On the number of distributive lattices

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Abstract

We investigate the numbers d_k of all (isomorphism classes of) distributive lattices with k elements, or, equivalently, of (unlabeled) posets with k antichains. Closely related and useful for combinatorial identities and inequalities are the numbers v_k of vertically indecomposable distributive lattices of size k. We present the explicit values of the numbers d_k and v_k for $k < 50$ and prove the following exponential bounds:

$$1.67^k < v_k < 2.33^k \quad \text{and} \quad 1.84^k < d_k < 2.39^k \quad (k \geqslant k_0).$$

Important tools are (i) an algorithm coding all unlabeled distributive lattices of height n and size k by certain integer sequences $0 = z_1 \leqslant \cdots \leqslant z_n \leqslant k-2$, and (ii) a "canonical 2-decomposition" of ordinally indecomposable posets into "2-indecomposable" canonical summands.

1 Vertical decompositions and additive functions

For the enumeration of classes of finite posets or lattices, so-called *ordinal* resp. *vertical decompositions* are of particular use (see, for example, [6, 7]). Roughly speaking, ordinal and vertical summation consists of placing the posets "above" each other, perhaps identifying extremal elements. As we are mainly interested in *unlabeled* (i.e. isomorphism classes of) posets and lattices, it suffices here to give the formal definitions only for sufficiently disjoint ground sets: The *ordinal sum* of two posets $P_1 = (X_1, \sqsubseteq_1)$ and $P_2 = (X_2, \sqsubseteq_2)$ with (o) $X_1 \cap X_2 = \emptyset$ can be defined as $P_1 \oplus P_2 = (X_1 \cup X_2, \sqsubseteq)$, where

$$x \sqsubseteq y \Longleftrightarrow x \sqsubseteq_1 y \text{ or } x \sqsubseteq_2 y \text{ or } (x, y) \in X_1 \times X_2.$$

Although this is also defined for lattices, one rather considers the *vertical sum* in that case, where the only difference to the former is that now the top element $\top_1$ of the lower summand and the bottom element $\bot_2$ of the upper summand are identified instead of becoming neighbours: If $L_1 = (X_1, \sqsubseteq_1)$ and $L_2 = (X_2, \sqsubseteq_2)$ are lattices with (v) $X_1 \cap X_2 = \{\top_1\} = \{\bot_2\}$, their vertical sum can be formally defined as the lattice $L = (X_1 \cup X_2, \sqsubseteq)$ with $\sqsubseteq$ as above. The ordinal [vertical] sum of two isomorphism classes is of course the isomorphism class of the sum of two representatives that fulfill (o) [(v)].

Now, a poset [lattice] is *ordinally* [*vertically*] *decomposable* if it is either empty [a singleton] or the ordinal [vertical] sum of two nonempty posets [non-singleton lattices], otherwise it is *ordinally* [*vertically*] *indecomposable.* The following facts are well known and easily verified.

Lemma 1 *Ordinal and vertical summation are associative (but clearly not commutative). Every finite poset* [*lattice*] *has a unique ordinal* [*vertical*] *decomposition into ordinally* [*vertically*] *indecomposable posets* [*lattices*]*. Vertical components of a lattice are intervals of that lattice.*

For graph theorists it may be of interest that the ordinal decomposition of a poset into indecomposable summands corresponds to the partition of the incomparability graph into connected components.

By Birkhoff's Theorem [3], the unlabeled finite posets are in one-to-one correspondence with the homeomorphism classes of finite T_0 spaces [1] and also with the unlabeled finite distributive lattices, by assigning to each poset P its topology (hence distributive lattice) $\mathcal{A}(P)$ of all *lower sets* (also known as *downsets, decreasing sets, lower segments, order ideals).* On the other hand, the latter are just the complements of *upper sets* (also known as *upsets, increasing sets, upper segments, order filters)*, and each upper, resp. lower set is generated by a unique antichain (in the finite case). Therefore, the cardinalities of the following entities are counted by the same number d_k:

— unlabeled distributive lattices with k elements,

— non-homeomorphic T_0 spaces with k open (closed) sets,

— unlabeled posets with k antichains (upper sets, lower sets).

The above one-to-one correspondence does not preserve ordinal sums, but instead sends the ordinal sum of P and Q to the vertical sum of $\mathcal{A}(P)$ and $\mathcal{A}(Q)$. Therefore, the same symbol v_k may denote the number of all

— vertically indecomposable unlabeled distributive lattices with k elements,

— non-homeomorphic T_0 spaces having no nonempty proper open subset comparable to all other open sets,

— ordinally indecomposable unlabeled posets with k antichains, upper sets, or lower sets, respectively.

From Lemma 1, we infer immediately (cf. [6, 7]):

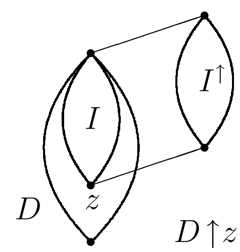
Corollary 2 *The numbers v_k are related to the numbers d_k by*

$$d_1 = 1,\ v_1 = 0,\ \textit{and } d_k = \sum_{j=1}^{k-1} v_{k-j+1} d_j \quad \textit{for } k \geqslant 2.$$

2 A useful representation of finite distributive lattices

We shall use a special case of A. Day's "doubling construction" [4] , generating larger lattices from given ones. Let $D = (k, \sqsubseteq)$, be a distributive lattice of height n, where we adopt the usual set-theoretic definition of natural numbers $k = \{0, 1, \ldots, k-1\}$. Consider an element $z \in D$ and the principal filter $I = \uparrow z := \{d \in D : z \sqsubseteq d\}$. Let $\psi : I^{\uparrow} \to I$ be the unique isomorphism from the distributive lattice $I^{\uparrow}$ with underlying set $\{k, \ldots, k+|I|-1\}$ onto I such that ψ is strictly increasing with respect to the usual order $\leqslant$ on the natural numbers. Define the order relation $\sqsubseteq^{\uparrow}$ on $k + |I|$ by

$$
\begin{aligned}
x \sqsubseteq^{\uparrow} y \Longleftrightarrow & \\
& x, y < k \text{ and } x \sqsubseteq y \\
\text{or} \quad & x, y \geqslant k \text{ and } \psi(x) \sqsubseteq \psi(y) \\
\text{or} \quad & x < k \leqslant y \text{ and } x \sqsubseteq \psi(y).
\end{aligned}
$$



Then $D\uparrow z := (k+|I|, \sqsubseteq^{\uparrow})$ is again a distributive lattice, and D is a retract of $D\uparrow z$ with retraction $y \mapsto y \wedge \bigvee D$ ($= \psi(y)$ for $y \in I^{\uparrow}$). This construction reflects the extensions of the corresponding poset P of $\vee$-irreducible (equivalently: $\vee$-prime) elements by one new maximal point n (see [5]): the join map from $\mathcal{A}(P)$ to D is an isomorphism, and for any $Z \in \mathcal{A}(P)$, there is a unique poset $P \cup \{n\}$ containing P as a subposet such that n becomes a maximal element generating the principal ideal $Z \cup \{n\}$. Now, the above isomorphism extends to one between $\mathcal{A}(P \cup \{n\})$ and $D\uparrow z$ where $z = \bigvee Z$. Any isomorphism $\varphi : D \to D' = (k, \sqsubseteq')$ extends uniquely to an isomorphism $\varphi^{\uparrow}$ between $D\uparrow z$ and $D'\uparrow\varphi(z)$ (mapping $y \in \uparrow k$ to $\varphi^{\uparrow}(y) = \psi'^{-1} \circ \varphi \circ \psi(y)$).

Since every poset of size $n+1$ arises from one of size n by the one-point extension process described above, every finite distributive lattice with more than one element is isomorphic to one of the form $D\uparrow z$. Directly, this can also be seen as follows. Any $\wedge$-prime element x in a finite distributive lattice E has a unique cover u, and there is a least element y not dominated by x. This y, henceforth denoted by $u \setminus x$, in turn is $\vee$-prime and covers a unique element z. The intervals $[z, x]$ and $[y, u]$ of E are isomorphic via transposition: $z = x \wedge y$, $u = x \vee y$. Moreover, E is the disjoint union of $\downarrow x = \{e \in E : e \sqsubseteq x\}$ and $\uparrow y = \{e \in E : y \sqsubseteq e\}$. Now, it is easy to verify that if x is a coatom in E and D is the principal ideal $\downarrow x$ then the whole lattice E is isomorphic to $D\uparrow z$.

This observation makes it possible to generate any finite distributive lattice up to isomorphism by a finite number of "doublings" of principal filters.

