ON THE NUMBER OF FIELD TOPOLOGIES
ON AN INFINITE FIELD

JOHN O. KILTINEN

ABSTRACT. K.-P. Podewski has recently proven that every count-able infinite field admits $2^{2^{\aleph_0}}$ different field topologies. Using methods of valuation theory, it is proven that every uncountable field, and more generally, every field F of infinite transcendence degree over some subfield, admits $2^{2^{|F|}}$ field topologies. By purely set theoretic considerations, it then follows that there are $2^{2^{|F|}}$ field topologies on any infinite field F, no two of which are topologically isomorphic. This latter result is then generalized to any infinite commutative ring without proper zero-divisors. A further aspect of Podewski's work on countable fields is generalized in a final theorem which states that a field F of infinite transcendence degree admits $2^{2^{|F|}}$ field topologies which fail to be suprema of locally bounded ring topologies.

1. **Introduction.** This paper extends to the uncountable case some recent results of K.-P. Podewski [7] on the number of field topologies on a countable infinite field. One then has that any infinite field F admits $2^{2^{|F|}}$ field topologies, the maximum number possible.

Our methods of proof follow Podewski's in identifying a class of $2^{|F|}$ topologies on F which are interrelated so that the suprema of distinct subclasses are distinct. However, for uncountable fields, and more generally for fields of infinite transcendence degree over a subfield, one is able to use a class of $2^{|F|}$ valuation topologies,[1] and the details of the proof become much easier than in the countable case. Valuation topologies are clearly not directly applicable to obtaining $2^{2^{|F|}}$ topologies which are not suprema of locally bounded ones, so to do this, we turn in §3 to the inductive ring topologies developed in [4], [5], and [6].

We state here two requisite known theorems—one from valuation theory and the other from set theory. We will consider valuations as mapping into the ordered multiplicative group of positive real numbers, with zero adjoined.

THEOREM 1.1. *Let B be a transcendence basis over a subfield for a field F. Then for any function f from B into R_+^*, the positive real numbers, there is a valuation v_f on F such that $v_f(x)=f(x)$ for every x in B.*

This theorem can be proven by transfinite induction using standard methods from valuation theory found in a reference such as [8].

DEFINITION. Let X be a set and let $\mathscr{S}$ be a subset of $\mathscr{P}(X)$, the power set of X. We will say that $\mathscr{S}$ has the *finite intersection property with complements* (f.i.p.c.) if for each pair of natural numbers m and n and each pairwise distinct collection of sets $S_1, S_2, \cdots, S_m, T_1, T_2, \cdots, T_n$ from $\mathscr{S}$, the set $(\bigcap_{i=1}^m S_i) \cap (\bigcap_{j=1}^n T_j^c)$ is not empty. Here T_j^c denotes the complement of T_j in X.

THEOREM 1.2 (HAUSDORFF [2]). *If X is an infinite set, then there exists a subset $\mathscr{S}$ of $\mathscr{P}(X)$ such that $\mathscr{S}$ has the f.i.p.c. and $|\mathscr{S}|=2^{|X|}$.*

2. The main results.

THEOREM 2.1. *Let F be an infinite field of transcendence degree $|F|$ over a subfield K. Then there are $2^{2^{|F|}}$ distinct field topologies on F, each of which is the supremum of a family of valuation topologies.*

PROOF. Let B be a transcendence basis for F over K, and let X be an index set such that $|X|=|B|$. Since B is infinite, we may choose countably many subsets B_i of B such that $B=\bigcup_{i=1}^\infty B_i$, $B_i \cap B_j=\varnothing$ if $i \neq j$, and $|B_i|=|X|$ for each i. For each i, let $\phi_i: X \to B_i$ be a bijection. Let $\mathscr{S}$ be a subset of $\mathscr{P}(X)$ with the f.i.p.c. such that $|\mathscr{S}|=2^{|X|}$.

Now for each $S \in \mathscr{S}$, let $f_S: B \to R_+^*$ be defined by

$$f_S(x) = 1/n \quad \text{if } x \in \phi_n(S),$$
$$= 1 \quad \text{otherwise.}$$

Then by Theorem 1.1, there is a valuation v_S on F which agrees on B with f_S. Let the topology on F induced by v_S be denoted by $\mathscr{T}_S$. Note that for any α in X, the sequence $(\phi_n(\alpha))$ in B converges to zero in $\mathscr{T}_S$ if $\alpha \in S$ and is bounded from zero otherwise.

