s}{t} + \left(\frac{1}{t} + \frac{2}{s(t-1)} \right) \binom{l-u-s}{t-1} \right).$$

As $u \geqq l/2$, and as we have assumed that $u \geqq s \geqq t+2$, it follows that

$$\frac{1}{st} \cdot \binom{l-u-s}{t} + \left(\frac{1}{t} + \frac{2}{s(t-1)} \right) \cdot \binom{l-u-s}{t-1}$$

$$\leqq \frac{1}{st} \cdot \binom{u}{t} + \left(\frac{1}{t} + \frac{2}{s(t-1)} \right) \cdot \binom{u}{t-1}$$

$$= \left(\frac{1}{st} + \frac{1}{u-t+1} + \frac{2t}{(u-t+1)s \cdot (t-1)} \right) \binom{u}{t}$$

$$\leqq \left(\frac{1}{8} + \frac{1}{3} + \frac{2t}{3 \cdot 4 \cdot (t-1)} \right) \binom{u}{t}$$

$$\leqq \binom{u}{t}.$$

By Lemma 10

$$N(G', K(1, s)) + N(G', H(s-1, 1)) \leqq \binom{l-u}{s}.$$

The last three inequalities imply

$$N(G_l, H) \leqq \binom{u}{s} \cdot \binom{l-u}{t} + \binom{u}{t} (N(G', K(1,s)) + N(G', H(s-1,1)))$$

$$\leqq \binom{u}{s}\binom{l-u}{t} + \binom{u}{t} \cdot \binom{l-u}{s}$$

$$\leqq f(l, s, t).$$

This completes the proof of the induction step for Case 2 and establishes Theorem 5. □

Theorem 5 determines $N(l, H(s,t))$ for every pair (s,t) $(s \geqq t \geqq 1,\ s \geqq 2)$ and for all $l \geqq 0$ precisely but not explicitly, since it is not clear for which v the maximum in the formula for $f(l,s,t)$ is attained, unless $(s-t)^2 < s+t$. (See Lemma 7.) The next two simple lemmas determine explicitly the asymptotic behaviour of $N(l, H(s,t))$ for every fixed pair (s,t), $s > t \geqq 1$, as l tends to infinity. For every such pair define

$$r_{s,t}(x) = (x^s + x^t)/(1+x)^{s+t},$$

$$h_{s,t}(x) = -t \cdot x^{s-t+1} + s \cdot x^{s-t} - s \cdot x + t.$$

We also let $M(s,t)$ denote the maximum of $r_{s,t}(x)$ in $[0,\infty)$. (This maximum exists and is attained in $(0,1]$, since $r_{s,t}(0) = 0$ and $r_{s,t}(x) = r_{s,t}(1/x)$ for all $x > 0$.)

Using this notation we can prove the following two lemmas, whose somewhat technical, rather straightforward proofs are omitted.

Lemma 12. *For every* $s > t \geqq 1$

$$f(l,s,t) = \frac{M(s,t)}{s! \cdot t!}\, l^{s+t} + O(l^{s+t-1}).$$

Lemma 13. (i) *If* $(s-t)^2 \leqq s+t$, *then*

$$M(s,t) = 1/2^{s+t-1}.$$

(ii) *If* $(s-t)^2 > s+t$, *then*

$$M(s,t) = \frac{x_0^s + x_0^t}{(1+x_0)^{s+t}},$$

where x_0 *is the unique zero of* $h_{s,t}(x)$ *in* $(0,1)$.

Remark 3. For $s > t \geqq 1$, let $x_0(s,t)$ denote the minimal zero of $h_{s,t}$ in $(0,1]$. One can easily check that

$$M(s,t)=r_{s,t}(x_0(s,t)) \qquad \text{for all } s>t\geqq 1, \quad \text{and} \quad x_0(s,t)=1$$

iff $(s-t)^2\leqq s+t$. It is easily checked that $x_0(s,t)\geqq t/s$ for all $s>t\geqq 1$, and we can prove that

$$\lim_{s\to\infty}\max_{1\leqq t\leqq s}|x_0(s,t)-t/s|=0,$$

and that

$$M(s,t)=s^st^t/(s+t)^{s+t}(1+o(1)) \qquad \text{if } (s-t)^2/(s+t)\to\infty.$$

We conclude this paper with a few remarks concerning Conjecture 1 stated in Section 3 and with another conjecture.

CONJECTURE 2. *If H is a disjoint union of stars, then for every $l>0$ (or at least for sufficiently large l), there exists a graph G_l which is a disjoint union of stars, such that*

$$N(l,H)=N(G_l,H).$$

Conjecture 2 holds trivially if H is $I(k)$ — a disjoint union of isolated edges — or if H is a star. It also holds if H is a disjoint union of two stars — by Theorem 5 — and if H is $HE(r,k)$, where $[k/r]\geqq r$ — by Theorem 3. By Theorem 4 the conjecture holds for all sufficiently large l if $H=H(s_1,\dots,s_r)$, where $s_1\geqq\cdots\geqq s_r>r$ and $(s_1-s_r)^2<s_1+s_r-2r$.

Very recently, Z. Füredi [3] proved that the conjecture holds for all sufficiently large l if H contains no stars of size 1.

Conjecture 1 holds for every a.e.c. graph H — by Theorem B — and for every disjoint union of stars — by Theorem 2. We can also prove that Conjecture 1 holds for the following graphs H.

(1) Every tree of diameter three without 2-valent vertices.

(2) Every graph H obtained by adding edges to a graph $T=H(s_1,s_2,\dots,s_r)$, where $s_1\geqq s_2\geqq\cdots\geqq s_r>r$ and $(s_1-s_r)^2<s_1+s_r-2r$ (see Theorem 4), provided that every additional edge contains at least one multi-valent vertex of T. (For example, every complete bipartite graph $K(r,s)$, where $s\geqq r^2+r$, is such an H.)

(3) Every tree with fewer than 6 edges.

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