Theorem 3 *Every distributive lattice $(D, \sqsubseteq)$ of finite cardinality $k > 1$ and height n is isomorphic to a lattice of the form $D_0 \uparrow z_1 \uparrow \ldots \uparrow z_n$ with $|D_0| = 1$ and a sequence $(z_1, \ldots, z_n) \in k^n$ with $0 = z_1 \leqslant z_2 \leqslant \cdots \leqslant z_n$.*

Figure 1: A handy network of distributive lattices of size ⩽ 8 or height ⩽ 4

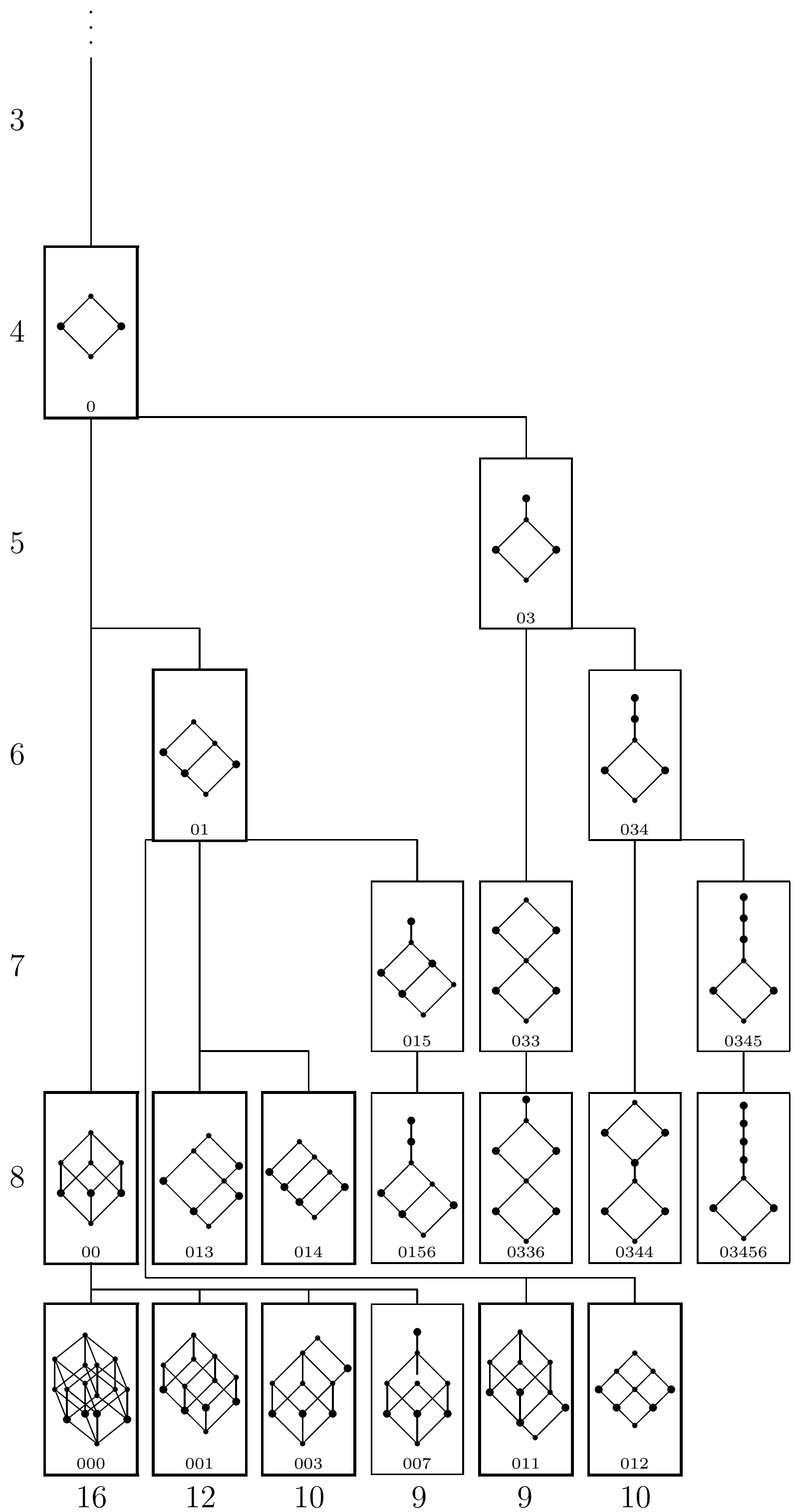
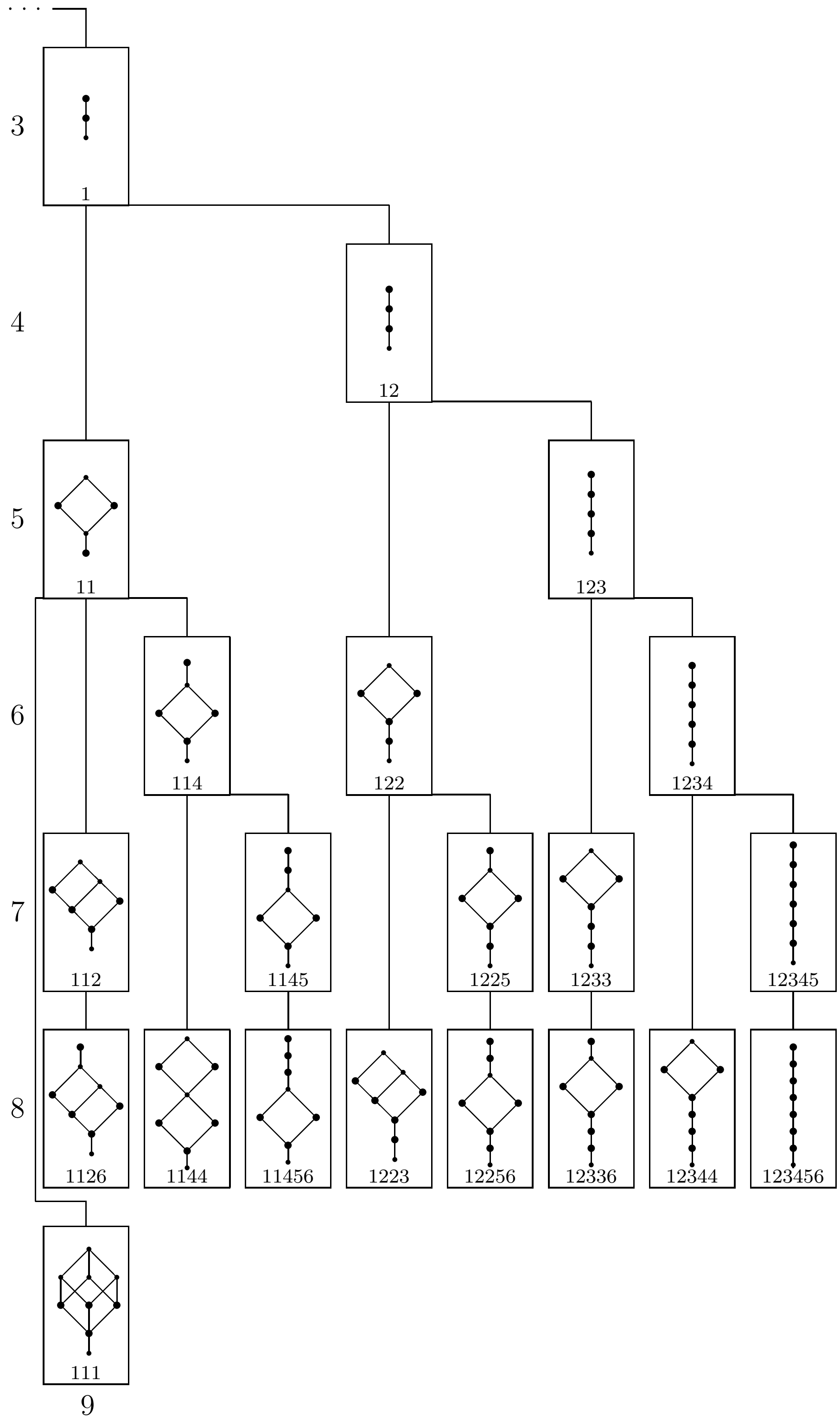


Figure 2: A handy network of distributive lattices (continued)



Proof. We recursively determine elements $x_i, y_i, z_i \in D$, distributive lattices $D_i = (k_i, \sqsubseteq_i)$ and isomorphisms $\varphi_i : \downarrow x_i \to D_i$, so that $x_0 \sqsubset x_1 \sqsubset \cdots \sqsubset x_n$ is a maximal chain in D, $y_1, \ldots, y_n$ are the $\vee$-irreducible elements of D, $z_1, \ldots, z_n$ are their unique lower covers, $u \sqsubseteq v$ implies $\varphi_i(u) \leqslant \varphi_i(v)$ (in the *natural* order), φ_i extends φ_{i-1}, and $D_i = D_{i-1} \uparrow \varphi_{i-1}(z_i)$ $(i > 0)$.

Let $x_0 = y_0 = z_0$ be the bottom element and D_0 the distributive lattice with underlying set $1 = \{0\}$. Then $\varphi_0 : \downarrow x_0 \to D_0$ is uniquely determined. If $x_{i-1}, y_{i-1}, z_{i-1}$ and φ_{i-1} have been defined and x_{i-1} is not the top of D, take for x_i one element among those covers u of x_{i-1} for which $\varphi_{i-1}(x_{i-1} \wedge (u \setminus x_{i-1}))$ is minimal in the *natural* order $\leqslant$ on D_{i-1}, and put $y_i = x_i \setminus x_{i-1}$, $z_i = x_{i-1} \wedge y_i$. Then y_i is $\vee$-irreducible and z_i is its unique lower cover. Moreover, the intervals $[z_i, x_{i-1}]$ and $[y_i, x_i]$ are isomorphic via transposition, and $\downarrow x_i = \downarrow x_{i-1} \cup [y_i, x_i]$. Hence, there exists an isomorphism $\varphi_i : \downarrow x_i \to D_i = D_{i-1} \uparrow \varphi_{i-1}(z_i)$ satisfying $u \sqsubseteq v \Rightarrow \varphi_i(u) \leqslant \varphi_i(v)$ and extending φ_{i-1}. Continuing the construction, we get an isomorphism $\varphi = \varphi_n$ between D and $D_n = D_0 \uparrow \varphi(z_1) \uparrow \ldots \uparrow \varphi(z_n)$.