We now let $\mathscr{T}_\mathscr{O}$ denote the supremum of the family $\{\mathscr{T}_S | S \in \mathscr{O}\}$ for each subset $\mathscr{O}$ of $\mathscr{S}$. We will show that each of these $2^{2^{|F|}}$ field topologies $\mathscr{T}_\mathscr{O}$ on F is distinct.

To see this, suppose $\mathscr{O}_i \subseteq \mathscr{S}$ for $i=1, 2$, and that $\mathscr{O}_1 \neq \mathscr{O}_2$. Let us suppose

in particular that there is a subset S_0 of X such that $S_0 \in \mathcal{O}_1$ and $S_0 \notin \mathcal{O}_2$. It will follow that $\mathscr{T}_{\mathcal{O}_1} \not\subseteq \mathscr{T}_{\mathcal{O}_2}$, and hence $\mathscr{T}_{\mathcal{O}_1} \neq \mathscr{T}_{\mathcal{O}_2}$.

Suppose to the contrary that $\mathscr{T}_{\mathcal{O}_1} \subseteq \mathscr{T}_{\mathcal{O}_2}$. Let $U = \{x \in F | v_{S_0}(x) < \tfrac{1}{2}\}$. Then U is a $\mathscr{T}_{S_0}$-, and hence also a $\mathscr{T}_{\mathcal{O}_i}$-neighborhood of zero, for $i = 1$ and 2. Hence, there are sets $S_1, S_2, \cdots, S_n$ in $\mathcal{O}_2$ and $\varepsilon > 0$ such that $\bigcap_{i=1}^{n} V_i \subseteq U$, where $V_i = \{x \in F | v_{S_i}(x) < \varepsilon\}$. Let m be an integer such that $1/m < \varepsilon$.

Now since $\mathscr{S}$ has the f.i.p.c., there is an element α in $(\bigcap_{i=1}^{n} S_i) \cap S_0^c$. For each i, $1 \leq i \leq n$, $\phi_m(\alpha) \in \phi_m(S_i)$, so $v_{S_i}(\phi_m(\alpha)) = 1/m$. Thus,

$$\phi_m(\alpha) \in \bigcap_{i=1}^{n} V_i.$$

However, since $\alpha \notin S_0$, $\phi_m(\alpha) \notin \phi_m(S_0)$, so $v_{S_0}(\phi_m(\alpha)) = 1$, and hence $\phi_m(\alpha) \notin U$. This is a contradiction, so we conclude that $\mathscr{T}_{\mathcal{O}_1} \not\subseteq \mathscr{T}_{\mathcal{O}_2}$.

Since an uncountable field F satisfies the conditions of Theorem 2.1, taking K to be the prime subfield, from Podewski's result and the above theorem, we have the following

COROLLARY 2.2. *Every infinite field F admits $2^{2^{|F|}}$ distinct field topologies.*

We note that in the proof of Theorem 2.1, the topologies $\mathscr{T}_{\mathcal{O}}$ distinguished themselves from each other by their behavior on B, and thus, their restrictions to any subring of F which contains B will remain distinct. This suggests the following generalization.

THEOREM 2.3. *If R is any infinite commutative ring without proper zero-divisors, then there are $2^{2^{|R|}}$ distinct ring topologies on R for which (if R has an identity element) multiplicative inversion is continuous everywhere that it is defined.*

PROOF. Let F denote a quotient field of R, let K be the prime subfield of F. It is easily seen that there exists a transcendence basis B for F over K such that $B \subseteq R$. If B is infinite (in particular, if R is uncountable) then as was observed above, the $2^{2^{|F|}} = 2^{2^{|R|}}$ distinct topologies on F presented in Theorem 2.1 have distinct restrictions to R. These relative topologies are, of course, ring topologies for which multiplicative inversion is continuous where it is defined.

