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On the Number of Complete Intersection Calabi-Yau Manifolds

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Abstract. The intersection numbers and the action of the Pontryagin class on the integral cohomology are used to distinguish between the many CICY manifolds that have the same Hodge numbers. It is shown by examining manifolds embedded in fewer than six projective spaces that at least 2590 of the manifolds are distinct.

Complete intersection Calabi–Yau manifolds are Calabi-Yau manifolds that can be realized as a complete intersection of polynomials in a product of projective spaces. The prototype of such a space is the manifold introduced by Tian and Yau [1]

$$\begin{matrix}\mathbb{P}_3\\ \mathbb{P}_3\end{matrix}\begin{pmatrix}3 & 0 & 1\\ 0 & 3 & 1\end{pmatrix}.$$

The notation denotes that three polynomials, with multidegrees corresponding to the columns of the matrix, act in $\mathbb{P}_3\times\mathbb{P}_3$.

A specific choice of such polynomials is:

$$\sum_{i=1}^{4} x_i^3=0 \quad \begin{pmatrix}3\\0\end{pmatrix},$$

$$\sum_{i=1}^{4} y_i^3=0 \quad \begin{pmatrix}0\\3\end{pmatrix},$$

$$\sum_{i=1}^{4} x_i y_i=0 \quad \begin{pmatrix}1\\1\end{pmatrix}.$$

This construction was generalized in [2] and [3] to the case that N polynomials p^α, $\alpha=1,\ldots,N$, have transverse intersection in the product of F projective spaces of total dimension $N+3$. Such manifolds can be specified by a configuration of the form

$$\begin{matrix} \mathbb{P}_{n_1} \\ \mathbb{P}_{n_2} \\ \vdots \\ \mathbb{P}_{n_F} \end{matrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{F_1} & a_{F_2} & \dots & a_{FN} \end{pmatrix},$$

where $a_{j\alpha} = \deg_j(\alpha)$ denotes the degree of p^α in the variables of the projective spaces. The rows of the matrix are subject to the condition

$$\sum_{\alpha=1}^{N} \deg_j(\alpha) = n_j + 1$$

corresponding to the vanishing of the first Chern class. Certain other restrictions are placed on the degrees to prevent the repetition of the same manifold in an infinite number of ways. In this way an exhaustive list of 7868 degree matrices was compiled [2].

The Hodge numbers (b_{21}, b_{11}) have been calculated for each of the matrices in the list in [4]. There are only 266 distinct pairs of values that occur; these are plotted in Fig. 1.

The natural question is how many matrices in the list actually correspond to distinct manifolds. By means of a calculation of more refined topological data we find that a great many of the manifolds of the list are indeed distinct.

Given a real 6-dimensional manifold (not necessarily Calabi-Yau) there is topological information in the quantities

$$\mu_{ijk} \stackrel{\text{def}}{=} \int_M e_i \wedge e_j \wedge e_k, \qquad \nu_i \stackrel{\text{def}}{=} \int_M P \wedge e_i,$$

where the e_i's are a basis for $H^2(M, \mathbb{Z})$ and P denotes the second Pontryagin class. In fact Wall has shown [5] that these quantities classify real, simply connected, 6-dimensional manifolds. It is not known how to extend Wall's classification to complex manifolds since (i) it is not known which manifolds admit a complex structure and (ii) a given real manifold could admit more than one deformation

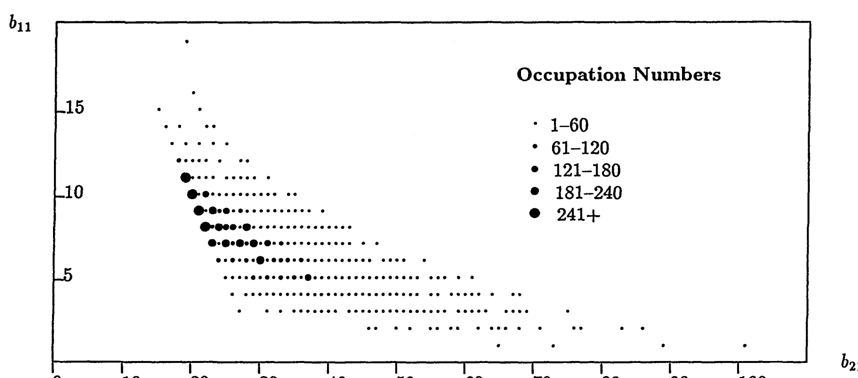

Fig. 1. The Hodge number distribution of CICY manifolds. The number of matrices that have a given value of the pair (b_{21}, b_{11}) is indicated by the heaviness of the type

class of complex structures. Wall's criteria are nevertheless very useful for distinguishing two manifolds that are different since if two manifolds are different as real manifolds they are certainly different as complex manifolds.