Thus, we see that D is uniquely determined, up to isomorphism, by the sequence $\varphi(z_1), \ldots, \varphi(z_n)$. Without loss of generality, let φ be the identity map. Finally, we show that the sequence $0 = z_1, \ldots, z_n$ is increasing. Assume $i < j$ but $z_j < z_i$. Since z_j is covered by y_j and $y_j \not\sqsubseteq x_{i-1} \sqsubset x_{j-1}$, it follows that $x_{i-1} = z_j \vee x_{i-1}$ is covered by ${x_i}' := y_j \vee x_{i-1}$. Moreover, in the interval $\downarrow x_i$,

$$ {y_i}' := {x_i}' \setminus x_{i-1} = \min\{d \in \downarrow {x_i}' : d \not\sqsubseteq x_{i-1}\} \sqsubseteq y_j $$

and ${z_i}' := x_{i-1} \wedge {y_i}' \sqsubseteq x_{j-1} \wedge y_j = z_j$, whence ${z_i}' \leqslant z_j < z_i$, contradicting the choice of x_i (making z_i minimal). □

Notice that in the above theorem several different sequences (e.g. $(0, 0, 1)$ and $(0, 0, 2)$) may describe the same isomorphism type, and that not every increasing sequence $(z_1, \ldots, z_n) \in k^n$ corresponds to a distributive lattice. For example, it is not difficult to see that the construction yields the following inequality:

Corollary 4 *If an integer sequence $z_1 \leqslant \cdots \leqslant z_n$ represents a distributive lattice $D_0 \uparrow z_1 \uparrow \ldots \uparrow z_n$ then*

$$ \sum_{i=1}^{j} z_i < 2^{j-1} \textit{ for } 1 \leqslant j \leqslant n, \textit{ in particular } z_1 = 0. $$

Proof. The lattices $D_i = \downarrow x_i = D_0 \uparrow z_1 \uparrow \ldots \uparrow z_i$ have height i and, therefore, size $k_i \leqslant 2^i$. Furthermore, $k_0 = 1$ and $k_i = |\downarrow x_{i-1}| + |[z_i, x_{i-1}]| \leqslant 2k_{i-1} - z_i$ for $i > 0$. Hence, $z_i \leqslant 2k_{i-1} - k_i$ and

$$ \sum_{i=1}^{j} z_i \leqslant 2 \sum_{i=1}^{j} k_{i-1} - \sum_{i=1}^{j} k_i = 2 + \sum_{i=1}^{j-1} k_i - k_j \leqslant 1 + \sum_{i=1}^{j-2} k_i < 2^{j-1}. $$

□

Another inequality immediately results from doubling one- or two-element intervals only:

Corollary 5 *The number d_k of distributive lattices with k elements is greater than or equal to the k-th Fibonacci number F_k (with $F_1 = 0$ and $F_2 = 1$).*

The previous construction may be used to generate a set of representatives (coded by finite sequences of natural numbers) for the isomorphism classes of finite distributive lattices with at least two elements. Define recursively such *representative d-sequences* as follows. The empty sequence is a representative d-sequence (for the 2-element chain). Assume $(z_2, \ldots, z_{n-1})$ is a representative d-sequence, representing a distributive lattice $D = D_0 \uparrow z_1 \uparrow \ldots \uparrow z_{n-1}$. If k is the size of D then for each integer z with $z_{n-1} \leqslant z \leqslant k-1$, the sequence $(z_2, \ldots, z_{n-1}, z)$ codes the distributive lattice $D \uparrow z$. Now, call $(z_2, \ldots, z_{n-1}, z_n)$ a representative d-sequence if z_n is minimal among all z for which $D \uparrow z$ is isomorphic to $D \uparrow z_n$. By our earlier remarks on the doubling construction, this selects from each isomorphism class of finite distributive lattices one representative which is coded by the (increasing) sequence $(z_2, \ldots, z_n)$. Indeed, if D is any distributive lattice of height n and size k then D is isomorphic to $D_0 \uparrow z_1 \uparrow \ldots \uparrow z_n$ for some sequence(s) of natural numbers $z_1 = 0, z_2, \ldots, z_n$. Taking the lexicographically smallest among these sequences, one obtains a representative d-sequence (proof by induction, using the unique extensions of isomorphisms from D_{i-1} to $D_i = D_{i-1} \uparrow z_i$). Similarly, one checks that different representative d-sequences represent non-isomorphic lattices. Figures 1 and 2 show how all distributive lattices with $\leqslant 8$ elements or height $\leqslant 4$ arise in this way, the vertically indecomposable ones being framed by bold lines.

3 A second ordinal decomposition of a poset

In this section we need a notion of canonicity adopted from [8, 9] which is useful for various kinds of ordered structures. For the sake of consistency with the forerunners, we prefer here a downward numbering of elements. Of course, an upward numbering would work as well.

Here, an n-poset is a poset P with underlying set $n = \{0, \ldots, n-1\}$. We write $i \prec j$ if j is a cover of i in P and define the *weight*

$$w_P = (w_P(0), \ldots, w_P(n-1))$$

of an n-poset P by setting

$$w_P(i) = \sum_{i \prec j} 2^j.$$

Since a finite poset is uniquely determined by its covering relation, the map $P \mapsto w_P$ is injective. Let P, Q be n-posets. Then we say that w_P is (lexicographically) smaller than w_Q if there is an $i \leqslant n-1$ such that $w_P(i) < w_Q(i)$ and $w_P(k) = w_Q(k)$ for all $k = 0, \ldots, i-1$. We call an n-poset C a *canonical* poset if there is no n-poset isomorphic to C that has a smaller weight. It was shown in [8, 9] that *for every canonical n-poset C the sequence w_C is increasing, i.e. $w_C(0) \leqslant \cdots \leqslant w_C(n-1)$.*

The set P_1 of all maximal elements in a finite poset P is called the *first level* of P. One recursively defines the *i-th level* P_i of P to be the first level of the subposet $P \setminus \bigcup_{j=1}^{i-1} P_j$.

It is well known and easy to see that an element $x \in P$ is contained in P_i iff i is the maximal cardinality of a chain in P with least element x, denoted by $d_P(x)$ (the *depth* of x). Notice that $x \sqsubset y$ implies $d_P(x) > d_P(y)$. The height of the poset P will be denoted by $h(P)$. The last nonempty level $\{x \in P : d_P(x) = h(P) + 1\}$ consists of minimal elements only, but there may also be minimal elements of P in higher levels. It was proven in [8, 9] that *every canonical poset P is level-monotone* (="levelized" in the cited papers), i.e. $d_P(x) \leqslant d_P(y)$ for all $x, y \in P$ with $x \leqslant y$.

Let p, q be natural numbers and let $P = (p, \sqsubseteq_P)$, $Q = (q, \sqsubseteq_Q)$ be canonical posets. Set

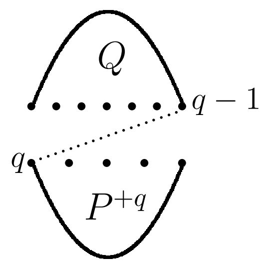
$$p^{+q} = (p+q) \setminus q = \{q, q+1, \dots, q+p-1\},$$

$$\sqsubseteq_P^{+q} = \{(x+q, y+q) : x \sqsubseteq_P y\},$$

$$P^{+q} = (p^{+q}, \sqsubseteq^{+q}),$$

$$\sqsubseteq = \sqsubseteq_Q \cup \sqsubseteq_P^{+q} \cup (p^{+q} \times q),$$

$$\sqsubseteq_2 = \sqsubseteq \setminus \{(q, q-1)\}.$$



Since P and Q are level-monotone, the element $q-1$ is minimal in Q and q is maximal in P^{+q}. Now, it is easy to verify that $\sqsubseteq$ and $\sqsubseteq_2$ are order relations on $p+q$. Also, it is not hard to see that the "canonical sum" $(p+q, \sqsubseteq)$ is the canonical representative for the ordinal sum $P \oplus Q$. More involved is the proof of the following property of the "canonical 2-sum" $P +_2 Q := (p+q, \sqsubseteq_2)$.

Theorem 6 *If $P = (p, \sqsubseteq_P)$ and $Q = (q, \sqsubseteq_Q)$ are ordinally indecomposable canonical posets then $R = P +_2 Q$ is also an ordinally indecomposable canonical poset.*

Proof. Let φ be a permutation of $p+q$ such that the poset $R' = (p+q, \{(x, y) : \varphi(x) \sqsubseteq_2 \varphi(y)\})$ is canonical. In order to prove that R is canonical, we have to verify that the vector $w_{R'} = (w_{R'}(0), \dots, w_{R'}(p+q-1))$ coincides with $w_R = (w_R(0), \dots, w_R(p+q-1))$, i.e., that φ is an automorphism of R.

Let $t, \dots, q-1$ be the minimal elements in Q and let $q, \dots, q+s$ be the maximal elements in P^{+q}. We shall only consider the case $t < q-1$, i.e. that Q has at least two minimal elements. Otherwise, it would follow from the ordinal indecomposability of Q that it has only one element. In that case some of the weights below have to be computed in a different way but the reader may easily check that all arguments stay correct. Since P and Q are canonical, they are level-monotone. Then R is also level-monotone since $d_R(x) = d_Q(x)$ for $x \in q$ and $d_R(y) = d_P(y) + h(Q) + 1$ for $y \in p^{+q} \setminus \{q\}$, while $d_R(q) \in \{h(Q)+1, h(Q)+2\}$.

If $d_R(q) = h(Q) + 2$ then the fact that the canonical poset R' is also level-monotone implies that $\varphi[q] = q$ and, since Q is canonical, that $\varphi|_q$ is an automorphism of Q, i.e. $w_{R'}(x) = w_R(x)$ for $x \in q$. Then

$$w_{R'}(\varphi^{-1}(q)) = \sum_{i=t}^{q-1} 2^i - 2^{\varphi^{-1}(q-1)} < \sum_{i=t}^{q-1} 2^i \leqslant w_{R'}(y)$$

for every element $y \in p^{+q} \setminus \{\varphi^{-1}(q)\}$. Since R' is canonical, $w_{R'}$ is increasing and, therefore, $\varphi(q) = q$. Now,

$$w_{R'}(q) = \sum_{i=t}^{q-1} 2^i - 2^{\varphi^{-1}(q-1)} \geqslant \sum_{i=t}^{q-2} 2^i = w_R(q)$$

implies $\varphi(q-1) = q-1$ and, therefore, $w_{R'}(q) = w_R(q)$.

If $d_R(q) = h(Q) + 1$ then $\varphi[q+1] = q+1$. In this case, $\{q-1\}$ is the last level of Q and $\{q-1, q\}$ constitutes a whole level in R and in R'. Since all covers of $q-1$ dominate q in R, it follows from the minimality of $w_{R'}$ that $\varphi(q-1) = q-1$ and $\varphi(q) = q$. Again, we see that $\varphi|_q$ is an automorphism of Q and $w_{R'}(q) = \sum_{i=t}^{q-2} 2^i = w_R(q)$.