The proof for the case where R and F are countable is based on the fact that the $2^{2^{\aleph_0}}$ topologies which Podewski produces on F actually distinguish themselves from each other by their behavior on a certain inductively defined set. (The set is the union over all $n \in N$ and all $f \in {}^n 2$ of $p^f(0, 1) \cap (\bigcap \{p^g(n, 0) | g \in {}^n 2$ and $g \neq f\}$. See the development after Theorem 12 in [7].) At the nth stage in the definition of this set, one chooses finitely many elements, subject to certain restrictions keeping

them out of certain finite subsets of F. Let us now impose the further restriction that these elements be chosen from R. It then follows that the $2^{2^{\aleph_0}}$ distinct topologies on F remain distinct when relativized to R.

We conclude this section with the observation that any infinite field F admits $2^{2^{|F|}}$ field topologies which are *really* distinct from each other in the sense of not being topologically isomorphic. This follows from Corollary 2.2 via purely set-theoretic considerations.

THEOREM 2.4. *If F is an infinite field, then there is a set Σ of $2^{2^{|F|}}$ field topologies on F such that if $\mathcal{T}_1$ and $\mathcal{T}_2$ are distinct members of Σ, then $(F, \mathcal{T}_1)$ and $(F, \mathcal{T}_2)$ are not topologically isomorphic. The analogous statement for commutative rings without proper zero-divisors is also true.*

PROOF. Let Σ' be the family of all field topologies on F. Then, by Corollary 2.2, $|\Sigma'| = 2^{2^{|F|}}$. Let Σ be a set which contains one topology out of each equivalence class on Σ' for the equivalence relation $\mathcal{T}_1 \sim \mathcal{T}_2$ iff $(F, \mathcal{T}_1)$ is topologically isomorphic to $(F, \mathcal{T}_2)$. Since there are at most $|F^F| = 2^{|F|}$ (see [9, p. 417]) field automorphisms of F, each equivalence class has at most $2^{|F|}$ elements in it. But then $2^{2^{|F|}} = |\Sigma'| \leq 2^{|F|} \cdot |\Sigma|$. The only way this cardinal inequality can be satisfied is if $2^{2^{|F|}} \leq |\Sigma|$. Clearly then $|\Sigma| = 2^{2^{|F|}}$.

3. **Suprema of nonlocally bounded topologies.** The $2^{2^{|F|}}$ field topologies produced on F in §2 are suprema of valuation topologies, which are, of course, locally bounded. We outline here how it can be shown that a field F of infinite transcendence degree over a subfield also admits $2^{2^{|F|}}$ field topologies which are not the suprema of locally bounded ring topologies.

We will use the same notation in this section as in §2. The basic idea will be to define a nonlocally bounded topology $\mathcal{T}'_S$ on F for each $S \in \mathcal{S}$ such that, for each $\alpha \in X$, $(\phi_n(\alpha))$ converges to zero in $\mathcal{T}'_S$ if $\alpha \in S$ and is bounded away from zero otherwise. One sees that this is precisely what made the proof of Theorem 2.1 work.

To get these topologies, we modify the inductive procedure of [5] somewhat. This procedure involved inductively building up a basis $V_0 \supseteq V_1 \supseteq \cdots$ of neighborhoods of zero for a ring topology $\mathcal{T}((a_n), (C_n))$ on F such that $a_n \in V_n$ for each n and such that $C_{n+1} \cdot V_{n+1} \subseteq V_n$ for each set C_{n+1} in a nested sequence $C_1 \subseteq C_2 \subseteq \cdots$ of subsets of F. This procedure can easily be generalized to yield a ring topology $\mathcal{T}((A_n), (C_n))$, where now the A_n's are subsets of F, and for each n, $A_n \subseteq V_n$.

In what follows, $K_0, K_1, \cdots$ will be a sequence of subfields of F defined by $K_0 = K$, and, for each $n \geq 0$, K_{n+1} equals the algebraic closure of $K_n(B_n)$

in F. We note that $K_n \subseteq K_{n+1}$, that $\bigcup_{n=0}^{\infty} K_n = F$, and that $\bigcup_{i=n}^{\infty} B_i$ is algebraically independent over K_n.

For each S in $\mathscr{S}$, we will take $\mathscr{T}'_S$ to be the inductive ring topology $\mathscr{T}((\phi_n(S)), (K_n))$. By this definition, it is automatically the case that for $\alpha \in S$, the sequence $(\phi_n(\alpha))$ converges to zero in $\mathscr{T}'_S$. The following lemma will assure that $(\phi_n(\alpha))$ is bounded from zero if α is not in S, and will assure that suprema of sets of these topologies are of the desired type.