There are two parts to the problem of distinguishing CICY manifolds, the first is to compute the data (μ_{ijk}, v_l) for each manifold in the list and the second is to compare the resulting quantities and to decide when they correspond to different manifolds. In general the computation of the quantities (μ_{ijk}, v_l) is quite complicated [6], however for about 2/3 of the matrices of the list, i.e. about 5000 matrices, we have that $b_{11}=F$ so that the (1,1)-cohomology of these manifolds is spanned by the Kähler-forms of the ambient projective spaces, we shall here say that such matrices are favourable. For these matrices the calculation is far simpler and lends itself to automation.

For the cases that we can represent the (1,1)-cohomology classes by Kähler-forms of the projective factors we employ a standard calculus to compute the cubic and the vector forms. The first Chern class $\xi(\alpha)$ of the normal bundle to the hypersurface $p^\alpha=0$ in the embedding space may be expressed in terms of the Kähler-forms h_j with $j=1,\dots,F$ of the factor spaces

$$\xi(\alpha)=\sum_{j=1}^{F}\deg_j(\alpha)h_j,$$

where, as above, $\deg_j(\alpha)$ denotes the degree of p^α in the variables of the j'^{th} factor space. We also have

$$c_2(\mathcal{M})=\frac{1}{2}\left(\sum_{\alpha=1}^{N}\xi^2(\alpha)-\sum_{j=1}^{F}(n_j+1)h_j^2\right).$$

The cubic and the vector forms can then be identified as the coefficients of $H\overset{\text{def}}{=}\prod_{j=1}^{F}h_j^{n_j}$ in the expansion of certain formal polynomials

$$h_ih_jh_k\prod_{\alpha=1}^{N}\xi(\alpha)=\mu_{ijk}H,\qquad h_ic_2(\mathcal{M})\prod_{\alpha=1}^{N}\xi(\alpha)=v_iH.$$

The fact that there is only one term in each of these expressions follows from the facts that $h_j^{n_j+1}=0$ and the degree of the polynomials on the left-hand side has the maximal value $\sum_{j=1}^{F}n_j$.

Having calculated the data for each of the favourable matrices we then have to decide when two sets of data (μ_{ijk}, v_i), (μ'_{ijk}, v'_i) computed for two matrices $\mathcal{M}$ and $\mathcal{M}'$ can correspond to the same topology. In other words we must decide if the difference between them corresponds merely to a change of basis. A necessary condition for the difference to correspond to a change of basis is that there exist a matrix R^i_j such that

$$\mu'_{ijk}=\mu_{lmn}R^l_iR^m_jR^n_k,\qquad v'_i=v_lR^l_i. \tag{1}$$

Our bases $\{h_j\}$, $\{h'_j\}$ do not necessarily correspond to an integral cohomology basis but the h_j are certainly in $H^2(\mathcal{M},\mathbb{Z})$ even if they do not form a basis. In any event the h_j form a basis for $H^2(\mathcal{M},\mathbb{Q})$ that is a basis for H^2 with rational coefficients. It follows that the R^i_j would have to be matrices of rational numbers.

Curiously, there seems to be no easy way to decide when two sets of data (μ_{ijk}, v_l), (μ'_{ijk}, v'_l) can be related by a matrix as in (1). Our method is therefore to transform the problem into one of minization. We define

$$\varphi(R)=\sum_{i,j,k}(\mu'_{ijk}(R)-\mu_{ijk})^2+\sum_i(v'_i(R)-v_i)^2,$$

with $\mu'_{ijk}(R)$ and $v'_i(R)$ as in (1). Clearly $\varphi(R)$ is positive and $\varphi(R)=0$ if and only if (1) is satisfied. In this form the problem lends itself to numerical solution. An R which solves (1) corresponds to a minimum of φ. Thus