Since either $\{q, \ldots, q+s\}$ or $X := \{q+1, \ldots, q+s\}$ is one level of R and of R' (or empty) and since $\varphi(q) = q$, we have $\varphi[X] = X$. All elements $x \in X$ have the same covers in R and R', namely $t, \ldots, q-1$, i.e. $w_{R'}(x) = \sum_{i=t}^{q-1} 2^i = w_R(x)$ for $x \in X$.

Let $s+1, \ldots, s+u$ be those elements in P which are covered by 0 only. Then $w_R(y) = 2^{q-1} + 2^q$ for $y \in Y := \{q+s+1, \ldots, q+s+u\}$ and $w_R(z) \geqslant 2^{q+1}$ for $z \in Z := \{q+s+u+1, \ldots, q+p-1\}$. Notice that for $z \in Z$, every cover of z in R or R' is contained in p^{+q}. From the lexicographic minimality of $w_{R'}$ it follows that $\varphi[Y] = Y$ and that $w_{R'}(y) = w_R(y)$ for $y \in Y$.

Consider the poset $\tilde{P} = (p, \{(x, y) : \varphi(x+q) \sqsubseteq_2 \varphi(y+q)\})$. If $w_{R'}$ were lexicographically smaller than w_R then the vector

$$\begin{aligned} w_{\tilde{P}} &= (w_{\tilde{P}}(0), \ldots, w_{\tilde{P}}(s), w_{\tilde{P}}(s+1), \ldots, w_{\tilde{P}}(s+u), w_{\tilde{P}}(s+u+1), \ldots, w_{\tilde{P}}(p-1)) \\ &= (0, \ldots, 0, 1, \ldots, 1, 2^{-q} w_{R'}(q+s+u+1), \ldots, 2^{-q} w_{R'}(q+p-1)) \end{aligned}$$

would be lexicographically smaller than

$$w_P = (0, \ldots, 0, 1, \ldots, 1, 2^{-q} w_R(q+s+u+1), \ldots, w_R(q+p-1)),$$

contradicting the canonicity of P.

Now, in order to prove that R is ordinally indecomposable, let us assume the contrary. Then there is a nonempty proper upper set S of R such that the relation $((p+q) \setminus S) \times S$ is contained in $\sqsubseteq_2$. Since $q \not\sqsubseteq_2 q-1$, we have $S \neq q$, whence $S \not\subseteq q$ or $q \not\subseteq S$. In the first case, $S \cap p^{+q}$ is a nonempty proper upper subset in P^{+q} with $(p^{+q} \setminus S) \times (S \cap p^{+q}) \subseteq \sqsubseteq_P{}^{+q}$, i.e., P^{+q} and P are ordinally decomposable. In the second case, $S \cap q$ is a nonempty proper upper set of Q and $(q \setminus S) \times (S \cap q) \subseteq \sqsubseteq_Q$, i.e. Q is ordinally decomposable, a contradiction. $\square$

The above theorem says that $+_2$ is an operation on the set of ordinally indecomposable canonical posets. It is not difficult to check from the definition that this operation is associative. If the canonical posets $P = (p, \sqsubseteq_P), Q = (q, \sqsubseteq_Q)$ have i and j antichains, respectively, then $P +_2 Q$ has $i+j$ antichains because every nonempty antichain of $P +_2 Q$ different from $\{q-1, q\}$ is either contained in Q or in P^{+q}, while the empty antichain is contained in both.

An ordinally indecomposable canonical poset R will be called *canonically 2-decomposable* if there are ordinally indecomposable canonical posets P, Q with $R = P +_2 Q$. We denote by w_k the number of canonically 2-indecomposable posets with k antichains.

If $R = (r, \sqsubseteq_R)$ is an ordinally indecomposable but canonically 2-decomposable poset then there is a smallest $p < r$ such that there are ordinally indecomposable posets $P = (p, \sqsubseteq_P), Q = (q, \sqsubseteq_Q)$ with $R = P +_2 Q$. Then, clearly, P and Q are unique, and associativity of $+_2$ assures that P is canonically 2-indecomposable. Hence the number of those posets which are ordinally indecomposable but canonically 2-decomposable, have k antichains, and whose first canonically 2-indecomposable summand has exactly i antichains, is $w_i \cdot v_{k-i}$. Since a nonempty poset has at least 2 antichains, it follows that

$$v_k = w_k + \sum_{i=2}^{k-2} w_i \cdot v_{k-i}.$$

Corollary 7 *The numbers w_k of canonically 2-indecomposable posets with k antichains are related to the numbers v_k of ordinally indecomposable posets with k antichains by the identities*

$$v_0 = 1, \quad w_1 = v_1 = 0, \quad \textit{and} \quad v_k = \sum_{j=0}^{k-1} w_{k-j} \cdot v_j \quad (k > 1).$$

It would be reasonable to call a poset (ordinally) 2-indecomposable if it is indecomposable and augmenting the order relation by one *arbitrary* pair never produces a decomposable poset. The number of such posets with k antichains is, of course, at most w_k. But, unfortunately, not every 2-decomposable poset is canonically 2-decomposable (consider the disjoint union of a singleton and a 3-chain) and, what is more important, there is no formula like that in the previous corollary for 2-indecomposable posets. A poset is 2-indecomposable if its incomparability graph is 2-edge-connected.

4 Exponential estimates for summatorial sequences

This section contains the necessary theoretical background for the intended (partly asymptotical) estimates of the numbers d_k and v_k. In what follows, $(a_k : k \geqslant 1)$ always designates a sequence of nonnegative real numbers, and

$$a(x) = \sum_{k=1}^{\infty} a_k x^k \quad \text{and} \quad a_{<m}(x) = \sum_{k=1}^{m-1} a_k x^k$$

the corresponding (formal) power series and its partial sums, regarded as polynomials. The "summatorial" sequence (s_k) and its partial sums are given by

$$s(x) = \sum_{k=0}^{\infty} s_k x^k = (1 - a(x))^{-1} = \sum_{k=0}^{\infty} a(x)^k, \quad s_{<m}(x) = \sum_{k=0}^{m-1} s_k x^k$$

and their coefficients are determined recursively by

$$s_0 = 1, \quad s_k = \sum_{j=0}^{k-1} a_{k-j} s_j = \sum_{j=1}^{k} a_j s_{k-j} \quad \text{for } k \geqslant 1.$$

We say that a proposition holds "eventually" when it holds for all k larger than some k_0.

Lemma 8 *The following statements are equivalent:*

(1) $s_k > 0$ *eventually.*

(2) *There is no integer* $m > 1$ *with* $a_k > 0 \Longrightarrow m|k$.

(3) $\gcd(m : a_m > 0) = 1$.

Proof. (1)$\Longrightarrow$(2): If $m|k$ for all k with $a_k > 0$ then the recursion for s_k yields $s_k = 0$ whenever $m \nmid k$.

(2)$\Longrightarrow$(3): Clear.

(3)$\Longrightarrow$(1): There exist indices $k_1, \ldots, k_u$ with $\gcd(k_1, \ldots, k_u) = 1$ and $a_{k_i} > 0$ for $i = 1, \ldots, u$. Hence, for each natural number k, there are integers $l_1, \ldots, l_u$ with $k_1 l_1 + \cdots + k_u l_u = k$, and if k is sufficiently large, then the l_i can be chosen nonnegative, whence

$$s_k \geqslant a_{k_1} s_{k-k_1} \geqslant a_{k_1}^2 s_{k-2k_1} \geqslant \cdots \geqslant \prod_{i=1}^{u} a_{k_i}^{l_i} > 0,$$

where we used the recursion formula $l_1 + \cdots + l_u$ times. $\square$

In the subsequent lemmas, we always assume that (1)–(3) are fulfilled. *Lower* exponential bounds for s_k are provided by

Lemma 9 *Suppose* $m \in \mathbb{N}$ *and* $\sigma > 0$ *are constants with* $a_{<m}(\frac{1}{\sigma}) > 1$. *Then there is a* $\tau > \sigma$ *and an* n *with* $m \leqslant n < 2m$ *and* $s_k \tau^{-k} \geqslant s_n \tau^{-n}$ *for all* $k \geqslant m$. *Hence, if* $s_k > 0$ *for* $k \geqslant m$,

$$\tau^k = O(s_k) \quad \textit{and} \quad \sigma^k = o(s_k).$$

Proof. By continuity, there is a $\tau > \sigma$ with $a_{<m}(\frac{1}{\tau}) > 1$. Put $\delta := \min\{s_j \tau^{-j} : m \leqslant j < 2m\}$, say $\delta = s_n \tau^{-n}$. Then $s_j \tau^{-j} \geqslant \delta$ for all j with $m \leqslant j < 2m$. Let $k \geqslant 2m$ and assume that $s_j \tau^{-j} \geqslant \delta$ has also been established for all j with $m \leqslant j < k$. Then

$$s_k \geqslant \sum_{j=1}^{m-1} a_j s_{k-j} \geqslant \sum_{j=1}^{m-1} a_j \delta \tau^{k-j} = \delta \tau^k a_{<m}(\tfrac{1}{\tau}) > \delta \tau^k.$$

Hence, by induction, $s_k\tau^{-k} \geqslant \delta = s_n\tau^{-n}$ for all $k \geqslant m$. $\square$

Let ϱ_s denote the radius of convergence for $s(x)$. If the series $1 - a(x)$ has a smallest positive root ϱ, then $\varrho = \varrho_s$, since by nonnegativity of the a_k and monotonicity of $a(x)$, the series $s(x)$ surely converges for $0 \leqslant x < \varrho$ and diverges for $x > \varrho$.

The criterion in Lemma 9 is not only sufficient but also necessary for the estimate $\sigma^k < \tau^k = O(s_k)$. More precisely:

Corollary 10 *For $\sigma > 0$, the following statements are equivalent:*

(a) $a(\frac{1}{\sigma}) > 1$ *(not excluding $a(\frac{1}{\sigma}) = \infty$).*

(b) $a_{<m}(\frac{1}{\sigma}) > 1$ *for some m.*

(c) $\tau^k = O(s_k)$ *for some $\tau > \sigma$.*

(d) *For some $\tau' > \sigma$, eventually $\sqrt[k]{s_k} > \tau'$.*

(e) $\limsup \sqrt[k]{s_k} > \sigma$.