LEMMA 3.1. *Let $\{V_n | n \geq 0\}$ denote the inductively generated basis for the inductive ring topology $\mathscr{T}'_S$. If x is in $V_n \cap (K_{m+1} \sim K_m)$, then x has the form $P(a_1, \cdots, a_k)$, where:*

(a) *$P(Y_1, \cdots, Y_k)$ is a polynomial with coefficients in K_m;*
(b) *The degree of P in any one of the indeterminates Y_i is at most 2^m; and*
(c) *The a_i's are all in $\bigcup_{j=1}^{m} \phi_j(S)$.*

PROOF. To define the topology $\mathscr{T}((\phi_n(S)), (K_n))$, one would modify the development in [5, §1] merely by putting an indeterminate for each element of $\phi_{k+1}(S)$ into the set W_{k+1}^{k+1} defined in [5, (1.10), p. 151], rather than just putting one indeterminate into this set. One may check that the lemmas in [5, §2], suitably modified, would still hold.

In particular, Lemma 2.2, (1.19), and (1.12) of [5] would assure that $x = P(a_1, \cdots, a_k)$, where, for some integer r, (i) $\{a_i | 1 \leq i \leq k\} \subseteq \bigcup_{j=1}^{r} \phi_j(S)$; (ii) $P(Y_1, \cdots, Y_k)$ has coefficients in K_r; (iii) the degree of P in any Y_i is at most 2^r; (iv) at least one a_i, say a_k, is in $\phi_r(S)$; and (v) the degree of $P(a_1, \cdots, a_{k-1}, Y_k)$ in Y_k is at least one. (Some of the details of the proof parallel those of [5, §4].) Thus, $\{a_i | 1 \leq i \leq k\} \subseteq \bigcup_{j=1}^{r} B_j$, while a_k and perhaps others of the a_i's are in B_r. It follows that $x \in K_{r+1}$. Since $a_k \in B_r$, and B_r is algebraically independent over K_r, it follows that $x \notin K_r$, for if it were then, by (v), $x = P(a_1, \cdots, a_k)$ would be an equation of algebraic dependence over K_r for elements in B_r, which is a contradiction. Thus, $x \in K_{r+1} \sim K_r$, so it follows that $r = m$. Statements (a)—(c) now follow from (i)—(iii).

As a consequence of Lemma 3.1, one has that if $\alpha \in X \sim S$, then $\phi_n(\alpha) \notin V_0$ for each n. Since $\phi_n(\alpha) \in B_n$, $\phi_n(\alpha) \in K_{n+1} \sim K_n$. Thus, if $\phi_n(\alpha) \in V_0$, then $\phi_n(\alpha) = P(a_1, \cdots, a_k)$, with P and the a_i's as specified in Lemma 3.1. Since $\phi_n(\alpha) \notin \bigcup_{j=1}^{n} \phi_j(S)$, (a) and (c) show that $\phi_n(\alpha) = P(a_1, \cdots, a_k)$ is a nontrivial equation of algebraic dependence for $\phi_n(\alpha)$, and perhaps other elements of B_n, over K_n, which is a contradiction.

THEOREM 3.2. *For distinct, nonempty subsets $\mathcal{O}$ of $\mathscr{S}$, the ring topologies $\mathscr{T}'_\mathcal{O} = \sup\{\mathscr{T}'_S | S \in \mathcal{O}\}$ are distinct, and fail to be suprema of locally bounded ring topologies on F. Thus, a field F of infinite transcendence degree over a subfield admits $2^{2^{|F|}}$ ring topologies of this latter type.*

PROOF. As we have noted, the fact that $(\phi_n(\alpha))$ converges to zero in $\mathcal{T}'_S$ if $\alpha \in S$ and is bounded from zero if $\alpha \notin S$ assures that the proof of Theorem 2.1 will adapt to prove the first assertion.

To prove the second assertion, we use a result of J. Heine [3] which says that $\mathcal{T}'_\mathcal{O}$ is not the supremum of a family of locally bounded ring topologies on F provided that there is a $\mathcal{T}'_\mathcal{O}$-neighborhood U of zero such that for every $\mathcal{T}'_\mathcal{O}$-neighborhood V of zero, there is an integer m such that $V^m \nsubseteq U$.