$$\frac{\partial\varphi(R)}{\partial R^a}=0, \tag{2}$$

where, in order to prevent a proliferation of indices, we take a to run over the $(b_{11})^2$ values of the pair (i,j). Equation (2) may be solved by the Newton-Raphson iteration. If R is an approximate solution and $R+\delta R$ the true solution then we have

$$0=\frac{\partial\varphi(R)}{\partial R^a}(R+\delta R)=\frac{\partial\varphi(R)}{\partial R^a}+\frac{\partial^2\varphi(R)}{\partial R^a\partial R^b}\delta R^b,$$

so

$$\delta R^a=-G^{ab}\frac{\partial\varphi(R)}{\partial R^a}$$

with G^{ab} the matrix inverse of

$$G_{ab}\stackrel{\text{def}}{=}\frac{\partial^2\varphi(R)}{\partial R^a\partial R^b}.$$

For the case in hand φ has the form

$$\varphi=\sum_I f_I^2,$$

where the f_I vanish at the minimum of φ. It is therefore sufficient to take

$$G_{ab}=2\sum_I\frac{\partial f_I}{\partial R^a}\frac{\partial f_I}{\partial R^b}.$$

A technical point is that at each iteration we must invert the matrix G_{ab}. The range of the indices is $(b_{11})^2$, so for CICY's corresponding to large matrices this becomes expensive in computer time. This however is not too serious since, when a solution exists, the convergence is rapid; typically 15 iterations give solutions accurate to 16 figures. A more serious problem is the need to consider all pairs of CICY's with the same Hodge numbers. The severity of this problem can be judged from Fig. 1. There is in fact one site in this diagram that is occupied more than 300 times.

There are 7890 CICY manifolds in the original list, of these 4858 manifolds have $b_{11}=F$. In Table 1 we show the result of applying our procedure to the 2590 matrices with $b_{11}\leqq 6$. This table shows that this part of the original list has at most 15% redundancy.

The following are the examples of the matrices which have the same cubic and vector forms. If for a given pair of manifolds it is the case that the generators of the

Table 1. The numbers of favourable matrices before and after identification by the intersection numbers and second Pontrjagin class

Number of factors	1	2	3	4	5	6
Number of favourable matrices	5	36	155	425	834	1135
Number found to be different	5	29	99	312	694	1068
Percentage reduction	0%	19%	36%	26%	17%	6%

projective spaces form a basis for $H^2(M,\mathbb{Z})$, then the two manifolds are diffeomorphic.

$$\begin{matrix}\mathbb{P}_1\\ \mathbb{P}_4\end{matrix}\begin{pmatrix}2&0\\1&4\end{pmatrix}^{2}_{-168} \sim \begin{matrix}\mathbb{P}_1\\ \mathbb{P}_5\end{matrix}\begin{pmatrix}1&1&0\\1&1&4\end{pmatrix}^{2}_{-168}$$

$$\begin{matrix}\mathbb{P}_1\\ \mathbb{P}_6\end{matrix}\begin{pmatrix}2&0&0&0\\1&2&2&2\end{pmatrix}^{2}_{-132} \sim \begin{matrix}\mathbb{P}_1\\ \mathbb{P}_7\end{matrix}\begin{pmatrix}1&1&0&0&0\\1&1&2&2&2\end{pmatrix}^{2}_{-132} \tag{3}$$

$$\begin{matrix}\mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_3\end{matrix}\begin{pmatrix}0&0&2\\2&0&0\\0&2&0\\1&2&1\end{pmatrix}^{4}_{-96} \sim \begin{matrix}\mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_4\end{matrix}\begin{pmatrix}1&1&0&0\\0&0&0&2\\0&0&2&0\\1&1&2&1\end{pmatrix}^{4}_{-96} \sim \begin{matrix}\mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_2\\ \mathbb{P}_5\end{matrix}\begin{pmatrix}1&1&0&0&0\\0&0&1&1&0\\0&0&0&0&2\\1&1&1&1&2\end{pmatrix}^{4}_{-96}$$

where the numbers on the upper and lower right corners are the b_{11}'s and Euler numbers for the manifolds respectively;

$$\begin{matrix}\mathbb{P}_1\\ \mathbb{P}_5\end{matrix}\begin{pmatrix}1&1&0\\2&2&2\end{pmatrix}^{2}_{-112} \sim \begin{matrix}\mathbb{P}_1\\ \mathbb{P}_5\end{matrix}\begin{pmatrix}2&0&0\\2&2&2\end{pmatrix}^{2}_{-112}$$