Proof. (a)$\Longleftrightarrow$(b) is clear since $\sup_m a_{<m}(\frac{1}{\sigma}) = a(\frac{1}{\sigma})$ (at least improperly).

(b)$\Longrightarrow$(c) follows from Lemma 9.

For (c)$\Longrightarrow$(d), choose $\varepsilon > 0$ with $\tau^k \leqslant \varepsilon s_k$ for all k, a τ' with $\sigma < \tau' < \tau$, and finally an n with $\varepsilon < (\frac{\tau}{\tau'})^n$; then each $k \geqslant n$ satisfies $\tau \leqslant \sqrt[k]{\varepsilon s_k} < \frac{\tau}{\tau'}\sqrt[k]{s_k}$, hence $\tau' < \sqrt[k]{s_k}$.

(d)$\Longrightarrow$(e): $\limsup \sqrt[k]{s_k} \geqslant \tau' > \sigma$.

(e)$\Longrightarrow$(a): Since $\varrho_s = (\limsup \sqrt[k]{s_k})^{-1} < \sigma^{-1}$, there is an $x < \sigma^{-1}$ for which $s(x)$ diverges. Thus, it cannot happen that $a(x)$ converges to a value < 1, because otherwise $s(x) = (1 - a(x))^{-1}$ were convergent. It follows that $a(\frac{1}{\sigma}) > a(x) \geqslant 1$. $\square$

Interestingly, the implication (e)$\Longrightarrow$(d) shows that the limes superior of the values $\sqrt[k]{s_k}$ is in fact a proper limit:

Corollary 11 *$\sqrt[k]{s_k}$ converges to $\frac{1}{\varrho_s}$.*

As another consequence of Corollary 10, we get

Corollary 12 *If $a(\frac{1}{\sigma}) > 1$ for some $\sigma > 0$ then $\sigma^k = o(s_k)$.*

Now we derive *upper* exponential bounds for s_k from those for a_k.

Lemma 13 *Suppose there are constants $m \in \mathbb{N}$, $\gamma > 0$, and $\sigma > \alpha > 0$ such that*

(1) $a_k \leqslant \gamma\alpha^k$ *for $k \geqslant m$,*

(2) $a_{<m}(\frac{1}{\sigma}) + \gamma(\frac{\alpha}{\sigma})^m \frac{\sigma}{\sigma-\alpha} < 1$.

Then there is a τ with $\alpha < \tau < \sigma$ and

(3) $s_k = O(\tau^k)$, *a fortiori* $s_k = o(\sigma^k)$.

If, in addition,

(4) $\alpha^{m-1}(\sigma - \alpha)s_{<m}(\frac{1}{\alpha}) \leqslant s_m$

then there exists an integer n *with* $m \leqslant n < 2m$ *and*

(5) $s_k\tau^{-k} \leqslant s_n\tau^{-n}$ *for all* $k \geqslant m$.

Proof. By continuity, there is a τ with $\alpha < \tau < \sigma$ such that (2) holds for τ instead of σ. Put

$$\delta := \max\{\tfrac{\tau-\alpha}{\alpha}(\tfrac{\alpha}{\tau})^m s_{<m}(\tfrac{1}{\alpha}), s_j\tau^{-j} : m \leqslant j < 2m\}.$$

Then $s_j \leqslant \delta\tau^j$ for $m \leqslant j < 2m$. Consider a $k \geqslant 2m$ such that $s_j \leqslant \delta\tau^j$ for all j with $m \leqslant j < k$. Then, by (1),

$$\begin{aligned} s_k &= \sum_{j=0}^{k-1} a_{k-j}s_j \leqslant \gamma\sum_{j=0}^{m-1}\alpha^{k-j}s_j + \gamma\delta\sum_{j=m}^{k-m}\alpha^{k-j}\tau^j + \delta\sum_{j=k-m+1}^{k-1} a_{k-j}\tau^j \\ &= \gamma\alpha^k s_{<m}(\tfrac{1}{\alpha}) + \gamma\delta\alpha^m\tau^m\sum_{j=0}^{k-2m}\alpha^{k-2m-j}\tau^j + \delta\sum_{j=1}^{m-1} a_j\tau^{k-j} \\ &= \gamma\alpha^k s_{<m}(\tfrac{1}{\alpha}) + \gamma\delta\alpha^m\tau^m\frac{\tau^{k-2m+1}-\alpha^{k-2m+1}}{\tau-\alpha} + \delta\tau^k a_{<m}(\tfrac{1}{\tau}) \\ &= \gamma\alpha^k\left(s_{<m}(\tfrac{1}{\alpha}) - \delta(\tfrac{\tau}{\alpha})^m\tfrac{\alpha}{\tau-\alpha}\right) + \delta\tau^k\left(a_{<m}(\tfrac{1}{\tau}) + \gamma(\tfrac{\alpha}{\tau})^m\tfrac{\tau}{\tau-\alpha}\right) \\ &\leqslant \gamma\alpha^k\cdot 0 + \delta\tau^k\cdot 1 = \delta\tau^k, \end{aligned}$$

using (2) (with τ for σ) and the definition of δ. Thus $s_k = O(\tau^k)$ and $s_k = o(\sigma^k)$. Under hypothesis (4), we get $\delta = s_n\tau^{-n}$ for some n with $m \leqslant n < 2m$, and $s_k \leqslant \delta\tau^k = s_n\tau^{k-n}$ for $k \geqslant m$. ☐

Again, it is not hard to see that the bounds provided by Lemma 13 cannot be improved essentially:

Corollary 14 *Assume* $0 < \alpha < \sigma$ *and* $a_k = O(\alpha^k)$. *Then the following statements are equivalent:*

(a) $a(\frac{1}{\sigma}) < 1$.

(b) $a_{<m}(\frac{1}{\sigma}) < 1 - \gamma\left(\frac{\alpha}{\sigma}\right)^m\frac{\sigma}{\sigma-\alpha}$ *for some* m *and* $\gamma \geqslant \sup_{k\geqslant m}\frac{a_k}{\alpha^k}$.

(c) $s_k = O(\tau^k)$ *for some* $\tau < \sigma$.

(d) *For some* $\tau' < \sigma$, *eventually* $\sqrt[k]{s_k} < \tau'$.

(e) $\limsup \sqrt[k]{s_k} < \sigma$.

Proof. For (a)$\Longrightarrow$(b), first find some $\gamma > 0$ so that $a_k \leqslant \gamma\alpha^k$ for all k. As $\lim a_{<m}(\frac{1}{\sigma}) = a(\frac{1}{\sigma}) < 1$, there exists an n such that for all $m \geqslant n$, we have $a_{<m}(\frac{1}{\sigma}) < \frac{1}{2}(1 + a(\frac{1}{\sigma}))$. Now choose an $m \geqslant n$ with $\gamma(\frac{\alpha}{\sigma})^m \frac{\sigma}{\sigma-\alpha} \leqslant \frac{1}{2}(1 - a(\frac{1}{\sigma}))$. Then

$$a_{<m}(\tfrac{1}{\sigma}) < \tfrac{1}{2}(1 + a(\tfrac{1}{\sigma})) = 1 - \tfrac{1}{2}(1 - a(\tfrac{1}{\sigma})) \leqslant 1 - \gamma(\tfrac{\alpha}{\sigma})^m \tfrac{\sigma}{\sigma-\alpha}.$$

(b)$\Longrightarrow$(c) follows from Lemma 13.

(c)$\Longrightarrow$(d): Choose $\varepsilon > 0$ with $\tau^k \geqslant \varepsilon s_k$ for all k, then τ' with $\sigma > \tau' > \tau$, and finally n with $\varepsilon > (\frac{\tau}{\tau'})^n$. Then each $k \geqslant n$ satisfies $\tau \geqslant \sqrt[k]{\varepsilon s_k} > \frac{\tau}{\tau'} \sqrt[k]{s_k}$, hence $\tau' > \sqrt[k]{s_k}$.

(d)$\Longrightarrow$(e): $\limsup \sqrt[k]{s_k} \leqslant \tau' < \sigma$.

(e)$\Longrightarrow$(a): Cauchy-Hadamard gives $\frac{1}{\sigma} < \varrho_s \leqslant \varrho_a$, hence $s(\frac{1}{\sigma})(1 - a(\frac{1}{\sigma})) = 1$ and therefore $a(\frac{1}{\sigma}) < 1$. ☐

Corollary 15 *If* $0 < \varrho$ *then* $a(\varrho) \substack{< \\ = \\ >} 1 \Longleftrightarrow \varrho \substack{< \\ = \\ >} \varrho_s$.