To get such a neighborhood U, we fix any $S \in \mathcal{O}$, and let $U = V_0$, the 0th neighborhood of zero in the inductively defined basis for $\mathcal{T}'_S$. Then U is also a $\mathcal{T}'_\mathcal{O}$-neighborhood of zero. If $V \in \mathcal{T}'_\mathcal{O}$ and $0 \in V$, then there are sets $S_1, S_2, \cdots, S_k$ in $\mathcal{O}$, and for each i a $\mathcal{T}'_{S_i}$-neighborhood W_i of zero such that $\bigcap_{i=1}^k W_i \subseteq V$. Since $\mathcal{S}$ has the f.i.p.c., there is an element α in $\bigcap_{i=1}^k S_i$. Then the sequence $(\phi_n(\alpha))$ converges to zero in each of the topologies $\mathcal{T}'_{S_i}$, so for a sufficiently large integer m, $\phi_m(\alpha) \in \bigcap_{i=1}^k W_i$.

We claim that $\phi_m(\alpha)^r \notin U = V_0$, where $r = 2^m + 1$. Suppose otherwise. Then since $\phi_m(\alpha) \in B_m$, $\phi_m(\alpha)^r \in V_0 \cap (K_{m+1} \sim K_m)$, so by Lemma 3.1, $\phi_m(\alpha)^r = P(a_1, \cdots, a_k)$, where P has coefficients in K_m, the a_i's are in K_m or in B_m, and the degree of P in each indeterminate is at most 2^m. This equation then gives a nontrivial equation of algebraic dependence for $\phi_m(\alpha)$, and perhaps other elements of B_m, over K_m, a contradiction.

We thus have that $V^r \nsubseteq U$, and the desired result is proven.

REMARK. We wish to note here that Theorem 3.2 can be strengthened to prove the existence of $2^{2^{|F|}}$ field topologies on F which are not suprema of locally bounded ring topologies. To do so, one needs to make more extensive modifications in the methods of [5]. Specifically, in order to build continuity for multiplicative inversion into an inductive topology, one must work with rational functions rather than polynomials, (such as appeared in Lemma 3.1). However, a presentation here of all the technical details is perhaps not justified for the sake of this relatively minor improvement in the result.

We conclude with two further questions of the type considered herein which to the writer's knowledge remain unanswered.

1. In the proof of Theorem 2.1, we displayed $2^{|F|}$ first countable, locally bounded field topologies (the maximum number possible) on a field F of infinite transcendence degree over a subfield. The only fields not of this type are those of finite transcendence degree over their prime subfields. How many first countable, locally bounded field topologies are there on a field of this latter type? It is known [5, Theorem 6.1, p. 159] that if F is algebraic over its finite prime subfield, then there are two, namely the discrete and indiscrete topologies. Otherwise, there are at least countably many determined by valuations. In particular, it would be of

interest to know how many there are on the rational numbers and on a simple transcendental extension of a field of prime order.

2. We have seen that on a field F of infinite transcendence degree, there are as many first countable, locally bounded field topologies as possible, namely $2^{|F|}$. Are there as many nonfirst countable, locally bounded ones as possible, namely $2^{2^{|F|}}$?

ADDED IN PROOF. Results close to those obtained herein have just recently been published by J. Heine [*Ringtopologien auf nichtalgebraischen Körpern*, Math. Ann. **199** (1972), 205–211]. By quite different means, he obtains, for any nonalgebraic field F, $2^{2^{|F|}}$ *ring* topologies which are, and $2^{2^{|F|}}$ *ring* topologies which are not suprema of locally bounded ones. Thus, this paper extends these results for most nonalgebraic fields by obtaining like numbers and types of *field* topologies. It does not appear that Heine's methods generalize to give field topologies. However, Podewski writes that he now has extended his methods to the uncountable case. Heine also points out in correspondence that question 2 above has an easy negative answer; since a locally bounded topology on F is uniquely determined by a bounded neighborhood of zero, there can be at most $2^{|F|}$ of them, whether or not they are first countable.

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DEPARTMENT OF MATHEMATICS, NORTHERN MICHIGAN UNIVERSITY, MARQUETTE, MICHIGAN 49855