$$\begin{matrix}\mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_5\end{matrix}\begin{pmatrix}1&1&0&0\\1&1&0&0\\1&1&2&2\end{pmatrix}^{3}_{-80} \sim \begin{matrix}\mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_5\end{matrix}\begin{pmatrix}2&0&0&0\\1&1&0&0\\1&1&2&2\end{pmatrix}^{3}_{-80}$$

$$\begin{matrix}\mathbb{P}_1\\ \mathbb{P}_3\\ \mathbb{P}_3\end{matrix}\begin{pmatrix}1&1&0&0\\1&1&1&1\\1&1&1&1\end{pmatrix}^{3}_{-88} \sim \begin{matrix}\mathbb{P}_1\\ \mathbb{P}_3\\ \mathbb{P}_3\end{matrix}\begin{pmatrix}2&0&0&0\\1&1&1&1\\1&1&1&1\end{pmatrix}^{3}_{-88} \tag{4}$$

$$\begin{matrix}\mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_2\\ \mathbb{P}_4\end{matrix}\begin{pmatrix}1&1&0&0&0&0\\0&0&1&1&0&0\\0&0&0&0&1&1\\0&0&1&0&1&1\\1&1&0&1&1&1\end{pmatrix}^{5}_{-72} \sim \begin{matrix}\mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_2\\ \mathbb{P}_4\end{matrix}\begin{pmatrix}1&1&0&0&0&0\\0&0&1&1&0&0\\0&0&0&0&2&0\\0&0&1&0&1&1\\1&1&0&1&1&1\end{pmatrix}^{5}_{-72}$$

The set (3) suggests the identity

$$\begin{matrix}\mathbb{P}_1\\ \mathbb{P}_n\end{matrix}\begin{pmatrix}2\\1\end{pmatrix}_{-2n} \sim \begin{matrix}\mathbb{P}_1\\ \mathbb{P}_{n+1}\end{matrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}_{-2n}, \tag{5}$$

and the set (4) suggests that

$$\begin{matrix}\mathbb{P}_1\\ X\end{matrix}\begin{pmatrix}1&1\\a&a\end{pmatrix} \sim \begin{matrix}\mathbb{P}_1\\ X\end{matrix}\begin{pmatrix}2&0\\a&a\end{pmatrix}, \tag{6}$$

where X denotes any product of projective spaces. We do not know of "identities" similar to (5) and (6) for the other identifications. For example,

$$\begin{matrix}\mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_2\end{matrix}\begin{pmatrix}1&1&0\\1&0&1\\1&0&1\\1&0&1\\1&1&1\end{pmatrix}^{5}_{-80} \sim \begin{matrix}\mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_2\end{matrix}\begin{pmatrix}1&1&0\\1&0&1\\1&0&1\\1&0&1\\0&1&2\end{pmatrix}^{5}_{-80}.$$

All the transformations with the real R matrix elements we found by the method described above are actually rational. For example, the transformation of the basis of the integral second cohomology group

$$R^i_j = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

relates the cubics and vectors of the following two configurations

$$\begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \\ \mathbb{P}_3 \end{matrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 & 0 \end{pmatrix}^{6}_{-36} \quad \text{and} \quad \begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_3 \end{matrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{pmatrix}^{6}_{-36} .$$

Further examples are listed here with the basis transformation matrix at right,

$$\begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_3 \end{matrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix}^{2}_{-168} \rightarrow \begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_4 \end{matrix} \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}^{2}_{-168} \qquad \begin{bmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{bmatrix}$$

$$\begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_3 \end{matrix} \begin{pmatrix} 0 & 2 \\ 2 & 0 \\ 3 & 1 \end{pmatrix}^{3}_{-144} \rightarrow \begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_2 \end{matrix} \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}^{3}_{-144} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{2}{3} & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_5 \end{matrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}^{4}_{-72} \rightarrow \begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_3 \end{matrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{pmatrix}^{4}_{-72} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \end{matrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}^{5}_{-48} \rightarrow \begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \end{matrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}^{5}_{-48} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Due to computer limitations and the diminishing returns manifested by the final row of the table, we have not performed the calculation for $b_{11} > 6$.

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