The practical application of our lemmas is based on the following

Proposition 16 *Let* m_0 *be a natural number so that* $a_{k_i} \neq 0$ *for some* $k_1, \ldots, k_u < m_0$ *with* $\gcd(k_1, \ldots, k_u) = 1$ *and* $a_k \leqslant \gamma\alpha^k$ *for all* $k \geqslant m_0$ *(with* $\alpha, \gamma > 0$ *fixed). Let*

$$\underline{a}_m(x) = x^{m-1} - \sum_{j=0}^{m-2} a_{m-j-1} x^j \quad \textit{and}$$
$$\overline{a}_m(x) = \underline{a}_m(x)(x - \alpha) - \gamma\alpha^m.$$

Then, for each $m \geqslant m_0$*, there is a unique solution* $\underline{\sigma}_m > 0$ *of the equation* $\underline{a}_m(x) = 0$ *and a unique solution* $\overline{\sigma}_m > \alpha$ *of* $\overline{a}_m(x) = 0$*. Furthermore,*

$$\underline{\sigma}_m^{\,k} \leqslant s_k \leqslant \overline{\sigma}_m^{\,k} \textit{ eventually,} \quad \textit{and}$$
$$\underline{\sigma}_m \leqslant \underline{\sigma}_{m+1} \leqslant \lim \underline{\sigma}_k = \lim \sqrt[k]{s_k} = \tfrac{1}{\varrho_s} \leqslant \overline{\sigma}_{m+1} \leqslant \overline{\sigma}_m.$$

Proof. For $m \geqslant m_0$, the polynomial $a_{<m}(x)$ is not zero. The equation $x^{1-m}\underline{a}_m(x) = 1 - a_{<m}(\frac{1}{x}) = 0$ has a unique positive solution $\underline{\sigma}_m$, as $1 - a_{<m}(\frac{1}{x})$ is strictly increasing in x, with $1 - a_{<m}(\frac{1}{x}) \to 1$ for $x \to \infty$ and $1 - a_{<m}(\frac{1}{x}) \to -\infty$ for $x \to 0^+$. Of course, $\underline{\sigma}_m$ is then also the unique positive root of $\underline{a}_m(x)$. Moreover,

$$\underline{a}_{m+1}(x) = x^m(1 - a_{<m}(\tfrac{1}{x})) - a_m$$

yields $\underline{a}_{m+1}(\underline{\sigma}_m) = -a_m \leqslant 0$, and as $\underline{a}_{m+1}(x) \to \infty$ for $x \to \infty$, it follows that $\underline{\sigma}_m \leqslant \underline{\sigma}_{m+1}$. For $0 < \sigma < \underline{\sigma}_m$, we have $a(\frac{1}{\sigma}) > 1$, so that from Corollary 12 we know that $\sigma^k \leqslant s_k$ eventually. Hence, $\sigma \leqslant \limsup \sqrt[k]{s_k} = \frac{1}{\varrho_s}$ and, taking the limit $\sigma \to \underline{\sigma}_m$, also $\underline{\sigma}_m \leqslant \frac{1}{\varrho_s}$. For the equation $\lim \underline{\sigma}_k = \frac{1}{\varrho_s}$, it remains to show that $s(x)$ converges for

$x < x_0 := (\lim \underline{\sigma}_k)^{-1} = \inf \frac{1}{\underline{\sigma}_k}$. Since $a_{<m}(x_0) \leqslant a_{<m}(\frac{1}{\underline{\sigma}_m}) = 1$ for all m and $a(x)$ is strictly increasing, we have $a(x) < a(x_0) \leqslant 1$, so that $s(x) = \sum a(x)^n$ converges to $(1 - a(x))^{-1}$.

Similarly, for $x > \alpha$, the function

$$h_m(x) = \frac{\overline{a}_m(x)}{x^m - \alpha x^{m-1}} = 1 - a_{<m}(\tfrac{1}{x}) - \gamma \left(\frac{\alpha}{x}\right)^m \frac{x}{x - \alpha}$$

is strictly increasing because $a_{<m}(\frac{1}{x})$, $\gamma(\frac{\alpha}{x})^m$ and $\frac{x}{x-\alpha}$ are strictly decreasing functions. As $h_m(x) \to -\infty$ for $x \to \alpha^+$ and $h_m(x) \to 1$ for $x \to \infty$, there is a unique solution $\overline{\sigma}_m > \alpha$ of $h_m(x) = 0$, and this is also the unique solution of $\overline{a}_m(x) = 0$ $(x > \alpha)$. Furthermore, substitution of $\overline{\sigma}_m$ for x in the equation

$$\begin{aligned} \overline{a}_{m+1}(x) &= x^m(1 - a_{<m}(\tfrac{1}{x}) - a_m x^{-m})(x - \alpha) - \gamma\alpha^{m+1} \\ &= x\overline{a}_m(x) + (\gamma\alpha^m - a_m)(x - \alpha) \end{aligned}$$

gives $\overline{a}_{m+1}(\overline{\sigma}_m) = (\gamma\alpha^m - a_m)(\overline{\sigma}_m - \alpha) \geqslant 0$ because of $a_m \leqslant \gamma\alpha^m$. As before, we conclude that $\overline{\sigma}_{m+1} \leqslant \overline{\sigma}_m$ since $\overline{a}_{m+1}(\alpha) = -\gamma\alpha^{m+1} < 0$. Now $\overline{a}_m(x) \to \infty$ for $x \to \infty$ implies for $\sigma > \overline{\sigma}_m$:

$$\begin{aligned} 0 < \overline{a}_m(\sigma) &= \underline{a}_m(\sigma)(\sigma - \alpha) - \gamma\alpha^m \\ &= \sigma^{m-1}(1 - a_{<m}(\tfrac{1}{\sigma}))(\sigma - \alpha) - \gamma\alpha^m, \end{aligned}$$

and as α lies between 0 and σ, the previous inequality is equivalent to (2) in Lemma 13, whence $s_k \leqslant \sigma^k$ eventually. Thus $\frac{1}{\varrho_s} = \limsup \sqrt[k]{s_k} \leqslant \sigma$ and finally also $\frac{1}{\varrho_s} \leqslant \overline{\sigma}_m$. $\square$

In all, we see that full information about the coefficients a_j $(j < m)$ provides a two-sided asymptotical estimate

$$\underline{\sigma}_m < \lim \sqrt[k]{s_k} < \overline{\sigma}_m.$$

If the numbers a_j are known even for $j < 2m$ then so are the numbers s_j, and one obtains from the proofs of Lemmas 9 and 13 concrete estimates

$$\begin{aligned} &\underline{\delta}_m \underline{\sigma}_m^{\,k} \leqslant s_k \leqslant \overline{\delta}_m \overline{\sigma}_m^{\,k} \text{ for } k \geqslant m, \text{ with} \\ &\qquad \underline{\delta}_m = \min\{s_j \underline{\sigma}_m^{-j} : m \leqslant j < 2m\}, \\ &\qquad \overline{\delta}_m = \max\{s_j \overline{\sigma}_m^{-j} : m \leqslant j < 2m\}, \end{aligned}$$

the upper bound requiring that $\alpha^{m-1}(\overline{\sigma}_m - \alpha)s_{<m}(\frac{1}{\alpha}) \leqslant s_m$. Note that, for the upper bound, the hardest part may often be to determine α and γ so that $a_k \leqslant \gamma\alpha^k$ at least for all $k \geqslant m$.

5 Distributive lattices with less than 50 elements

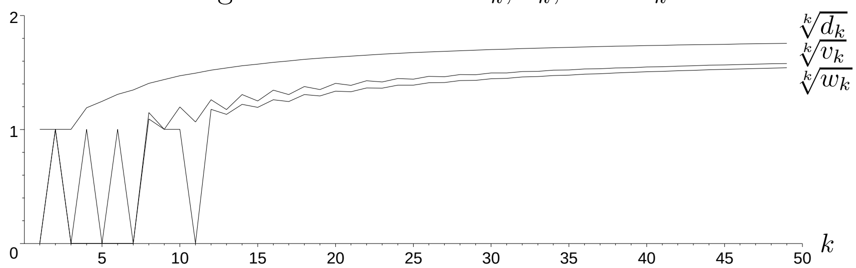
For efficient applications of the theory developed in the previous sections, one needs sufficiently many of the numbers v_k and w_k for small k. We determined v_k for $k \leqslant 49$ with the orderly algorithm described in [8, 9]. The numbers d_k and w_k are then obtained by Corollaries 2 and 7, the results are shown in Table 1 and Figure 3. Seeking a good fit, one may approximate these values in the following form:

Proposition 17 *For $k < 50$,*

$$\begin{array}{rcll} d_k &=& 1.8439^{k-4}(1+\delta_k), & 0 \leqslant \delta_k < (7/k)^3, \\ v_k &=& 1.7250^{k-8}(1+(-1)^k\gamma_k), & 0 \leqslant \gamma_k < (14/k)^3, \\ w_k &=& 1.6765^{k-8}(1-\beta_k), & 0 \leqslant \beta_k < (23/k)^3, \end{array}$$

except for $k \in \{2, 8\}$ in the third case.

Figure 3: Growth of d_k, v_k, and w_k



6 Lower and upper bounds for v_k and d_k

We are now going to apply the general results established in Section 4 to the two cases that concern us here, viz.

(1) $a_k = v_{k+1}$, the number of all ordinally indecomposable posets with k nonempty antichains, or, equivalently, the number of all vertically indecomposable distributive lattices with $k+1$ elements, and

$s_k = d_{k+1}$, the number of all posets with k nonempty antichains, respectively, of all distributive lattices with $k+1$ elements.

(2) $a_k = w_k$, the number of all canonically 2-indecomposable posets with k antichains, and

$s_k = v_k$, the number of all ordinally indecomposable posets with k antichains.

Before we turn to numerical evaluation, let us note a few qualitative results that do not require any concrete calculation of the involved numbers.

We know that, in both cases, (s_k) is the summatorial sequence of (a_k). Thus, $s(x) = (1-a(x))^{-1}$, and as $a(x) = xv(x)$ and $d(x) = 1 + xs(x)$ in the first case, we get

$$d(x) = 1 + \frac{x}{1 - xv(x)},$$

and in the second case,

$$v(x) = \frac{1}{1 - w(x)}.$$

Table 1: Numbers of [vertically indecomposable / canonically 2-indecomposable] distributive lattices

k	d_k	v_k	w_k
1	1	0	0
2	1	1	1
3	1	0	0
4	2	1	0
5	3	0	0
6	5	1	0
7	8	0	0
8	15	3	2
9	26	1	1
10	47	6	1
11	82	2	0
12	151	16	7
13	269	8	5
14	494	42	16
15	891	28	14
16	1 639	112	40
17	2 978	93	41
18	5 483	311	120
19	10 006	295	131
20	18 428	869	321
21	33 749	939	402
22	62 162	2 454	901
23	114 083	2 931	1 210
24	210 189	7 032	2 590
25	386 292	9 036	3 621
26	711 811	20 301	7 371
27	1 309 475	27 701	10 841
28	2 413 144	58 929	21 178
29	4 442 221	84 413	32 222
30	8 186 962	172 104	61 273
31	15 077 454	255 919	95 408
32	27 789 108	504 637	177 384
33	51 193 086	773 511	282 405
34	94 357 143	1 484 392	515 174
35	173 859 936	2 331 180	833 295
36	320 462 062	4 378 773	1 500 030
37	590 555 664	7 009 288	2 455 337
38	1 088 548 290	12 944 347	4 372 535
39	2 006 193 418	21 039 961	7 229 231
40	3 697 997 558	38 328 890	12 761 691
41	6 815 841 849	63 067 623	21 260 746
42	12 563 729 268	113 651 785	37 286 778
43	23 157 428 823	188 831 922	62 483 221
44	42 686 759 863	337 361 112	109 014 426
45	78 682 454 720	564 890 985	183 542 099
46	145 038 561 665	1 002 268 019	318 906 720
47	267 348 052 028	1 688 673 026	538 889 399
48	492 815 778 109	2 979 703 035	933 361 886
49	908 414 736 485	5 045 200 597	1 581 666 042

Theorem 18 *For $k \geqslant 8$, there are at least 1.678^{k-10} many unlabeled vertically indecomposable distributive lattices of size k and dimension $\leqslant 3$.*

Proof. We inductively define systems $\mathcal{V}_k$ of k-element subsets of the distributive lattice $(\omega, \leqslant)^3$. Let $[a, b] = \{a, a+1, \ldots, b\}$, and write (x_A, y_A, z_A) for the pointwise maximum of a subset $A \subseteq \omega^3$ (if it exists). Put

$$\mathcal{V}_6 = \{[0,1] \times [0,2] \times [0,0], [0,2] \times [0,1] \times [0,0]\},$$

and $\mathcal{V}_k = \emptyset$ for $k \in \{1,2,3,4,5,7\}$. For $k \geqslant 8$, let $\mathcal{V}_k$ be the smallest system such that

$$A^{(\xi,\eta,\zeta)} = A \cup ([x_A - 1, x_A + \xi] \times [y_A - 1, y_A + \eta] \times [z_A, z_A + \zeta]) \in \mathcal{V}_k$$

whenever (i) $\xi, \eta, \zeta \in \omega$, (ii) $(\xi + \eta)(\eta + \zeta)(\zeta + \xi) > 0$ or $\xi + \eta + \zeta = 1$, and (iii) $A \in \mathcal{V}_{k-\delta}$, where $\delta = (\xi + 2)(\eta + 2)(\zeta + 1) - 4$. In other words, we construct larger subsets from smaller ones by replacing the top square $[x_A - 1, x_A] \times [y_A - 1, y_A] \times \{z_A\}$ with some larger cuboid. As is easily seen, condition (ii) assures that, for each $A' \in \mathcal{V}_k$, there is exactly one quadruple (A, ξ, η, ζ) with $A' = A^{(\xi,\eta,\zeta)}$. By construction, each $A \in \mathcal{V}_k$ is a sublattice of $(\omega, \leqslant)^3$, hence distributive (cf. Section 2). The unique other lattice $A' \in \mathcal{V}_k$ that is isomorphic to A is the lattice $A' = \{(y, x, z) : (x, y, z) \in A\}$. For $\delta = 2$, 4, 5, 8, 11, or 12, there are exactly 2, 1, 1, 4, 2, or 6 possibilities for (ξ, η, ζ), respectively, so that

$$|\mathcal{V}_k| \geqslant 2|\mathcal{V}_{k-2}| + |\mathcal{V}_{k-4}| + |\mathcal{V}_{k-5}| + 4|\mathcal{V}_{k-8}| + 2|\mathcal{V}_{k-11}| + 6|\mathcal{V}_{k-12}|$$

for $k \geqslant 13$, and $|\mathcal{V}_k| \geqslant 2 \cdot 1.678^{k-10}$ for $14 \leqslant k \leqslant 25$. Hence also $|\mathcal{V}_k| \geqslant 2 \cdot 1.678^{k-10}$ for $k \geqslant 26$. For $8 \leqslant k \leqslant 13$, the proposition is verified directly. □

The representation of an isomorphism type by an increasing sequence $0 = z_1 \leqslant z_2 \leqslant \cdots \leqslant z_n \leqslant k - 2$ (for $k \geqslant 2$) instantly provides us with an exponential *upper* bound on d_k. Making the sequences strictly increasing,

$$1 \leqslant z_2 + 1 < z_3 + 2 < \cdots < z_n + (n-1) \leqslant (k-2) + (n-1),$$

we get

$$d_k \leqslant \sum_{n=1}^{k} \binom{k+n-3}{n-1} = \binom{2k-2}{k-1} < 4^{k-1}.$$

One can improve this upper bound by considering vertically indecomposable lattices first. Such lattices don't have "knots", i.e. nonextremal elements comparable to all other elements; thus, each step of the doubling construction must give at least two new elements. Hence, $2n \leqslant k$, $z_2 = 0$, and $z_n \leqslant k - 4$. Therefore, putting $\ell = \lfloor k/2 \rfloor$, v_k satisfies

$$v_k \leqslant \sum_{n=2}^{\ell} \binom{k+n-6}{n-2} = \binom{k+\ell-5}{\ell-2} \quad (k \geqslant 3).$$

This easily gives the following exponential bound:

$$v_k \leqslant \frac{\alpha^k}{25\sqrt{k}} = o(\alpha^k) \quad \text{for } \alpha = \frac{3}{2}\sqrt{3} < 2.6,\ k \neq 2.$$

But we can do better:

Theorem 19 *The numbers $v_{\leqslant k}$ of vertically indecomposable distributive lattices with at most k elements satisfy the inequalities*

$$v_k \leqslant v_{\leqslant k} \leqslant \sum_{t=1}^{\lfloor k/2 \rfloor - 1} \binom{k-4}{t-1} \binom{\lfloor k/4 + t/2 \rfloor}{t} < 2.33^{k-4},$$

and $v_{\leqslant k} = o(2.33^k)$.

Proof. We know that the vertically indecomposable distributive lattices of height n and size $\leqslant k$ may be coded by certain integer sequences $(z_1, \ldots, z_n)$ with

$$0 = z_1 = z_2 \leqslant \cdots \leqslant z_n \leqslant k - 4.$$

Moreover, if $z_i = z_{i+1}$ then the interval doubled at step $i > 1$ is doubled again at step $i+1$, so that at least four elements must be added in the latter case. (More generally, if $z_i = z_{i+1} = \cdots = z_{i+r}$ then at least 2^{j+1} elements have to be added at step $i+j$, $j \leqslant r$). Denoting by s the number of indices i with $z_i = z_{i+1}$, we finally have generated at least $2 + 2(n-1) + 2s$ elements, i.e.

$$2n + 2s \leqslant k. \quad (\star)$$

There are $\binom{n-2}{s}$ possibilities to choose s places i with $1 < i < n$ and $z_i = z_i + 1$. For the remaining $t = n - 1 - s$ indices $i_1 < \cdots < i_t$, there are $\binom{k-4}{t-1}$ many strictly increasing sequences $0 = z_{i_1} < \cdots < z_{i_t} \leqslant k - 4$. By $(\star)$, we have the inequalities $t < n \leqslant \frac{k}{2} - s = \frac{k}{2} - n + 1 + t$, hence

$$0 < t < n \leqslant \frac{k}{4} + \frac{t}{2} + \frac{1}{2}.$$

In all, this gives

$$\begin{aligned}
v_k \leqslant v_{\leqslant k} &\leqslant \sum_{t=1}^{\lfloor k/2 \rfloor - 1} \binom{k-4}{t-1} \sum_{n=t+1}^{\lfloor k/4 + t/2 + 1 \rfloor} \binom{n-2}{n-t-1} \\
&= \sum_{t=1}^{\lfloor k/2 \rfloor - 1} \binom{k-4}{t-1} \binom{\lfloor k/4 + t/2 \rfloor}{t} \\
&\leqslant \sum_{t=1}^{\lfloor k/2 \rfloor - 1} \frac{1}{2} \binom{k}{t} \binom{\lfloor k/4 + t/2 \rfloor}{t} \\
&< \frac{k}{4} \max \left\{ \binom{k}{t} \binom{\lfloor k/4 + t/2 \rfloor}{t} : 1 \leqslant t \leqslant \frac{k}{2} - 1 \right\}.
\end{aligned}$$

Now, the inequalities

$$\binom{\lfloor \alpha t\rfloor}{t} \leqslant \min\left\{1, \sqrt{\frac{\alpha}{2\pi(\alpha-1)t}}\right\}(\alpha^\alpha(\alpha-1)^{1-\alpha})^t,$$

which follow from the known estimate $n! = (2\pi n)^{-1/2}(\frac{n}{e})^n e^{\frac{1}{12n+\varepsilon}}$ with $0 \leqslant \varepsilon \leqslant \frac{1}{4}$ (cf. [10], p. 355), yield for $\alpha = \frac{k}{t} > 2$:

$$\binom{k}{t}\binom{\lfloor k/4+t/2\rfloor}{t} \leqslant \min\left\{1, \frac{g(\alpha)}{k}\right\} f(\alpha)^k$$

with

$$\begin{aligned} f(\alpha) &= \left(\alpha^\alpha(\alpha-1)^{1-\alpha}\left(\frac{\alpha}{4}+\frac{1}{2}\right)^{\frac{\alpha}{4}+\frac{1}{2}}\left(\frac{\alpha}{4}-\frac{1}{2}\right)^{\frac{1}{2}-\frac{\alpha}{4}}\right)^{\frac{1}{\alpha}} \\ &= \alpha(\alpha-1)^{\frac{1}{\alpha}-1}\left(\frac{\alpha}{4}+\frac{1}{2}\right)^{\frac{1}{2\alpha}+\frac{1}{4}}\left(\frac{\alpha}{4}-\frac{1}{2}\right)^{\frac{1}{2\alpha}-\frac{1}{4}} \quad \text{and} \\ g(\alpha) &= \alpha\sqrt{\frac{\alpha}{2\pi(\alpha-1)}}\sqrt{\frac{\alpha+2}{2\pi(\alpha-2)}} = \frac{\alpha}{2\pi}\sqrt{\frac{\alpha(\alpha+2)}{(\alpha-1)(\alpha-2)}}. \end{aligned}$$

Numerical evaluation yields

$$f(\alpha) < 2.3295 < 2.33 \quad \text{for all } \alpha > 2,$$

providing already the asymptotical result

$$v_{\leqslant k} = o(2.33^k).$$

To obtain the explicit estimate $v_{\leqslant k} \leqslant 2.33^{k-4}$ for all k, one has to be more careful. Putting

$$\begin{aligned} f_l &:= f(0.27^{-1}) = \max\{f(\alpha) : \alpha \geqslant 0.27^{-1}\}, \\ f_r &:= f(0.41^{-1}) = \max\{f(\alpha) : 2 \leqslant \alpha \leqslant 0.41^{-1}\}, \\ \overline{f}_n &:= \max\{f(\alpha) : \tfrac{100}{n+1} \leqslant \alpha \leqslant \tfrac{100}{n}\} \quad (n < 50), \\ \overline{g}_n &:= \max\{g(\alpha) : \tfrac{100}{n+1} \leqslant \alpha \leqslant \tfrac{100}{n}\} \quad (n < 50), \end{aligned}$$

we get

$$\begin{aligned} v_{\leqslant k} &\leqslant \frac{1}{2}\left(\sum_{t<0.27k} {f_l}^k + \sum_{n=27}^{41} \sum_{\frac{nk}{100}\leqslant t\leqslant\frac{(n+1)k}{100}} \overline{f}_n^{\,k}\, \overline{g}_n\, k^{-1} + \sum_{0.42k\leqslant t<0.5k} {f_r}^k\right) \\ &\leqslant 0.135k \cdot 2.267^k + \frac{1}{200}\sum_{n=27}^{41} \overline{f}_n^{\,k}\, \overline{g}_n + 0.04k \cdot 2.262^k. \end{aligned}$$

Hence, for $k \geqslant 400$,

$$\begin{aligned} 2.33^{-k} v_{\leqslant k} &\leqslant 54 \left(\tfrac{2.267}{2.33}\right)^{400} + \tfrac{1}{200} \sum_{n=27}^{41} \left(\tfrac{\overline{f}_n}{2.33}\right)^{400} \cdot \overline{g}_n + 16 \left(\tfrac{2.262}{2.33}\right)^{400} \\ &\leqslant 0.001 + 0.031 + 0.001 = 0.033 < 2.33^{-4}. \end{aligned}$$

For $k < 400$, the upper bound 2.33^{k-4} may be checked directly. □

With more effort, this upper bound can be improved considerably (at least to 2.28^k) by taking into account the remark about the increment 2^{j+1} at step $i+j$, which restricts the possibilities for the coding sequences enormously. However, even 2.28^k seems to be a very rough upper bound, since $v_k < 1.8^k$ for all $k < 50$ (see also Proposition 17). Nevertheless, we shall need the bound 2.33^{k-4} (not only 2.33^k) below for an estimate of the d_k. We apply Proposition 16 to the following data:

(1) $a_k = v_{k+1}$, $m_0 = 2$ $(v_2 = 1)$, $\alpha = 2.33$, and $\gamma = 2.33^{-4}$.

(2) $a_k = w_k$, $m_0 = 10$ $(w_2 = w_9 = 1, \gcd(2,9) = 1)$.

In case (1), we get:

Theorem 20 *Let m be a fixed natural number, and denote by $\underline{\sigma}_m$ and $\overline{\sigma}_m$ the unique positive roots of the equations*

$$x^{m-1} - \sum_{j=2}^{m} v_j x^{m-j} = 0 \quad \textit{resp.} \quad (x^{m-1} - \sum_{j=2}^{m} v_j x^{m-j})(x - \alpha) = \gamma \alpha^m.$$

Then $\underline{\sigma}_m^k = O(d_k)$, $d_k = O(\overline{\sigma}_m^k)$, and $\underline{\sigma}_m \leqslant \sqrt[k]{d_k} \leqslant \overline{\sigma}_m$ eventually. Moreover, $\lim \sqrt[k]{s_k} = \lim \underline{\sigma}_k$.

More explicitly, put

$$\begin{aligned} \underline{\delta}_m &= \min\{d_j \underline{\sigma}_m^{-j} : m < j \leqslant 2m\} \quad \text{and} \\ \overline{\delta}_m &= \max\{d_j \overline{\sigma}_m^{-j} : m < j \leqslant 2m\}. \end{aligned}$$

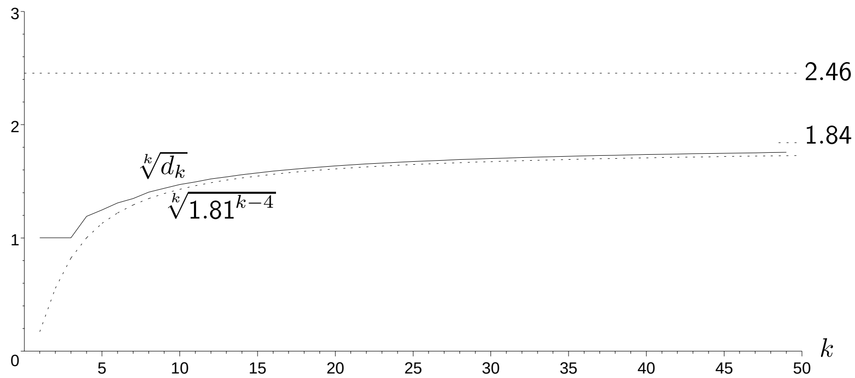
Then $d_k \geqslant \underline{\delta}_m \underline{\sigma}_m^k$ for $k \geqslant m$, and under the proviso that

$$d_{m+1} \geqslant \alpha^{m-1}(\overline{\sigma}_m - \alpha) \sum_{j=1}^{m} d_j \alpha^{-j+1}$$

(which holds for $m \leqslant 4$), $d_k \leqslant \overline{\delta}_m \overline{\sigma}_m^k$. Numerical evaluation yields

$$\begin{array}{ll} \underline{\sigma}_{24} > 1.81,\ \underline{\delta}_{24} > \tfrac{1}{4}, & \text{hence } d_k > \tfrac{1}{4} 1.81^k \text{ for } k \geqslant 24, \\ \underline{\sigma}_{49} > 1.8388, & \\ \underline{\sigma}_{60} > 1.84 \text{ (using } v_k \geqslant \sum_{j=k-49}^{k-1} w_{k-j} v_j \text{ for } k \geqslant 50), & \\ \overline{\sigma}_4 < 2.46,\ \overline{\delta}_4 < 0.34, & \text{hence } d_k < 0.34 \cdot 2.46^k, \\ \overline{\sigma}_{49} < 2.385 < 2.39, & \text{hence } d_k = o(2.39^k). \end{array}$$

Figure 4: Exponential bounds on d_k



Corollary 21 *(Fig. 4)*
$1.81^{k-4} < d_k < 2.46^{k-1}$ *for all* k*, and* $1.84^k < d_k < 2.39^k$ *eventually.*

Similarly, in case (2) we obtain

Theorem 22 *Let* $\underline{\tau}_m$ *denote the unique positive root of the equation*

$$x^{m-1} - \sum_{j=0}^{m-2} w_{m-j-1} x^j.$$

and put $\nu_m = \min\{v_j \underline{\tau}_m^{-j} : m \leqslant j < 2m\}$*. Then* $v_k \geqslant \nu_m \underline{\tau}_m^k$ *for* $k \geqslant m$*. Hence, for fixed* m*,* $\underline{\tau}_m^k = O(v_k)$ *and* $\sqrt[k]{v_k} \geqslant \underline{\tau}_m$ *eventually.*

Numerical evaluation yields

$$\begin{array}{ll} \underline{\tau}_{24} > 1.54, & \text{hence } v_k \geqslant v_{24} \cdot 1.54^{k-24} \text{ for } k \geqslant 26, \\ \underline{\tau}_{50} > 1.66 > \underline{\tau}_{49}, & \text{hence } 1.66^k = o(v_k) \end{array}$$

Corollary 23 *(Fig. 5)*
$v_k \geqslant \lfloor 1.5^{k-9} \rfloor$ *for all* k*,* $v_k \geqslant 1.54^{k-4}$ *for* $k \geqslant 24$*, and* $v_k \geqslant 1.66^k$ *eventually.*

As we see, the base of the lower bound obtained here does not exceed the one from Theorem 18. However, the w_k will probably serve for better bounds when more numerical material will be known.

Corollary 24 $\left(\frac{5}{3}\right)^{k-10} \leqslant v_k \leqslant \left(\frac{7}{3}\right)^{k-4}$ *for all* $k \geqslant 8$*.*

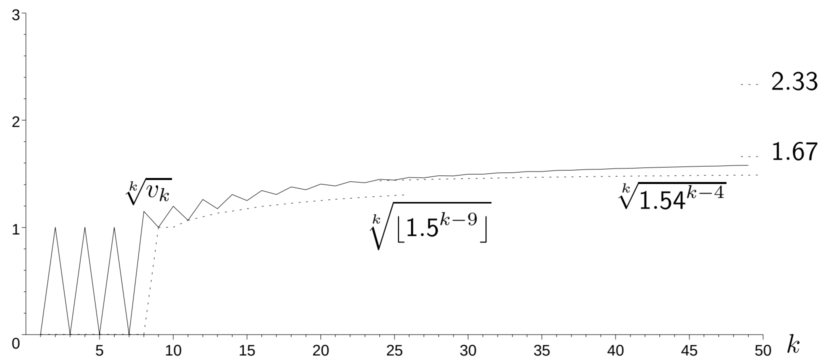
We conclude with some open questions:

(1) Is it true that $v_k = o(d_k)$ and $w_k = o(v_k)$?

(2) Can one even show that $\lim \sqrt[k]{w_k} < \lim \sqrt[k]{v_k} < \lim \sqrt[k]{d_k}$?

(3) How far can the upper bound be improved, e.g., is $d_k \leqslant 2^k$?

Figure 5: Exponential bounds on v_k



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